BINARY FORMS AND HARMONIC POLYNOMIALS

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In this talk, we describe an explicit isomorphism between the space of harmonic polynomials in three variables of degree $n$ and binary forms of degree $2n$.

1. Harmonic polynomials

Let $S_n(\mathbb{R}^3)$ be the space of homogeneous polynomial of degree $n$ in $x, y, z$ over $\mathbb{R}$. Then $SO(3)$ acts on $S_n(\mathbb{R}^3)$ by sending $p$ to $p \circ g^{-1}$.

The Laplacian $\triangle$ is $SO(3)$-equivariant in the sense that

$$\triangle(fg^{-1}) = (\triangle(f))g^{-1}$$

for $g \in SO(3)$. This gives an $SO(3)$-equivariant morphism

$$\triangle : S_n(\mathbb{R}^3) \to S_{n-2}(\mathbb{R}^3)$$

with kernel $H_n(\mathbb{R}^3)$, the space of harmonic polynomials of degree $n$.

The representation $H_n(\mathbb{R}^3)$ is irreducible, and every irreducible representation of $SO(3)$ is isomorphic to some $H_n(\mathbb{R}^3)$. There is an $SO(3)$-invariant inner product on $S_n(\mathbb{R}^3)$:

$$\langle p_1, p_2 \rangle_n = (\triangle_{\alpha, \beta})^n p_1(X_\alpha)p_2(X_\beta) = \sum_{i,j,k} i! j! k! (p_1)_{i,j,k}, (p_2)_{i,j,k}$$

where $\triangle_{\alpha, \beta}$ is the polarization of the Laplacian operator, namely

$$\triangle_{\alpha, \beta} := \frac{\partial}{\partial x^\alpha} \frac{\partial}{\partial x^\beta} + \frac{\partial}{\partial y^\alpha} \frac{\partial}{\partial y^\beta} + \frac{\partial}{\partial z^\alpha} \frac{\partial}{\partial z^\beta}.$$ 

We have

$$\langle \triangle p_1, p_2 \rangle_n = \langle p_1, qp_2 \rangle_{n+2}$$

for $q = x^2 + y^2 + z^2$. We deduce thus that $H_n(\mathbb{R}^3)$ is the orthogonal complement of $qS_{n-2}(\mathbb{R}^3)$, so that

$$\dim H_n(\mathbb{R}^3) = \dim S_n(\mathbb{R}^3) - \dim S_{n-2}(\mathbb{R}^3) = 2n + 1.$$ 

Theorem 1.1. Every $p \in S_n(\mathbb{R}^3)$ admits a unique decomposition of the form

$$p = p_0 + qp_1 + \cdots + q^r p_r$$

where $r = \lfloor \frac{n}{2} \rfloor$ and where the $p_i$ are harmonic.
2. COMPLEXIFICATION

We extend scalars to \( \mathbb{C} \). On \( S_n(\mathbb{C}^3) \) we have an action of the complexification

\[
\text{SO}(3, \mathbb{C}) = \{ M \in M_3(\mathbb{C}) | M^tM = I, \det M = 1 \}.
\] (2.1)

We can also complexify our scalar product into a (hermitian) inner product

\[
\langle p_1, p_2 \rangle = \Delta_{\alpha, \beta}^n p_1(X_\alpha) \overline{p}_2(X_\beta) = \sum i! j! k! (p_1)_{i,j,k} (\overline{p}_2)_{i,j,k}
\] (2.2)

which now loses its good invariance properties.

Let \( \text{SL}_2(\mathbb{C}) = \{ \gamma \in M_2(\mathbb{C}) | \det \gamma = 1 \} \). We get the corresponding Lie algebra \( \mathfrak{sl}_2(\mathbb{C}) = \{ M \in M_2(\mathbb{C}) | \text{tr}(M) = 0 \} \).

A triple \((x, y, z)\) gives rise to the matrix

\[
M(x, y, z) = \begin{pmatrix} -z & x + iy \\ x - iy & z \end{pmatrix} \in \mathfrak{sl}_2(\mathbb{C}).
\] (2.3)

Note that \( \det(M) = -(x^2 + y^2 + z^2) \) and \( M(x)M(y)M(x) = 2\langle x, y \rangle \) \( i.e \ M : \mathbb{C}^3 \rightarrow \text{End}(\mathbb{C}^2) \) induces a representation of the Clifford algebra \( \text{Cl}_3(\mathbb{C}) \). Now consider the adjoint action of \( \text{SL}_2(\mathbb{C}) \) on \( \mathfrak{sl}_2(\mathbb{C}) \), which gives a map \( \text{SL}_2(\mathbb{C}) \rightarrow \text{End}(\mathfrak{sl}_2(\mathbb{C})) \) given by

\[
\pi : g \rightarrow \text{Ad}_g : M \mapsto gMg^{-1}.
\] (2.4)

Thus we can realize the universal cover of \( \text{SO}_3(\mathbb{C}) \) as \( \pi : \text{SL}_2(\mathbb{C}) \rightarrow \text{SO}_3(\mathbb{C}) \).

From any representation \( \rho \) of \( \text{SO}(3, \mathbb{C}) \) we obtain in this way a representation \( \rho \pi \) of \( \text{SL}_2(\mathbb{C}) \).

We can restrict this construction to

\[
\text{SU}(2) = \{ \gamma \in M_2(\mathbb{C}) | \overline{\gamma}^tM = 1 \text{ and } \det(M) = 1 \}
\] (2.5)

Using \( M \) restricted to \( \mathfrak{su}(2) \) \( i.e \) taking \( x, y, z \) purely imaginary, which correspond to \( \text{Pauli matrices} \), we thus get an arrow

\[
\pi : \text{SU}(2) \rightarrow \text{SO}(3, \mathbb{R}),
\] (2.6)

which is the universal cover of \( \text{SO}(3, \mathbb{R}) \).

The space \( \mathbb{C}^2 \) has a canonical volume form \( \omega(\xi_1, \xi_2) = \det(\xi_1, \xi_2) \). This gives an isomorphism \( \mathbb{C}^2 \rightarrow (\mathbb{C}^2)^\ast \) that sends \( \xi \) to \( \xi^\omega : \eta \mapsto \omega(\xi, \eta) \). Write \( \xi = (u, v)^t \) so that \( \xi^\omega = (-v, u) \). There is then a map

\[
\varphi : \mathbb{C}^2 \rightarrow \mathfrak{sl}_2(\mathbb{C})
\]

\[
\xi \mapsto \xi \xi^\omega = \begin{pmatrix} -uv & u^2 \\ -v^2 & uv \end{pmatrix}
\] (2.7)

We have \( \det(\xi \xi^\omega) = \text{tr}(\xi \xi^\omega) = 0 \). So the image of \( \varphi \) is contained in the subspace of \( \mathfrak{sl}_2(\mathbb{C}) \) where additionally the determinant is 0.

**Lemma 2.1.** The morphism \( \varphi \) is equivariant; \( \varphi(g \xi) = \text{Ad}_g \varphi(\xi) \) for all \( g \in \text{SL}_2(\mathbb{C}) \).
We also get a morphism
\[ \varphi^* : S_n(\mathbb{C}^3) \rightarrow S_{2n}(\mathbb{C}^2) \]
\[ p \mapsto p \circ \varphi \tag{2.8} \]

**Theorem 2.2.** The morphism \( \varphi^* \) is an equivariant isomorphism between \( H_n(\mathbb{C}^3) \) and \( S_{2n}(\mathbb{C}^2) \).

To see this, we have to show that \( \varphi^* : H_n(\mathbb{C}^3) \rightarrow S_{2n}(\mathbb{C}^2) \) is injective (because \( H_n(\mathbb{C}^3) \) and \( S_{2n}(\mathbb{C}^2) \) have same dimension). Consider the kernel of \( \varphi^* \). These are exactly the polynomials of the form \( qS_{n-2}(\mathbb{C}^3) \). So suppose that \( p \) is harmonic and \( p = qr \) for \( r \in S_{n-2}(\mathbb{C}^3) \). Then \( p \) is in fact zero, which concludes the proof.

**Remark 2.3.** Of course, if we admit that \( H_n(\mathbb{C}^3) \) and \( S_{2n}(\mathbb{C}^2) \) are irreducible representations of \( \text{SL}_2(\mathbb{C}) \), there is nothing to prove but that \( \varphi^* \) is not zero. Anyway, it is pedagogical to understand that, in this construction, \( \mathbb{C}^2 \) is a parametrization of the isotropic cone in \( \mathbb{C}^3 \) and that an harmonic polynomial on \( \mathbb{C}^3 \) which vanishes on the isotropic cone vanishes identically.

### 3. Explicit Isomorphisms

**From an harmonic polynomial \( H \) to a binary form \( f \).** The conversion is trivial. Just put
\[ f(u,v) = H \left( \frac{u^2 - v^2}{2}, \frac{u^2 + v^2}{2i}, uv \right). \tag{3.1} \]

**From a binary form \( f \) to an harmonic polynomial \( H \).** Let \( f \in S_{2n}(\mathbb{C}^2) \). Set
\[ u^{2n-k}v^k \mapsto \begin{cases} z^k(x + iy)^{n-k} & \text{if } 0 \leq k \leq n \\ z^{2n-k}(-x + iy)^{k-n} & \text{if } n \leq k \leq 2n. \end{cases} \tag{3.2} \]

We get \( \varphi^*(P) = f \). The solution is \( P_0 \), the orthogonal projection of \( P \) onto \( H_n(\mathbb{C}^3) \).

**Real harmonic polynomials.** Now let \( P \in H_n(\mathbb{R}^3) \subset H_n(\mathbb{C}^3) \). To which binary form does it correspond? If \( \varphi^*(P) = f \) and
\[ f(u,v) := \sum_{k=0}^{n} a_{2n-k}u^{2n-k}v^k, \]
we have \( a_{2n-k} = (-1)^{n-k}a_k \) or equivalently \( \tilde{f}(-v,u) = (-1)^nf(u,v) \).

**Remark 3.1.** The transvectant preserves this (real) subspace of \( S_{2n}(\mathbb{C}^2) \) corresponding to real harmonic polynomials.