Covariant algebra of the binary nonic and the binary decimic

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Théorie effective des invariants

Marseille, November 2014

Motivation

A famous result by Hilbert [Hil1890] states that the algebra of invariants for linear reductive group is finite.

For the action of $\mathrm{SL}_2(\mathbb{C})$ on binary forms, the result was already known since Gordan [Gor1868].

Many (heavy) calculations have been done at this time to compute these algebras, but after few decades, it became clear that it was hopeless to deal by hand with forms of degree > 8.

One century later, with the help of computers, we might think that it is possible to go further on the subject.

Notations

- A group $G \subset \mathrm{SL}_2(\mathbb{C})$ acts on $\mathbb{C}^2$ in the natural action:

  i.e. if $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in G$, then $g.(x,y) = (ax+by, cx+dy)$.

- Let S_n be the space of homogeneous polynomial functions of degree n in (x, y) .

- The group G acts on S_n: if $f \in S_n$ then $g.f$ is defined by

$$(g.f)(x,y) = f(g^{-1}.(x,y))\,.$$

Invariants

Definition

A homogeneous polynomial function $I : S_n \to \mathbb{C}$ is an invariant if for all $g \in G$ and all $f \in S_n$,

$$I(g.f) = I(f).$$

We note $\mathcal{I}_n$ the algebra of invariants $\mathbb{C}[S_n]^{\mathrm{SL}_2(\mathbb{C})}$.

Few examples

- S_2 : $\{a\,x^2 + 2b\,xy + c\,y^2 \;\; : a, b, c \in \mathbb{C}\}$, and

$$\mathcal{I}_2 = \mathbb{C}[b^2 - ac]\,.$$

- S_3 : $\{ax^3 + 3\,bx^2y + 3\,cxy^2 + dy^3 \;\; : a, b, c, d \in \mathbb{C}\}$, and

$$\mathcal{I}_3 = \mathbb{C}[(ad - bc)^2 - 4(ac - b^2)(bd - c^2)]\,.$$

- S_4 : $\{ax^4 + 4\,bx^3y + \cdots + ey^4 \;\; : a, \ldots, e \in \mathbb{C}\}$, and

$$\mathcal{I}_4 = \mathbb{C}[I, J] \text{ with } I = ae - 4bd + 3c^2,\;\; J = \begin{vmatrix} a & b & c \\ b & c & d \\ c & d & e \end{vmatrix}.$$

- S_5 : $\mathcal{I}_5 = \mathbb{C}[I_4, I_8, I_{12}, I_{18}]$ with the relation of degree 36,

$$16I_{18}^2 = I_4 I_8^4 + 8I_8^3 I_{12} - 2I_4^2 I_8^2 I_{12} - 72 I_4 I_8 I_{12}^2 - 432 I_{12}^3 + I_4^3 I_{12}^2\,.$$

Covariants

Definition

$C : S_n \oplus \mathbb{C}^2 \to \mathbb{C}$ is a covariant of degree d and *order* $o > 0$ if C is a homogeneous function on $S_n \oplus \mathbb{C}^2$ of degree (d, o) s.t.

$\forall g \in G$, $\forall f \in S_n$ and $\forall (x, y) \in \mathbb{C}^2$, we have

$$C(g.f, g.(x, y)) = C(f, (x, y))$$

The generic form $\sum_{i=0}^{n} a_i x^i y^{n-i}$ is a cov. of ord. n and deg. 1.

We note $\mathcal{C}_n$ the algebra of covariants $\mathbb{C}[S_n \oplus \mathbb{C}^2]^{\mathrm{SL}_2(\mathbb{C})}$.

Covariants of S_3

$\mathcal{C}_3$ is generated by 4 forms :

- The generic form, $f = ax^3 + 3\,bx^2y + 3\,cxy^2 + dy^3$.
- The hessian of f,
$$H = \frac{1}{36}\begin{vmatrix} f''_{x^2} & f''_{xy} \\ f''_{xy} & f''_{y^2} \end{vmatrix} = (ac - b^2)x^2 + (ad - bc)xy + (bd - c^2)y^2 \,.$$
- The Jacobian of f and H,
$$T = \frac{1}{3}(f'_x H'_y - f'_y H'_x) = (a^2d - 3abc + 2b^3)x^3 + ..$$
- The discriminant of H,
$$\Delta = (ad - bc)^2 - 4(ac - b^2)(bd - c^2)\,.$$

And we have the relation $\Delta f^2 = T^2 + 4H^3$.

The algebra of covariants

Since [Gor1868], it is known that $\mathcal{I}_n$ and $\mathcal{C}_n$ are finitely generated.

- $\mathcal{I}_n$ and $\mathcal{C}_n$ are known for $n \leqslant 6$ since the 19^{th} century.
- $\mathcal{I}_7$ is known since [vGall1888], and $\mathcal{C}_7$ since [Bed09].
- $\mathcal{I}_8$ is known since [Syl1879, vGall1880], and $\mathcal{C}_8$ since [Bed08].
- $\mathcal{I}_9$ and $\mathcal{I}_{10}$ known by [Cro02, BrPo10a, BrPo10b].

In this talk, we focus on $\mathcal{C}_9$ and $\mathcal{C}_{10}$.

Transvectants

Let $\Omega_{\alpha,\beta} : S_p \otimes S_q \to S_{p-1} \otimes S_{q-1}$ be the Cayley operator,

$$\Omega_{\alpha,\beta} = \begin{vmatrix} \frac{\partial}{\partial x_\alpha} & \frac{\partial}{\partial x_\beta} \\ \frac{\partial}{\partial y_\alpha} & \frac{\partial}{\partial y_\alpha} \end{vmatrix} = \frac{\partial^2}{\partial x_\alpha \partial y_\beta} - \frac{\partial^2}{\partial y_\alpha \partial x_\beta}\,.$$

Let μ be the trace operator

$$\mu\left(C(\,(x_\alpha, y_\alpha), (x_\beta, y_\beta))\,\right) = C((x, y), (x, y))\,.$$

Definition

The transvectant of level r of two forms $f \in S_n$ and $g \in S_p$ is

$$\{f, g\}_r := \frac{(n-r)!}{n!}\,\frac{(p-r)!}{p!}\,\mu \circ \Omega^r_{\alpha,\beta}(\,f(x_\alpha, y_\alpha)\,g(x_\beta, y_\beta)\,)\,.$$

$$\{f, g\}_0 = f\,g \in S_{n+p}\,,$$

$$\{f, g\}_1 = \frac{1}{np}\left(\frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial y}\right) \in S_{n+p-2}\,,$$

$$\{f, g\}_2 = \frac{(n-2)!}{n!}\frac{(p-2)!}{p!}\times \left(\frac{\partial^2 f}{\partial x^2}\frac{\partial^2 g}{\partial x^2} - 2\,\frac{\partial^2 f}{\partial x \partial y}\frac{\partial^2 g}{\partial x \partial y} + \frac{\partial^2 f}{\partial y^2}\frac{\partial^2 g}{\partial y^2}\right) \in S_{n+p-4}\,,$$

etc.

Starting from f, on can compute

$$\{f, f\}_2, \{f, f\}_4, \{f, f\}_6, \ldots \{f, \{f, f\}_{2i}\}_j,\ \textit{etc.}$$

The covariants of f are all linear combinations of these transvectants [Hil1893, Str1888] .

A first algorithm [Olver99, p. 144]

Input: d_{max}, a bound on the degree.

1. $\mathcal{G}_1 \leftarrow \{f\}$
2. For $d = 2, \ldots, d_{max}$:
   1. $\mathcal{G}_d \leftarrow \{\}$
   2. For each $C = \prod_{c \in \mathcal{G}_i} c$ s.t. $\deg C = d$:
      - If $C \notin \langle \mathcal{G}_d \rangle$ then $\mathcal{G}_d \leftarrow \mathcal{G}_d \cup \{C\}$
   3. $$B_{d-1} = \{C \mid C = \prod_{c \in \mathcal{G}_i,\ \operatorname{ord} c \neq 0,\ \deg c \geqslant 2} c \quad \text{and } \deg C = d-1\}$$
   4. For each $B \in B_{d-1}$, and each possible level r :
      - If $\{B, f\}_r \notin \langle \mathcal{G}_d \rangle$ then $\mathcal{G}_d \leftarrow \mathcal{G}_d \cup \{\{B, f\}_r\}$

For d_{max} "large enough", $\langle \cup \mathcal{G}_i \rangle = \mathcal{C}_d$,

but... none tight bound on d_{max} is known !

The (possibly incomplete) algebra $\mathcal{C}_9$

d/o	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	21	22	#	*Cum*
1	—	—	—	—	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	1	1
2	—	—	1	—	—	—	1	—	—	—	1	—	—	—	1	—	—	—	—	—	—	—	4	5
3	—	—	—	1	—	1	—	1	—	2	—	1	—	1	—	1	—	1	—	—	1	—	10	15
4	2	—	—	—	2	—	2	—	3	—	2	—	2	—	2	—	1	—	1	—	—	1	18	33
5	—	1	—	3	—	4	—	4	—	3	—	4	—	2	—	3	—	—	—	1	—	—	25	58
6	—	—	4	—	4	—	6	—	6	—	3	—	4	—	—	—	1	—	—	—	—	—	28	86
7	—	4	—	7	—	8	—	7	—	6	—	1	—	1	—	—	—	—	—	—	—	—	34	120
8	5	—	8	—	10	—	10	—	4	—	2	—	—	—	—	—	—	—	—	—	—	—	39	159
9	—	9	—	14	—	10	—	7	—	1	—	—	—	—	—	—	—	—	—	—	—	—	41	200
10	5	—	15	—	15	—	3	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	39	239
11	—	17	—	16	—	7	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	41	280
12	14	—	23	—	4	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	42	322
13	—	25	—	10	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	36	358
14	17	—	13	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	31	389
15	—	26	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	27	416
16	21	—	3	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	24	440
17	—	7	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	7	447
18	25	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	25	472
19	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	473
20	2	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	2	475
21	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	475
22	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	476
Tot	92	90	67	52	36	31	23	20	14	13	8	6	6	4	3	4	2	1	1	1	1	1	476	

Proven methods (in a nutshell)

- Hilbert approach :
  - Exhibit a Homogeneous System Of Parameters (HSOP) for the algebra (not easy !).
  - Check the dimensions up to some bound deduced from the Hilbert series of the algebra correctly read thanks to the HSOP.
- Gordan algorithm :
  - Derive from a covariant algebra of smaller degree ($\mathcal{C}_6$ for $\mathcal{C}_9$), transvectants to be computed,
  - Keep transvectants that correspond to minimal solutions of some linear integer equations.
  - Compute by linear algebra a basis for the spaces of given degree/order defined by these transvectants.

Gordan algorithm for $\mathcal{C}_9$

Compute by induction basis A_0, A_1, A_2 and A_3 for covariants that are complete up to the covariants of "grade" 2, 4, 6 and 8.

- Start from $A_0 = \{f\}$, set $H = \{f, f\}_2$.
- Consider transvectants of the form $\{f^a H^b, H^c\}_r$, find that

$$A_1 = \{f,\ H,\ T\} \text{ with } T = \{H, H\}_1 \text{ generate them .}$$

- A_2 is (similarly) given by

Cov.	f	$h_{10} := \{f, f\}_4$	$h_{14} := \{f, f\}_2$	$\{f, h_{10}\}_2$	$\{f, h_{10}\}_1$	$\{f, h_{14}\}_1$	$\{h_{10}, h_{14}\}_1$
Ord.	9	10	14	15	17	21	22
Deg.	1	2	2	3	3	3	4

Gordan algorithm for $\mathcal{C}_9$

- Start from gen. of $\mathcal{C}_6$. Let B_3 be its $26 - 5 = 21$ cov. evaluated at

$$C_{2,6} = \{f, f\}_6 \in S_6$$

- A_3 is then a basis for the transvectants of the form

$$\left\{ \prod_{h \in A_2} h^a, \prod_{C \in B_3} C^b \right\}_r$$

This is the difficult part !

- Few additional transvect. with $\{f, f\}_8$, and we finally get $\mathcal{C}_9$.

d/o	0	2	4	6	8	10	12	#	*Cum*
1	–	–	–	1	–	–	–	1	1
2	1	–	1	–	1	–	–	3	4
3	–	1	–	1	1	–	1	4	8
4	1	–	1	1	–	1	–	4	12
5	–	1	1	–	1	–	–	3	15
6	1	–	–	2	–	–	–	3	18
7	–	1	1	–	–	–	–	2	20
8	–	1	–	–	–	–	–	1	21
9	–	–	1	–	–	–	–	1	22
10	1	1	–	–	–	–	–	2	24
11	–	–	–	–	–	–	–	–	24
12	–	1	–	–	–	–	–	1	25
13	–	–	–	–	–	–	–	–	25
14	–	–	–	–	–	–	–	–	25
15	1	–	–	–	–	–	–	1	26
Tot	5	6	5	5	3	1	1	26	

Linear integer systems

From the constraints on the order of the operands of the transvect.

$$\left\{ \prod_{h_i \in A_2} h_i^{a_i}, \prod_{C_i \in B_3} C_i^{b_i} \right\}_r ,$$

we can associate an integer system $(\mathcal{S})$ $(a_i,\ b_i,\ u,\ v,\ r \geqslant 0)$

$$A_2 : 9\,a_1 + 10\,a_2 + 14\,a_3 + 15\,a_4 + 17\,a_5 + 21\,a_6 + 22\,a_7 = u + r$$

$$B_3 : 2\,(b_1 + b_2 + b_3 + b_4 + b_5 + b_6) + 4\,(b_7 + b_8 + b_9 + b_{10} + b_{11}) + 6\,(b_{12} + b_{13} + b_{14} + b_{15} + b_{16}) + 8\,(b_{17} + b_{18} + b_{19}) + 10\,b_{20} + 12\,b_{21} = v + r$$

Solutions form an additive monoïd: they can be given as linear combinations of a finite set of minimal solutions.

Only transvectants associated to these minimal solutions are useful to compute A_3.

Available tools for solving linear integer systems

Numerous works on the subject [CF90, CD91]

Further, there exist optimized and reliable implementations :

- NORMALIZ: used in MACAULAY,
  `http://www.home.uni-osnabrueck.de/wbruns/normaliz/`
- 4TI2: especially, the so-called command HILBERT,
  `http://www.4ti2.de/`

In our case, only 2 linear equations, the command HILBERT performs better than NORMALIZ.

But the system for $\mathcal{C}_9$ has so many minimal solutions that it is not over !

(we had to abort calculations after several days of computation)

Reduced linear integer systems

Let us regroup variables with same coefficients in $(\mathcal{S})$, *i.e.*

$$(\mathcal{B}) \begin{cases} \beta_1 = b_1 + b_2 + b_3 + b_4 + b_5 + b_6, \\ \beta_2 = b_7 + b_8 + b_9 + b_{10} + b_{11}, \\ \quad \dots \end{cases}$$

and consider the "reduced" system

$$(\tilde{\mathcal{S}}) \begin{cases} 9\,a_1 + 10\,a_2 + 14\,a_3 + 15\,a_4 + 17\,a_5 + 21\,a_6 + 22\,a_7 = u + r \\ 2\,\beta_1 + 4\,\beta_2 + 6\,\beta_3 + 8\,\beta_4 + 10\,\beta_5 + 12\,\beta_6 = v + r \end{cases}$$

Lemma

- *If $((a_i), (b_i), u, v, r)$ is a minimal solution of (S), then $((a_i), (\sum b_j)_i, u, v, r)$ is a minimal solution of $(\tilde{\mathcal{S}})$.*
- *Conversely, if $((a_i), (\beta_i), u, v, r)$ is a minimal solution for $(\tilde{\mathcal{S}})$, then $((a_i), (b_i), u, v, r)$ is a minimal solution of (S) for any solution (b_i) of $(\mathcal{B})$*

Application at $\mathcal{C}_9$

Computing minimal solutions to $(\tilde{\mathcal{S}})$ with the soft. 4TI2/HILBERT took only 25 seconds on a laptop.

We found 7338 solutions.

Lemma

Given some $\beta \geqslant 0$, the number of solutions $(b_i) \geqslant 0$ of

$$\beta = b_1 + b_2 + \ldots + b_\ell,$$

is equal to the binomial coefficient $\binom{\beta}{\beta+\ell-1}$.

So, the 7338 solutions of $(\tilde{\mathcal{S}})$ yield

... 58 525 823 minimal solutions of $(\mathcal{S})$.

(easily computed with a MAGMA script)

Spaces span by Gordan transvectants

The 58 525 823 minimal solutions yield transvectants that we can gather by degree d and order o : it covers 1836 couples (d, o).

d/o	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	...
1	–	–	–	–	–	–	–	–	–	✓	–	–	–	–	–	...
2	–	–	–	–	–	–	✓	–	–	–	✓	–	✓	–	–	...
3	–	–	–	✓	–	✓	–	✓	–	✓	–	✓	–	✓	–	...
4	–	–	–	–	✓	–	✓	–	✓	–	✓	–	✓	–	✓	...
5	–	✓	–	✓	–	✓	–	✓	–	✓	–	✓	–	✓	–	...
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	...
268	✓	–	✓	–	–	–	–	–	–	–	–	–	–	–	–	...
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	...
502	✓	–	–	–	–	–	–	–	–	–	–	–	–	–	–	...
506	✓	–	–	–	–	–	–	–	–	–	–	–	–	–	–	...
510	✓	–	–	–	–	–	–	–	–	–	–	–	–	–	–	...

(the last row corresponds to the min. solution/transvectant $\{C^2_{3,21}, C^{21}_{24,2}\}_{42}$)

Hilbert series

Let us consider the bigraded decomposition

$$\mathcal{C}_n = (\mathcal{C}_n)_{0,0} + (\mathcal{C}_n)_{1,0} + \ldots + (\mathcal{C}_n)_{d,o} + \ldots$$

Theorem ([Spr80])

The dimension of $(\mathcal{C}_n)_{d,o}$ is equal to the $\lfloor (nd-o)/2 \rfloor$-th coefficient of the power series expansion of

$$\frac{(1-q^n)\,(1-q^{n+1})\ldots(1-q^{n+d})}{(1-q^2)\ldots(1-q^d)}\,.$$

Very easy (and fast) to evaluate this series on modern computers.

In our case, we find for instance

$$\dim(\mathcal{C}_9)_{501,0} = 14\ 510\,116\,319$$

which is far too large to be checked !

Some bounds

Now, several bounds help to simplify a lot the computations.

1. By a result given in [GY1903], we can restrict for $\mathcal{C}_9$ to

$$\text{orders} \leqslant 22$$

2. We can derive from Hilbert series
   - & the HSOP known for $\mathcal{I}_9$ by [BrPo10a]
   - & the Cohen-Macaulay behavior of the sub-algebras of covariants of orders $\leqslant 22$,

   the following degree bounds

Ord.	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22
Deg.	66	61	64	63	62	63	64	63	62	65	64	63	62	63	64	63	62	63	64	63	62	63	62

So, $(\mathcal{C}_9)_{64,18}$ is one of the largest case that remains.

$$\dim(\mathcal{C}_9)_{64,18} = 1\,576\,149$$

is much smaller than $\dim(\mathcal{C}_9)_{501,0}$, but still too large...

Relations between covariants of B_3

Let us order the covariants of B_3 (first by deg. then by ord., inv. last) ,

$$C_{24,2} > C_{20,2} > C_{18,4} > C_{16,2} > \dots > C_{4,8} > C_{2,6} >$$
$$C_{30,0} > C_{10,0} > C_{12,0} > C_{8,0} > C_{4,0} \,.$$

We found 18 probable relations of the form

$$C_{d,o}^{e} = \sum \prod_{C < C_{d,o}} C$$

(where the power e goes from 2 for $C_{24,2}$ up to $e = 9$ for $C_{6,8}$)

and for any $C1 > C2$ several hundred probable relations of the form

$$C1^{e_1} \times C2^{e_2} = \sum \prod_{C < C2} C \,.$$

To fill A_3, we can thus discard any transvectant of the form

$$\left\{ \prod_{h_i \in A_2} h_i^{a_i}, C1^{e_1} \times C2^{e_2} \prod_{C_i \in B_3} C_i^{b_i} \right\}_r \,.$$

Dimensions

Thanks to the degree/order bounds and the relations for B_3,

only 235 493 transvectants remain (vs. 58 525 823 previously).

It decreases the number of spaces to be $(\mathcal{C}_9)_{d,o}$ to be tested to 633 (vs. 1836 previously).

The largest one is $(\mathcal{C}_9)_{60,14}$, about 2 times smaller than $(\mathcal{C}_9)_{66,18}$,

$$\dim(\mathcal{C}_9)_{60,14} = 872\,368,$$

but still too large...

Quotient with free invariants

An additional (classical) improvement can simplify a lot the calculations.

Let j_4, A_4 and j_8 be the first three invariants for $\mathcal{I}_9$ as defined in [BrPo10a], and instead of $(\mathcal{C}_9)_{d,o}$, consider the quotient

$$\mathcal{Q}_{d,o} := (\mathcal{C}_9)_{d,o} \,/\, \left(j_4\,(\mathcal{C}_9)_{d-4,o} + A_4\,(\mathcal{C}_9)_{d-4,o} + j_8\,(\mathcal{C}_9)_{d-8,o}\right).$$

Note that working in $\mathcal{Q}_{d,o}$ amounts to evaluate invariants at forms f that zeroify j_4, A_4 and j_8.

Furthermore,

$$\dim \mathcal{Q}_{d,o} = \dim(\mathcal{C}_9)_{d,o} \;-\; 2\dim {\mathcal{C}_9}_{d-4,o} + 2\dim {\mathcal{C}_9}_{d-12,o} \;-\; \dim {\mathcal{C}_9}_{d-16,o}\,.$$

Typically, for $(\mathcal{C}_9)_{60,14}$, we find

$$\dim(\mathcal{Q})_{60,14} = 33\,360\,,$$

which *seems* to be affordable

Compute dimensions

To check that the dimension of some $(\mathcal{C}_9)_{d,o}$ is equal to some D, we proceed as follows.

1. Select $(1 + 10\%) \times D$ random covariants $(\mathfrak{c}_i)$ in $(\mathcal{C}_9)_{d,o}$.
2. Evaluate these $\mathfrak{c}_i$ at $D/(o+1) + O(1)$ forms $(\mathfrak{f}_j)$ chosen at random (in the zero set of j_4, A_4 and j_8),
   this yields a matrix $\mathcal{M} = (\, \mathfrak{c}_i(\mathfrak{f}_j)\,)_{i,j}$.
3. Compute the "parity-check" matrix $\mathcal{M}^\top$ of $\mathcal{M}$.
   ($i.e$ $\mathbf{c} \times \mathcal{M}^\top = \mathbf{0}$ if $\mathbf{c} = \mathcal{M} \times \mathbf{v}$ for some vector $\mathbf{v}$)
4. While the rank of $\mathcal{M}$ is $< D$,
   1. look for a $\mathfrak{c} \in (\mathcal{C}_9)_{d,o}$ s.t. $\mathcal{M}^\top \times (\, \mathfrak{c}(\mathfrak{f}_j)\,) \neq 0$,
   2. update $\mathcal{M}^\top$.

Implementation tricks

Computations can be done modulo a small prime p.
Set $p = 65521$ did it for $\mathcal{C}_9$.

(if the images of the invariants under reduction mod p are independent, then the invariants are independent)

The algorithm is straightforward enough to be easily implemented and optimized in language C.

Most of the algorithm is highly parallelizable: give one covariant $\mathfrak{c}$ to each node, both

- while computing $\mathcal{M}$,
- or looking for a covariant $\mathfrak{c}$ s.t. $\mathcal{M}^\top \times (\, \mathfrak{c}(\mathfrak{f}_j)\,) \neq 0$.

Focus on $(\mathcal{C}_9)_{60,14}$

There are 1 271 052 031 in covariants in $(\mathcal{C}_9)_{60,14}$!

With 36 700 covariants randomly selected in $(\mathcal{C}_9)_{60,14}$ in Phase 1,

- the rank of the matrix $\mathcal{M}$ is much smaller than expected,

$$\operatorname{rank} \mathcal{M} \simeq 25\,000 \,,$$

- many new covariants are found in Phase 2 and most of the time is spent to update $\mathcal{M}^\top$ (which is not parallelized).

We aborted the computation after one week !

To overcome this issue, we restrict the choice of random covariants in Step 1 to products that do not contain invariants,

$$\mathfrak{c} = \prod_{\substack{c \in (\mathcal{C}_9)_{d',o'},\ o' \neq 0 \\ d' \leqslant d\,,\ o' \leqslant o}} c$$

For $(\mathcal{C}_9)_{60,14}$, there are "only" 28 402 307 such covariants...

...and with this heuristic, we find $\operatorname{rank} \mathcal{M} \simeq 33\,350$.

Our basis for $\mathcal{C}_9$ is minimal and complete

Finally, we are left with 633 spaces $(\mathcal{C}_9)_{d,o}$, the dimension of which must be checked vs. what is predicted by the Hilbert series.

These dimensions goes from 1 for $(\mathcal{C}_9)_{1,9}$ to 33 360 for $(\mathcal{C}_9)_{60,14}$.

The whole computation took less than one day on a DELL computer with 32 processors (1400MHz AMD Opteron)

For instance, for $(\mathcal{C}_9)_{60,14}$, it took 3 hours on one processor :

- 2 hours to compute the matrix $\mathcal{M}^\top$ in Phase 1 (its rank was 33 359),
- and 1 hour to find a free covariant in Phase 2.

$\mathcal{C}_{10}$

We proved also that this table is complete for $\mathcal{C}_{10}$.

Again, the main difficulty is the computation of A_3, except that we have to deal with $69 - 9 = 60$ covariants for $\mathcal{C}_8$, of order 2, 4, 6, 8, 10, 12, 14, 18 (instead of the 21 cov. of $\mathcal{C}_6$)

d/o	0	2	4	6	8	10	12	14	16	18	20	22	24	26	#	*Cum*
1	–	–	–	–	–	1	–	–	–	–	–	–	–	–	1	1
2	1	–	1	–	1	–	1	–	1	–	–	–	–	–	5	6
3	–	1	–	2	1	1	2	1	1	1	1	–	1	–	12	18
4	1	–	3	1	3	3	2	3	1	2	1	1	–	1	22	40
5	–	3	3	4	5	4	5	2	4	–	2	–	–	–	32	72
6	4	2	5	8	6	8	2	4	–	1	–	–	–	–	40	112
7	–	7	10	8	12	2	4	–	1	–	–	–	–	–	44	156
8	5	8	11	15	4	7	–	1	–	–	–	–	–	–	51	207
9	5	13	19	8	7	–	1	–	–	–	–	–	–	–	53	260
10	8	20	13	13	–	1	–	–	–	–	–	–	–	–	55	315
11	8	18	21	–	1	–	–	–	–	–	–	–	–	–	48	363
12	12	30	1	2	–	–	–	–	–	–	–	–	–	–	45	408
13	15	16	2	–	–	–	–	–	–	–	–	–	–	–	33	441
14	13	17	–	–	–	–	–	–	–	–	–	–	–	–	30	471
15	19	–	1	–	–	–	–	–	–	–	–	–	–	–	20	491
16	5	3	–	–	–	–	–	–	–	–	–	–	–	–	8	499
17	5	–	–	–	–	–	–	–	–	–	–	–	–	–	5	504
18	1	1	–	–	–	–	–	–	–	–	–	–	–	–	2	506
19	2	–	–	–	–	–	–	–	–	–	–	–	–	–	2	508
20	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	508
21	2	–	–	–	–	–	–	–	–	–	–	–	–	–	2	510
Tot	106	139	90	61	40	27	17	11	8	4	4	1	1	1	510	

Some details for $\mathcal{C}_{10}$

The integer system $(\mathcal{S})$ is

$$A_2 : 10\,a_1 + 12\,a_2 + 16\,a_3 + 18\,a_4 + 20\,a_5 + 24\,a_6 + 26\,a_7 = u + r$$

$$\begin{aligned} B_3 : 2\,(b_1 + \ldots + b_{14}) + 4\,(b_{15} + \ldots + b_{27}) + 6\,(b_{28} + \ldots + b_{39}) + \\ 8\,(b_{40} + \ldots + b_{45}) + 10\,(b_{46} + \ldots + b_{52}) + 12\,(b_{53} + b_{54} + b_{55}) + \\ 14\,(b_{56} + b_{57} + b_{58}) + 18\,(b_{59} + b_{60}) = v + r \end{aligned}$$

It took $\simeq 3$ mn to find the 8985 min. solutions of $(\tilde{\mathcal{S}})$, which in return yield 1 345 290 951 minimal solutions for $(\mathcal{S})$.

But, using the bound that we have for $\mathcal{C}_{10}$,

we finally arrive at 588 spaces $(\mathcal{C}_{10})_{d,o}$ to be checked.

The largest one is $(\mathcal{C}_{10})_{46,20}$, which is only of dimension 26323 if we work modulo the inv. of deg. 2, 4, 6 and 6 given in [BrPo10b].

All in all, it was easier than $\mathcal{C}_9$,

. . . it took less than one day on a laptop.

Thank you !

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