STABILITY OF THE TANGENT BUNDLE OF G/P IN POSITIVE CHARACTERISTICS

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ABSTRACT. Let G be an almost simple simply-connected affine algebraic group over an algebraically closed field k of characteristic $p > 0$. If G has type B_n, C_n or F_4, we assume that $p > 2$, and if G has type G_2, we assume that $p > 3$. Let $\mathsf{P} \subset \mathsf{G}$ be a parabolic subgroup. We prove that the tangent bundle of G/P is Frobenius stable with respect to the anticanonical polarization on G/P.

1. INTRODUCTION

Let us recall the notion of slope stability of a sheaf over a polarized projective scheme. The slope of a sheaf $\mathscr{F}$ is defined as the quotient of its degree by its rank: it will be denoted by $\mu(\mathscr{F})$. A sheaf $\mathscr{F}$ is called *stable* (respectively, *semi-stable*) if for any strict subsheaf $\mathscr{G}$ we have $\mu(\mathscr{G}) < \mu(\mathscr{F})$ (respectively, $\mu(\mathscr{G}) \leq \mu(\mathscr{F})$). Throughout, (semi)stability will mean slope (semi)stability.

Let G be an almost simple simply-connected affine algebraic group over an algebraically closed field k, and let $\mathsf{P} \subset \mathsf{G}$ be a parabolic subgroup. If the characteristic char(k) is zero, then it is known that the tangent bundle of G/P is stable with respect to the anticanonical polarization on G/P.

In fact, if G/P is a Hermitian symmetric space, the result goes back to the sixties [Ram66]. In the complex case, it was proved long ago that this bundle admits a Kähler-Einstein metric (see [Ko55] or [Be87, Chapter 8]), which implies polystability. Simplicity of this bundle was proved in [AB10], proving the stability; A. Boralevi proved stability of $T(\mathsf{G}/\mathsf{P})$ when G is of type ADE [Bor12, Theorem C].

Our aim here is to address stability of $T(\mathsf{G}/\mathsf{P})$ in the case where char(k) is positive. If G is of type B_n, C_n or F_4, we assume that char$(k) > 2$; if G is of type G_2, we assume that char$(k) > 3$. The main Theorem of this note says that under the above assumption, the tangent bundle of G/P and all its iterated Frobenius pull-backs are stable with respect to the anticanonical polarization on G/P.

The method of proof of the main Theorem is as follows. We prove that the stability of $T(\mathsf{G}/\mathsf{P})$ is equivalent to a certain statement on the quotient Lie(G)/Lie(P) considered as a P–module. The statement in question is shown to be independent of the characteristic of k (as long as the above assumptions hold). Finally, the main Theorem follows from the fact that $T(\mathsf{G}/\mathsf{P})$ is stable if char$(k) = 0$.

A natural question to ask is whether $T(\mathsf{G}/\mathsf{P})$ remains stable with respect to polarizations on G/P other than the anticanonical one. A. Boralevi gave a negative answer to this question. She constructed examples of G/P and polarizations on them with respect to which $T(\mathsf{G}/\mathsf{P})$ is not even semi-stable [Bor12, Theorem D].

Another natural question is to understand what happens when one relaxes the hypothesis on the characteristic. We are quite far from having a complete answer to this question. However, in the last section, we give an example of a homogeneous space in type C_n and in characteristic 2 whose tangent bundle is not stable, but semi-stable. We have not been able to find any example of a tangent bundle which is not semi-stable.

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2. NOTATIONS AND STATEMENT OF THE MAIN THEOREM

Let G be an almost simple simply-connected affine algebraic group defined over an algebraically closed field k. Let $\mathsf{P} \subsetneq \mathsf{G}$ be a parabolic subgroup. The Lie algebra of G, P will be denoted by $\mathfrak{g}, \mathfrak{p}$. The nilpotent ideal of $\mathfrak{p}$ will be denoted by $\mathfrak{n}$.

Fix a maximal torus $\mathsf{T} \subset \mathsf{G}$ and a Borel subgroup B. Assume $\mathsf{T} \subset \mathsf{B} \subset \mathsf{P}$. Let R denote the set of roots of $\mathfrak{g}$. The set of positive (respectively, negative) roots of $\mathfrak{g}$ will be denoted by R^+ (respectively, R^-). The eigenspace corresponding to any $\alpha \in R$ will be denoted by $\mathfrak{g}^\alpha$.

A subsheaf $E \subset T(\mathsf{G}/\mathsf{P})$ is called G-*stable* if it is preserved by the left action of G on $T(\mathsf{G}/\mathsf{P})$. Since the left translation action of G on G/P is transitive, any G-stable subsheaf of $T(\mathsf{G}/\mathsf{P})$ is a subbundle. Sheaves with a G-action are called linearized sheaves. For a coherent sheaf on G/P, there is an open subset where it is free as a $\mathcal{O}_{\mathsf{G}/\mathsf{P}}$-module. Therefore, linearized sheaves are locally free. Moreover, we recall the well-known correspondence between G-linearized sheaves and P-modules (details can be found in [Bri09, 2.1] and [Jan03, 5.9]):

Proposition 2.1. *There is an equivalence of categories between* G*-linearized vector bundles on* G/P *and* P*-modules. On the one hand, to a* G*-linearized vector bundle* E*, one can associate the fiber* E_o *at the* P*-stable point. On the other hand, to a* P*-module* M*, one can associate the bundle* $(\mathsf{G} \times M)/\mathsf{P}$ *over* G/P.

If E is the G-linearized vector bundle corresponding to a P-module M, since this correspondence is functorial, it induces a correspondence between G-linearized subbundles of E and P-submodules of M.

We now impose the following assumptions on the characteristic of the field k ($\alpha^\vee$ is the coroot corresponding to α):

Working assumption.

- *The characteristic* $\mathrm{char}(k)$ *of* k *is positive, and*
- $\mathrm{char}(k)$ *is bigger than all the coefficients* $|\langle \alpha^\vee, \beta \rangle|$ *for all roots* α, β *of* G *with* $\alpha \neq \pm\beta$.

In other words, if the root system of G is simply-laced, then $\mathrm{char}(k)$ is only assumed to be positive; if G is any of B_n, C_n and F_4, we assume that $\mathrm{char}(k) > 2$; if $\mathsf{G} = G_2$, we assume that $\mathrm{char}(k) > 3$. Recall that a bundle is said to be *Frobenius stable* with respect to a given polarization if it is stable and all its iterated Frobenius pull-backs are again stable.

Main Theorem. *Under the previous assumption, the tangent bundle* $T(\mathsf{G}/\mathsf{P})$ *is Frobenius stable with respect to the anticanonical polarization on* G/P.

The rest of the article is devoted to the proof of this theorem. This is essentially given by reduction to characteristic zero. This reduction is achieved using the following construction: let $\mathsf{G}_{\mathbb{Z}}$ be the split simply-connected Chevalley group scheme over $\mathbb{Z}$ having the same root system as G. By the theory of reductive algebraic group schemes, as the root system characterizes simply-connected groups up to isomorphism, we have $\mathsf{G} \simeq \mathsf{G}_{\mathbb{Z}} \otimes \mathit{Spec}\, k$. On the other hand, we denote $\mathsf{G}_{\mathbb{Z}} \otimes \mathit{Spec}\, \mathbb{C}$ by $\mathsf{G}_{\mathbb{C}}$, and we denote by $\mathfrak{g}_{\mathbb{C}}$ its Lie algebra. There exists a parabolic group $\mathsf{P}_{\mathbb{Z}} \subset \mathsf{G}_{\mathbb{Z}}$ such that $\mathsf{P}_{\mathbb{Z}} \otimes \mathit{Spec}\, k$ is conjugate to P. The parabolic subgroup $\mathsf{P}_{\mathbb{Z}} \otimes \mathit{Spec}\, \mathbb{C}$ of $\mathsf{G}_{\mathbb{C}}$ will be denoted by $\mathsf{P}_{\mathbb{C}}$.

3. PROOF OF THE MAIN RESULT

The set of roots α such that $\mathfrak{g}^\alpha \subset \mathfrak{p}$ will be denoted by $I(\mathsf{P})$. Let $x_0 \in \mathsf{G}/\mathsf{P}$ denote the base point. We have

$$T_{x_0}(\mathsf{G}/\mathsf{P}) \simeq \mathfrak{g}/\mathfrak{p} \simeq \bigoplus_{\alpha \in R \setminus I(\mathsf{P})} \mathfrak{g}^\alpha.$$

Thus, the vector space $\bigoplus_{\alpha \in R \setminus I(\mathsf{P})} \mathfrak{g}^\alpha$, has a natural P-module structure, which is the one we consider in the following lemma.

Lemma 3.1. *Let $I \subset R \setminus I(\mathsf{P})$ be a set of negative roots. Then the sum $M(I) := \bigoplus_{\alpha \in I} \mathfrak{g}^\alpha$ is a P-stable submodule of $\bigoplus_{\alpha \in R \setminus I(\mathsf{P})} \mathfrak{g}^\alpha$ if, and only if,*

$$\forall\, \beta \in I(\mathsf{P}),\ \forall\, \alpha \in I,\ \alpha + \beta \in R \setminus I(\mathsf{P}) \Longrightarrow \alpha + \beta \in I. \tag{1}$$

Proof. Take $\alpha \in I$ and $\beta \in I(\mathsf{P})$ such that $\alpha + \beta \in R \setminus I(\mathsf{P})$. In particular, we have $\beta \neq \pm\alpha$. Since G is simply-connected, $\mathfrak{g}$ is the Lie algebra defined by Serre's relations (this is explained for example in [CR10, Remark 2.2.3]), so we can choose a basis of $\mathfrak{g}$ such that the coefficients of the Lie bracket are those of the Chevalley basis [Ca72]. Consider the biggest integer p such that $\alpha - p\beta \in R$. This p is smaller than the length of the β-string of roots through α minus 1 (since $\alpha + \beta \in R$), and thus, by the working Assumption, we have $p \leq \mathrm{char}(k) - 2$. This implies that $p + 1 < \mathrm{char}(k)$. It now follows from [Ca72, Theorem 4.2.1] that $[\mathfrak{g}^\beta, \mathfrak{g}^\alpha] = \mathfrak{g}^{\alpha+\beta}$. Assuming that $M(I)$ is P-stable, we have it to be $\mathfrak{p}$-stable, and therefore $\alpha + \beta \in I$.

On the other hand, let $U_\beta \subset \mathsf{G}$ be the one-parameter additive subgroup corresponding to the root β. Since $U_\beta \cdot \mathfrak{g}^\alpha \subset \bigoplus_{k \geq 0} \mathfrak{g}^{\alpha + k\beta}$, from (1) it follows that $M(I)$ is U_β-stable for any root β in $I(\mathsf{P})$, and thus $M(I)$ is P-stable. □

The anticanonical line bundles of G/P and $\mathsf{G}_\mathbb{C}/P_\mathbb{C}$ are ample. Fix the anticanonical polarization on G/P and also on $\mathsf{G}_\mathbb{C}/P_\mathbb{C}$.

Proposition 3.2. *Let $E \subset T(\mathsf{G}/\mathsf{P})$ be a G-stable subbundle of $T(\mathsf{G}/\mathsf{P})$. There exists a subbundle $E_\mathbb{C} \subset T(\mathsf{G}_\mathbb{C}/\mathsf{P}_\mathbb{C})$ such that $\mathrm{rk}(E_\mathbb{C}) = \mathrm{rk}(E)$ and $\deg(E_\mathbb{C}) = \deg(E)$.*

Proof. Under the correspondence of Proposition 2.1, let M be the P-submodule of $\bigoplus_{\alpha \notin I(\mathsf{P})} \mathfrak{g}^\alpha$ corresponding to E. Since M is a T-stable subspace of $\bigoplus_{\alpha \notin I(\mathsf{P})} \mathfrak{g}^\alpha$, there is a subset $I(M) \subset R \setminus I(\mathsf{P})$ such that $M = \bigoplus_{\alpha \in I(M)} \mathfrak{g}^\alpha$. By Lemma 3.1, we have

$$\forall\, \beta \in I(\mathsf{P}),\ \forall\, \alpha \in I(M),\ \alpha + \beta \in R \setminus I(\mathsf{P}) \Longrightarrow \alpha + \beta \in I(M).$$

Thus, $M_\mathbb{C} := \bigoplus_{\alpha \in I(M)} \mathfrak{g}^\alpha_\mathbb{C}$ is a $\mathsf{P}_\mathbb{C}$-submodule of $\bigoplus_{\alpha \notin I(\mathsf{P})} \mathfrak{g}^\alpha_\mathbb{C}$ and the subbundle $E_\mathbb{C} \subset T(\mathsf{G}_\mathbb{C}/\mathsf{P}_\mathbb{C})$ corresponding to $M_\mathbb{C}$ has the same rank as E.

Note that there is a corresponding vector bundle $E_\mathbb{Z}$ over $\mathsf{G}_\mathbb{Z}/\mathsf{P}_\mathbb{Z}$, since the P-module $\bigoplus_{\alpha \in I(M)} \mathfrak{g}^\alpha$ is defined over $\mathbb{Z}$. Since this is a flat bundle, we get that $\deg(E_\mathbb{C}) = \deg(E)$. □

Lemma 3.3. *The tangent bundle $T(\mathsf{G}/\mathsf{P})$ is polystable.*

Proof. Let E be the first term of the Harder-Narasimhan filtration of $T(\mathsf{G}/\mathsf{P})$. First assume $E \neq T(\mathsf{G}/\mathsf{P})$, so

$$\mu(E) > \mu(T(\mathsf{G}/\mathsf{P})). \tag{2}$$

Since the anticanonical polarization of G/P is fixed by G, from the uniqueness of the Harder-Narasimhan filtration it follows that E is G-stable. By Proposition 3.2 and stability of $T(\mathsf{G}_\mathbb{C}/\mathsf{P}_\mathbb{C})$ in characteristic 0 [AB10, Theorem 2.1], we thus have $\mu(E) < \mu(T(\mathsf{G}/\mathsf{P}))$ which contradicts (2). So $T(\mathsf{G}/\mathsf{P})$ is semi-stable.

We can then similarly argue with the polystable socle (cf. [HL97, page 23, Lemma 1.5.5]) of $T(\mathsf{G}/\mathsf{P})$ to deduce that $T(\mathsf{G}/\mathsf{P})$ is polystable. □

Since $T(\mathsf{G}/\mathsf{P})$ is polystable, it is isomorphic to

$$\bigoplus_{i=1}^{r} E_i^{\oplus m_i},$$

such that

- each E_i is stable with $\mu(E_i) = \mu(T(\mathsf{G}/\mathsf{P}))$,
- $m_i \geq 1$, and
- $E_i \neq E_j$ if $i \neq j$.

We note that the isomorphism classes of $E_1, \ldots, E_r$ are unique up to permutations of $\{1, \ldots, r\}$. Let

$$Hom(E_i, T(\mathsf{G}/\mathsf{P})) = H^0(\mathsf{G}/\mathsf{P}, T(\mathsf{G}/\mathsf{P}) \otimes E_i^\vee)$$

be the space of homomorphisms. Now consider the natural homomorphism

$$\bigoplus_{i=1}^{r} Hom(E_i, T(\mathsf{G}/\mathsf{P})) \otimes E_i \longrightarrow T(\mathsf{G}/\mathsf{P}) \tag{3}$$

that sends any $s \otimes v$, where $s \in Hom(E_i, T(\mathsf{G}/\mathsf{P}))$ and $v \in (E_i)_x$ to $s(x)(v) \in T(\mathsf{G}/\mathsf{P})_x$. Since $Hom(E_i, E_j) = 0$ if $i \neq j$, and $Hom(E_i, E_i) = k$, it follows that the homomorphism in (3) is an isomorphism.

Lemma 3.4. *Take any $g \in \mathsf{G}$ and integer $1 \leq j \leq r$. Then $g^* E_j \simeq E_j$ as vector bundles on G/P.*

Proof. Let $\phi : \mathsf{G} \times (\mathsf{G}/\mathsf{P}) \longrightarrow \mathsf{G}/\mathsf{P}$ be the left-translation action. Let $p_2 : \mathsf{G} \times (\mathsf{G}/\mathsf{P}) \longrightarrow \mathsf{G}/\mathsf{P}$ be the projection to the second factor. The action ϕ produces an isomorphism of vector bundles

$$\Phi : \bigoplus_{i=1}^{r} Hom(E_i, T(\mathsf{G}/\mathsf{P})) \otimes \phi^* E_i = \phi^* T(\mathsf{G}/\mathsf{P}) \longrightarrow p_2^* T(\mathsf{G}/\mathsf{P}) = \bigoplus_{i=1}^{r} Hom(E_i, T(\mathsf{G}/\mathsf{P})) \otimes p_2^* E_i. \tag{4}$$

For $i \neq \ell$, as E_i and E_l are stable of the same slope, we have

$$Hom((\phi^* E_i)|_{\{e\} \times \mathsf{G}/\mathsf{P}}, (p_2^* E_\ell)|_{\{e\} \times \mathsf{G}/\mathsf{P}}) = Hom(E_i, E_\ell) = 0.$$

Hence, using the semi-continuity of the function $(g_1, g_2) \mapsto \dim\ Hom((\phi^* E_i)|_{\{g_1\} \times \mathsf{G}/\mathsf{P}}, (p_2^* E_\ell)|_{\{g_2\} \times \mathsf{G}/\mathsf{P}})$, we get

$$Hom(\phi^* E_i, p_2^* E_\ell) = 0. \tag{5}$$

From (5) it follows immediately that Φ in (4) takes $Hom(E_i, T(\mathsf{G}/\mathsf{P})) \otimes \phi^* E_i$ to itself for every $1 \leq i \leq r$. In particular, we have $Hom(E_j, T(\mathsf{G}/\mathsf{P})) \otimes \phi^* E_j \simeq Hom(E_j, T(\mathsf{G}/\mathsf{P})) \otimes p_2^* E_j$. Fix $g \in \mathsf{G}$: restricting to $\{g\} \times \mathsf{G}/\mathsf{P}$, we get

$$Hom(E_j, T(\mathsf{G}/\mathsf{P})) \otimes g^* E_j \simeq Hom(E_j, T(\mathsf{G}/\mathsf{P})) \otimes E_j. \tag{6}$$

Since E_j is stable, we know that $g^* E_j$ is indecomposable. Now in view of the uniqueness of the decomposition into a direct sum of indecomposable vector bundles (see [At56, p. 315, Theorem 2]), from (6) we conclude that $g^* E_j \simeq E_j$. $\square$

Lemma 3.5. *For all $j \in [1, r]$, the vector bundle E_j is G-linearized.*

Proof. Fix an integer $1 \leq j \leq r$. We now introduce the group of symmetries of the vector bundle E_j. Let $\widetilde{G}$ denote the set of pairs (g, h), where $g \in \mathsf{G}$ and $h \in Aut(E_j)$, such that the diagram

$$\begin{tikzcd}
E_j \arrow[r, "h"] \arrow[d] & E_j \arrow[d] \\
\mathsf{G}/\mathsf{P} \arrow[r, "g"'] & \mathsf{G}/\mathsf{P}
\end{tikzcd}$$

commutes. Since E_j is simple, $Aut_{\mathsf{G}/\mathsf{P}}(E_j) \simeq \mathbb{G}_m$, and therefore we get a central extension

$$1 \longrightarrow \mathbb{G}_m \longrightarrow \widetilde{G} \xrightarrow{pr_1} \mathsf{G} \longrightarrow 1.$$

By Lemma 3.4, the above homomorphism pr_1 is surjective. This $\widetilde{G}$ is an algebraic group. To see this, consider the direct image $p_{2*} \mathscr{I}so(\phi^* E_j, p_2^* E_j)$, where ϕ and p_2 are the projections in the proof of Lemma 3.4, and $\mathscr{I}so(\phi^* E_j, p_2^* E_j)$ is the sheaf of isomorphisms between the two vector bundles $\phi^* E_j$ and $p_2^* E_j$. This direct image is a principal $\mathbb{G}_m$-bundle over G/P. The total space of this principal $\mathbb{G}_m$-bundle is identified with $\widetilde{G}$.

We consider the derived subgroup $[\widetilde{G}, \widetilde{G}]$. Since G is simple and not abelian, we have $[\mathsf{G}, \mathsf{G}] = \mathsf{G}$, so $\pi([\widetilde{G}, \widetilde{G}]) = \mathsf{G}$. The unipotent radical of $\widetilde{G}$ is trivial. Indeed, the unipotent radical is mapped to the trivial subgroup of G since G is simple. Therefore it is included in $\mathbb{G}_m$ and so the unipotent radical is trivial.

Since $\widetilde{G}$ is reductive, $[\widetilde{G},\widetilde{G}]$ is semi-simple, hence a proper subgroup of $\widetilde{G}$ (the radical of $\widetilde{G}$ contains $\mathbb{G}_m$ hence $\widetilde{G}$ is not semi-simple). Thus the restriction of pr_1 to $[\widetilde{G},\widetilde{G}]$ is an isogeny. Since G is simply-connected, the restriction of pr_1 to $[\widetilde{G},\widetilde{G}]$ is an isomorphism. Consequently, the tautological action of $[\widetilde{G},\widetilde{G}]$ on E_j makes it a G-linearized bundle. □

Lemma 3.6. *The integer r in* (3) *is* 1.

Proof. Since $Hom(E_1, T(\mathsf{G}/\mathsf{P})) \otimes E_1$ is a direct summand of $T(\mathsf{G}/\mathsf{P})$ (see (3)), from Lemma 3.3 we know that the slope of $Hom(E_1, T(\mathsf{G}/\mathsf{P})) \otimes E_1$ coincides with the slope of $T(\mathsf{G}/\mathsf{P})$. In the proof of Lemma 3.5 we saw that $Hom(E_1, T(\mathsf{G}/\mathsf{P})) \otimes E_1$ is a G-equivariant direct summand of $T(\mathsf{G}/\mathsf{P})$. As $T(\mathsf{G}_{\mathbb{C}}/\mathsf{P}_{\mathbb{C}})$ is stable, [AB10, Theorem 2.1], from Proposition 3.2 it now follows that $Hom(E_1, T(\mathsf{G}/\mathsf{P})) \otimes E_1 = T(\mathsf{G}/\mathsf{P})$. □

The following proposition holds without any restriction on the characteristic.

Proposition 3.7. *Let M_1, M_2 be two G-modules such that $H^0(\mathsf{G}/\mathsf{P}, T(\mathsf{G}/\mathsf{P})) = M_1 \otimes M_2$ as G-modules. Then either $M_1 = k$ or $M_2 = k$.*

Proof. Let θ be the highest root of $\mathfrak{g}$. We claim that θ is a maximal weight of $H^0(\mathsf{G}/\mathsf{P}, T(\mathsf{G}/\mathsf{P}))$ in the sense that $\theta + \alpha$ is not a weight of $H^0(\mathsf{G}/\mathsf{P}, T(\mathsf{G}/\mathsf{P}))$ for any positive root α. To prove this, first note that if $H^0(\mathsf{G}/\mathsf{P}, T(\mathsf{G}/\mathsf{P})) = \mathfrak{g}$, then this is in fact the definition of the highest root. By [De77, Théorème 1], there are only three cases where $H^0(\mathsf{G}/\mathsf{P}, T(\mathsf{G}/\mathsf{P})) \neq \mathfrak{g}$:

(1) $\mathsf{G} = \mathrm{Sp}(2n)$ of type C_n with G/P a projective space and $H^0(\mathsf{G}/\mathsf{P}, T(\mathsf{G}/\mathsf{P})) = \mathfrak{sl}(2n)$,
(2) $\mathsf{G} = \mathrm{SO}(n+2)$ of type B_n with G/P a spinor variety and $H^0(\mathsf{G}/\mathsf{P}, T(\mathsf{G}/\mathsf{P})) = \mathfrak{so}(2n+2)$, and
(3) $\mathsf{G} = G_2$ with G/P a quadric and $H^0(\mathsf{G}/\mathsf{P}, T(\mathsf{G}/\mathsf{P})) = \mathfrak{so}(7)$.

In these three cases, we have exceptional automorphisms that account for additional vector fields and we have $H^0(\mathsf{G}/\mathsf{P}, T(\mathsf{G}/\mathsf{P})) = \mathfrak{g} \oplus V$, where V has a unique highest weight which is not higher than θ. For example, if $\mathsf{G} = \mathrm{Sp}(2n)$, then $\mathsf{G}/\mathsf{P} = \mathrm{SL}(2n)/\mathsf{P}_{\mathrm{SL}(2n)}$ is a projective space of dimension $2n-1$, so that $H^0(\mathsf{G}/\mathsf{P}, T(\mathsf{G}/\mathsf{P}))$ is $\mathfrak{sl}(2n)$. Then V is a module with unique highest weight $\epsilon_1 + \epsilon_2$, whereas $\theta = 2\epsilon_1$ (in the notation of [Bou05, Chap VI, Planches]). So the claim is proved.

As θ is a maximal weight of $H^0(\mathsf{G}/\mathsf{P}, T(\mathsf{G}/\mathsf{P})) = M_1 \otimes M_2$, there are maximal weights ω_1 and ω_2 of M_1 and M_2 respectively, such that

$$\theta = \omega_1 + \omega_2 . \tag{7}$$

Since ω_1 and ω_2 are maximal, they are dominant. In all types except A_n and C_n, we have θ to be a fundamental weight. Therefore, from the equality in (7) it follows that either $\omega_1 = 0$ or $\omega_2 = 0$, hence the proposition is proved in these cases.

For the remaining cases of A_n and C_n, assume that $\omega_1 \neq 0$ and $\omega_2 \neq 0$. Let ϖ_i denote the i-th fundamental weight. In case of A_n, we have $\theta = \varpi_1 + \varpi_n$, so up to a permutation, $\omega_1 = \varpi_1$ and $\omega_2 = \varpi_n$. Since the Weyl group orbits of both ϖ_1 and ϖ_n have $n+1$ elements, it follows that $\dim M_1 \geq n+1$ and $\dim M_2 \geq n+1$. This implies that $\dim H^0(\mathsf{G}/\mathsf{P}, T(\mathsf{G}/\mathsf{P})) \geq (n+1)^2$ which is a contradiction. In case of C_n, we have $\theta = 2\varpi_1$, so similarly we get $\omega_1 = \omega_2 = \varpi_1$, and $\dim H^0(\mathsf{G}/\mathsf{P}, T(\mathsf{G}/\mathsf{P})) \geq (2n)^2$. This is again a contradiction. □

Lemma 3.8. $\dim Hom(E_1, T(\mathsf{G}/\mathsf{P})) = 1$.

Proof. From Lemma 3.6 we have $H^0(\mathsf{G}/\mathsf{P}, T(\mathsf{G}/\mathsf{P})) = Hom(E_1, T(\mathsf{G}/\mathsf{P})) \otimes H^0(\mathsf{G}/\mathsf{P}, E_1)$. Since $T(\mathsf{G}/\mathsf{P})$ is globally generated, so is E_1 and thus $\dim H^0(\mathsf{G}/\mathsf{P}, E_1) > 1$. Thus, as E_1 is G-linearized, the lemma follows from Proposition 3.7. □

From equation (3) and Lemma 3.6, we get that $T(\mathsf{G}/\mathsf{P}) \simeq Hom(E_1, T(\mathsf{G}/\mathsf{P})) \otimes E_1$. By Lemma 3.8, $Hom(E_1, T(\mathsf{G}/\mathsf{P})) \simeq k$, thus $T(\mathsf{G}/\mathsf{P}) \simeq E_1$ and it is stable.

The following lemma completes the proof of the main Theorem.

Lemma 3.9. *Let E be a semi-stable (respectively, stable) G-linearized vector bundle on G/P. Then E is Frobenius semi-stable (respectively, Frobenius stable).*

Proof. The absolute Frobenius morphism on G/P will be denoted by F. First assume that E is semi-stable. Let W be the first term of the Harder-Narasimhan filtration of F^*E. We use the correspondence between vector bundles on G/P and P-modules given in Proposition 2.1. Thus W corresponds to a P-stable subspace of $(F^*E)_{x_0}$, the fiber of F^*E at the base point in G/P. This is the same as an $F^*\mathsf{P}$-stable subspace S of E_{x_0}. Since $F : \mathsf{P} \longrightarrow \mathsf{P}$ is bijective, this S is also a P-submodule of E_{x_0}. Thus, there exists a subbundle $W' \subset E$ of slope $\frac{\mu(W)}{\mathrm{char}(k)} \geq \frac{\mu(F^*E)}{\mathrm{char}(k)} = \mu(E)$ such that $W = F^*W'$. By semi-stability of E, we have $W' = E$. Thus we get that $W = F^*E$.

Assume now that E is stable. So, as shown above, F^*E is semi-stable. Let $W \subset F^*E$ be a subbundle with $\mu(W) = \mu(F^*E)$. We consider the Cartier connection $F^*E \longrightarrow F^*E \otimes \Omega^1_{\mathsf{G}/\mathsf{P}}$. By [Ka76, Theorem 5.1], the subbundle W is a Frobenius pull-back if and only if its image under the composition

$$W \longrightarrow F^*E \longrightarrow F^*E \otimes \Omega^1_{\mathsf{G}/\mathsf{P}}$$

is contained in $W \otimes \Omega^1_{\mathsf{G}/\mathsf{P}}$. Let $f : W \longrightarrow F^*E \otimes \Omega^1_{\mathsf{G}/\mathsf{P}}$ be the above composition.

To prove that $f(W) \subset W \otimes \Omega^1_{\mathsf{G}/\mathsf{P}}$, consider the composition

$$W \xrightarrow{f} F^*E \otimes \Omega^1_{\mathsf{G}/\mathsf{P}} \longrightarrow ((F^*E)/W) \otimes \Omega^1_{\mathsf{G}/\mathsf{P}}. \tag{8}$$

Since E is Frobenius semi-stable, the pullback F^*E is Frobenius semi-stable. As $\mu(W) = \mu(F^*E)$, and F^*E is Frobenius semi-stable, it follows that both W and $(F^*E)/W$ are Frobenius semi-stable with

$$\mu(W) = \mu((F^*E)/W). \tag{9}$$

Now, since $\Omega^1_{\mathsf{G}/\mathsf{P}}$ is also Frobenius semi-stable, it follows that $((F^*E)/W) \otimes \Omega^1_{\mathsf{G}/\mathsf{P}}$ is semi-stable [RR84, p. 285, Theorem 3.18]. From (9) and the fact that $\mu(\Omega^1_{\mathsf{G}/\mathsf{P}}) < 0$ it follows that

$$\mu(W) > \mu(((F^*E)/W) \otimes \Omega^1_{\mathsf{G}/\mathsf{P}}).$$

So the composition in (8) being $\mathcal{O}_{\mathsf{G}/\mathsf{P}}$-linear has to vanish, meaning we have $f(W) \subset W \otimes \Omega^1_{\mathsf{G}/\mathsf{P}}$. Therefore, let $W' \subset E$ be such that $W = F^*W'$. We have $\mu(W') = \mu(E)$. By stability of E, we get that $W' = E$ and hence $W = F^*E$. □

Remark 3.10. It is not true that for any semistable vector bundle V on a smooth projective variety, the pullback of V by the Frobenius map of the variety is semistable. In fact, there are stable vector bundles on curves whose Frobenius pullback is are semistable; see [LP08].

4. An example in small characteristic

We give an example of a tangent bundle which is semi-stable but not stable. We do not know if there are some tangent bundles to homogeneous spaces which are not semi-stable in positive characteristic.

The example is that of $\mathsf{G}/\mathsf{P} = \mathbb{G}_\omega(n, 2n)$, the Grassmannian of Lagrangian spaces in a symplectic space of dimension $2n$, and we assume that k has characteristic 2. Namely, G is $\mathrm{Sp}(2n)$ and P is the maximal parabolic subgroup corresponding to the long simple root. Let U denote the universal bundle on G/P, of rank n and degree -1. Then $T(\mathsf{G}/\mathsf{P})$ is a subbundle of $U^* \otimes U^*$; in fact if S^2U denotes the symmetric quotient of $U \otimes U$, then $T(\mathsf{G}/\mathsf{P}) \simeq (S^2U)^*$.

We will implicitly use the correspondence between P-modules and G-linearized homogeneous bundles on G/P (Proposition 2.1). Note that the reductive quotient of P is $GL(U)$. Since $\mathrm{char}\, k = 2$, we have an additive map $U \to S^2U, x \mapsto x^2$, which can be seen as a linear application $F^*U \to S^2U$ (recall that F denotes the Frobenius morphism). It is $GL(U)$-equivariant, so this defines an exact sequence of bundles on G/P:

$$0 \to F^*U \to S^2U \to K \to 0 \tag{10}$$

It follows that there is a subbundle $K^* \subset T(\mathsf{G}/\mathsf{P})$. Since $\mu(F^*U) = \mu(S^2U) = 2\mu(U)$, we get $\mu(K^*) = \mu(T(\mathsf{G}/\mathsf{P}))$ and $T(\mathsf{G}/\mathsf{P})$ is not stable. However since F^*U is the only $GL(U)$-invariant subspace in S^2U, K^* is the only linearized subbundle in $T(\mathsf{G}/\mathsf{P})$. Thus the semi-stability inequality holds for this subbundle. Arguing as in the proof of Lemma 3.3, we deduce that $T(\mathsf{G}/\mathsf{P})$ is semi-stable.

For general homogeneous spaces G/P, we face two difficulties:

- There are linearized subbundles in $T(\mathsf{G}/\mathsf{P})$ which do not lift to characteristic 0, and contrary to the above example, they are numerous in general.
- The stability of $T(\mathsf{G}/\mathsf{P})$ for characteristic 0 says nothing about $\mu(E)$ of such a subbundle $E \subset T(\mathsf{G}/\mathsf{P})$. It is difficult to compute the $(\dim(\mathsf{G}/\mathsf{P})-1)$-th power of the anticanonical polarization to be able to show the semi-stability inequality for E.

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