

# BCOV INVARIANTS OF CALABI–YAU MANIFOLDS AND DEGENERATIONS OF HODGE STRUCTURES

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ABSTRACT. Calabi–Yau manifolds have risen to prominence in algebraic geometry, in part because of mirror symmetry and enumerative geometry. After Bershadsky–Cecotti–Ooguri–Vafa (BCOV), it is expected that genus 1 curve counting on a Calabi–Yau manifold is related to a conjectured invariant, only depending on the complex structure of the mirror, and built from Ray–Singer holomorphic analytic torsions. To this end, extending work of Fang–Lu–Yoshikawa in dimension 3, we introduce and study the BCOV invariant of Calabi–Yau manifolds of arbitrary dimension. To determine it, knowledge of its behaviour at the boundary of moduli spaces is imperative. To address this problem, we prove general results on degenerations of  $L^2$  metrics on Hodge bundles and their determinants, refining the work of Schmid. We express the singularities of these metrics in terms of limiting Hodge structures, and derive consequences for the dominant and subdominant singular terms of the BCOV invariant.

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## 1. INTRODUCTION

In this article we introduce and study a real valued invariant for Calabi–Yau manifolds, depending only on the complex structure, expected to be birationally invariant and to encode genus 1 curve counting on a mirror Calabi–Yau manifold. The invariant has as its origin the remarkable theoretical physics article of Bershadsky–Cecotti–Ooguri–Vafa [BCOV94]. It is a genus 1 counterpart of the Yukawa coupling studied by Candelas–de la Ossa–Green–Parkes [CdLOGP91] in their work on mirror symmetry and genus 0 Gromov–Witten invariants. The physics theory of [BCOV94] received a mathematical treatment by Fang–Lu–Yoshikawa [FLY08], where they defined what is nowadays called the BCOV invariant for three dimensional Calabi–Yau manifolds, in the strict sense. They confirmed the predictions in [BCOV94] for the mirror of the quintic Calabi–Yau 3-fold. An important and open question, already raised in [BCOV94, Sec. 5.8], is to extend these constructions and results to general dimensions, and this is the purpose of our study.

**1.1. The BCOV invariant of a Calabi–Yau manifold.** Let  $Z$  be a Calabi–Yau manifold of dimension  $n$ , meaning a compact Kähler manifold with trivial canonical bundle. Given a choice of Ricci flat Kähler metric  $\omega$  on  $Z$ , the invariant  $T(Z, \omega)$  proposed by [BCOV94] is a combination of Ray–Singer holomorphic torsions, that can be written as follows:

$$(1.1) \quad T(Z, \omega) = \prod_{p,q=1}^n (\det \Delta_{\bar{\partial}}^{(p,q)})^{(-1)^{p+q} pq}$$

where  $\det \Delta_{\bar{\partial}}^{(p,q)}$  refers to the zeta regularized determinant of the  $\bar{\partial}$ -Laplacian acting on  $(p, q)$ -forms on  $Z$ . This depends on the Kähler form, as opposed to the mirror symmetry principle that it should be an invariant of the complex structure in the  $B$ -model. An intrinsic definition in dimension 3, when  $h^{0,1} = h^{0,2} = 0$ , was provided by [FLY08], and was accomplished by multiplying by normalizing factors. A similar renormalization was noted in [PW05]. The correction term seems to be specific to dimension 3 and does not offer an obvious extension to higher dimensions. To our surprise, a general normalization was suggested by Kato’s formalism of heights of motives [Kat14].

In our approach, the BCOV invariant is in fact realized as the quotient of two natural metrics on the so-called BCOV complex line. This can be thought of as a weighted product of determinants of Hodge summands, defined for a compact complex manifold  $Z$  as

$$\lambda_{BCOV}(Z) = \bigotimes_{p,q} \det H^q(Z, \Omega_Z^p)^{(-1)^{p+q} p}.$$

On the one hand,  $\lambda_{BCOV}(Z)$  can be equipped with the Quillen-BCOV metric  $h_{Q,BCOV}$  introduced in [FLY08], and independent of any choice of Kähler structure. On the other hand, in the current article we exhibit a renormalized  $L^2$  metric on the BCOV complex line, that we call the  $L^2$ -BCOV metric. Precisely, for a Calabi–Yau manifold  $Z$  with Kähler form  $\omega$ , we define the  $L^2$ -BCOV norm of an element  $\sigma$  of the BCOV line by

$$h_{L^2,BCOV}(\sigma, \sigma) = h_{L^2}(\sigma, \sigma) \prod_{k=1}^{2n} \text{vol}_{L^2}(H^k(Z, \mathbb{Z}), \omega)^{(-1)^{k+1} k/2}.$$

Here  $h_{L^2}$  is the product of  $L^2$  metrics on the BCOV line provided by Hodge theory (harmonic representatives with respect to  $\omega$ ) and  $\text{vol}_{L^2}(H^k(Z, \mathbb{Z}), \omega)$  is the square of the covolume of the lattice  $H^k(Z, \mathbb{Z})_{\text{nt}} \subset H^k(Z, \mathbb{R})$  with respect to the  $L^2$  scalar product, where  $H^k(Z, \mathbb{Z})_{\text{nt}}$  is the maximal torsion free quotient of  $H^k(Z, \mathbb{Z})$ . We adopt the convention that the volume equals 1 for those factors with  $H^k(Z, \mathbb{R}) = 0$ . In Proposition 5.6 we show that the  $L^2$ -BCOV metric thus defined is also independent of the choice of Kähler metric. The BCOV invariant of the Calabi–Yau manifold  $Z$  is then defined as the proportionality factor comparing the Quillen-BCOV and  $L^2$ -BCOV metrics.

**Definition.** *Let  $Z$  be a Calabi–Yau manifold. Then we let*

$$\tau_{BCOV}(Z) := h_{Q,BCOV} / h_{L^2,BCOV} \in \mathbb{R}_{>0}.$$

*We refer to  $\tau_{BCOV}(Z)$  as the BCOV invariant of  $Z$ .*

Note that since the two metrics are independent of the choice of Kähler structure, the invariant only depends on the complex structure of  $Z$ . It generalizes the construction for strict Calabi–Yau 3-folds in [FLY08]<sup>1</sup> to general Calabi–Yau manifolds of arbitrary dimension.

The writing of the invariant as a quotient of metrics lends itself to computing the second variation of  $\tau_{BCOV}(Z)$  as the complex structure is deformed. If  $f: X \rightarrow S$  is a Kähler morphism of connected complex manifolds whose fibers are Calabi–Yau  $n$ -folds, we prove in Proposition 5.10 that the function  $s \mapsto \log \tau_{BCOV}(X_s)$  satisfies the differential equation

$$(1.2) \quad dd^c \log \tau_{BCOV} = \sum_{k=0}^{2n} (-1)^k \omega_{H^k} - \frac{\chi}{12} \omega_{WP}.$$

Here,  $\chi$  is the topological Euler characteristic of a general fiber, and  $\omega_{WP}$  and  $\omega_{H^k}$  are the Weil–Peterson and Hodge type forms for the family  $f: X \rightarrow S$ . The equation (1.2) is a higher dimensional version of the holomorphic anomaly equation at genus 1 in the mirror symmetry literature.

In general, the BCOV invariant is a non-trivial function. In some cases, such as abelian varieties of dimension at least two or hyperkähler varieties, it is constant in complex moduli (cf. Proposition 5.12). In the case of moduli of polarized Calabi–Yau manifolds, it is equal to the BCOV torsion in (1.1), up to a constant which depends on the polarization.

**1.2. Asymptotic behaviour of the BCOV invariant.** The only known strategy to approach the BCOV predictions in mirror symmetry consists in seeing the BCOV invariant as a function on a moduli space of Calabi–Yau varieties, then exploiting the holomorphic anomaly equation (1.2). We refer the reader to e.g. [HKQ09] for a discussion in the mathematical physics literature.

Since moduli spaces are in general non-compact, the differential equation determines  $\log \tau_{BCOV}$  at most up to a generally non-constant pluriharmonic function. To fix this indeterminacy, known

<sup>1</sup>Our definition differs by an explicit power of  $2\pi$  due to different normalizations.

as the holomorphic ambiguity, it is essential to know the limiting behaviour of the BCOV invariant as one approaches the boundary of the moduli space. We provide a general answer to this question, by identifying an explicit topological expression for such boundary conditions for one-parameter normal crossings degenerations (see Theorems A, B below).

Denote by  $\mathbb{D}$  the complex unit disc. Suppose  $f : X \rightarrow \mathbb{D}$  is a projective morphism of complex manifolds, with Calabi–Yau  $n$ -fold fibers outside the origin.<sup>2</sup> In Theorem 6.5 and Proposition 6.8 we prove that the BCOV invariant behaves as

$$(1.3) \quad \log \tau_{BCOV}(X_t) = \kappa_f \log(|t|^2) + \varrho_f \log(\log|t|^{-1}) + \text{continuous},$$

for constants  $\kappa_f, \varrho_f \in \mathbb{Q}$ , as  $t \rightarrow 0$ . The rationality of  $\kappa_f$  was established in the three-dimensional case by Yoshikawa [Yos15]. Under further assumptions on the degeneration, we obtain general expressions for  $\kappa_f$  and  $\varrho_f$ , in terms of the geometry of the special fiber and the limiting mixed Hodge structures. In this introduction, we focus on some relevant situations, when  $\kappa_f$  and  $\varrho_f$  take a particularly pleasant form.

To state the first such result, let  $f : X \rightarrow \mathbb{D}$  be a projective morphism of complex manifolds with Calabi–Yau  $n$ -fold fibers outside the origin, and such that the special fiber  $X_0 = \sum_i D_i$  is a simple normal crossings divisor. Introduce the notation  $D(k) = \bigsqcup_I \bigcap_{i \in I} D_i$  where the union is over index subsets  $I$  of cardinality  $k$ . Denote by  $\chi(D(k))$  the topological Euler characteristic of  $D(k)$ .

**Theorem A.** *In the above situation, suppose also that  $f$  is Kulikov, i.e.  $K_X \simeq \mathcal{O}_X$ . Then*

$$\kappa_f = \sum_{k=1}^{n+1} (-1)^k \frac{k(k-1)}{24} \chi(D(k)).$$

*In the case of strict Calabi–Yau 3-folds (i.e.  $h^{0,1} = h^{0,2} = 0$ ), this expression can be further simplified to*

$$\kappa_f = \frac{\chi(D(2)) - 6Q}{12}$$

*where  $Q = \#D(4)$  is the number of quadruple points.*

The case of dimension three in the theorem confirms a conjecture of Liu–Xia [LX19, Conj. 0.5] and is found in Corollary 7.9. The theorem, which follows from Proposition 7.1, can thus be seen as a far-reaching generalization thereof.

Another important example of generic singularities for several Calabi–Yau families is the case of ordinary double points, often called conifold singularities. For these singularities we prove in Theorem 7.3:

**Theorem B.** *Let  $f : X \rightarrow \mathbb{D}$  be a projective morphism of complex manifolds, with smooth Calabi–Yau  $n$ -fold fibers outside the origin. If the special fiber  $X_0$  has at most ordinary double point singularities, then*

- *if  $n$  is odd,*

$$\kappa_f = \frac{n+1}{24} \#\text{sing}(X_0) \quad \text{and} \quad \varrho_f = \#\text{sing}(X_0).$$

- *if  $n$  is even,*

$$\kappa_f = -\frac{n-2}{24} \#\text{sing}(X_0) \quad \text{and} \quad \varrho_f = 0.$$

*Here  $\#\text{sing}(X_0)$  denotes the number of singular points in the fiber  $X_0$ .*

<sup>2</sup>Actually a slightly stronger technical condition is required, that  $f$  extends to a projective morphism  $\tilde{X} \rightarrow S$  of compact complex manifolds. See §1.5.

For  $\kappa_f$ , the case  $n = 3$  of the statement was already established by Fang–Lu–Yoshikawa [FLY08], and it was a key point to their proof of the BCOV conjecture for the mirror quintic family. For  $n = 4$  the theorem corroborates an expectation suggested by work of Klemm–Pandharipande in [KP08, Sec. 4 & 6]. The case of general  $n$  confirms a conjecture of Fang–Lu–Yoshikawa from 2004 (private communication). In the theoretical physics literature this is usually referred to as the conifold gap condition at conifold points. The refined information for  $\varrho_f$  had not been considered before, and is one of the applications of our approach.

In dimensions three and four we have more general formulas for  $\kappa_f$  only assuming a smooth total space (cf. Theorem 7.6 and Theorem 7.13). These rely on a careful study of Serre duality on Kähler extensions due to T. Saito [Sai04]. In these cases  $\kappa_f$  involves in particular the total dimension of the vanishing cycles of the family.

In Section 7.4 we derive some consequences concerning the number of singular fibers for families admitting at most ordinary double point singularities over a complete curve. It would be interesting to investigate the same type of global-local interaction more generally, following *e.g.* the approaches in [GGK09] or [LX19].

**1.3. Asymptotic behaviour of  $L^2$  metrics and monodromy eigenvalues.** In order to generally study the asymptotics of the BCOV invariant (cf. (1.3)) one needs a precise control on the asymptotics of Quillen and  $L^2$  metrics. The singularities of the Quillen-BCOV metric were already dealt with in our previous paper [EFiMM18], itself relying on results of Yoshikawa [Yos07]. As for  $L^2$  metrics on Hodge bundles, we elaborate on the work of Schmid [Sch73] and Peters [Pet84], and obtain explicit asymptotics in terms of limiting Hodge structures.

The framework of our analysis is a projective morphism  $f: X \rightarrow \mathbb{D}$  of complex manifolds, with normal crossings special fiber. Denote by  $\Omega_{X/\mathbb{D}}^p(\log)$  the sheaf of relative differential  $p$ -forms with (not necessarily reduced) logarithmic poles along the special fiber. Then the higher direct image sheaves  $R^q f_* \Omega_{X/\mathbb{D}}^p(\log)$  are locally free. Given a choice of rational Kähler structure, the vector bundles  $R^q f_* \Omega_{X/\mathbb{D}}^p(\log)$  carry corresponding  $L^2$  metrics, possibly singular at the origin. In Theorem 4.4 we determine the singularities of the induced metrics on  $\det R^q f_* \Omega_{X/\mathbb{D}}^p(\log)$ . The statement involves the limiting Hodge structure on  $H^{p+q}(X_\infty)$ , the cohomology of a general fiber. Recall that this is a mixed Hodge structure with a Hodge type filtration  $F_\infty^\bullet$  and a weight filtration  $W_\bullet$ . The semi-simple part of the monodromy, denoted  $T_s$ , acts upon the whole mixed Hodge structure and admits a logarithm  ${}^\ell \log T_s$  with eigenvalues in  $2\pi i \cdot \{\mathbb{Q} \cap (-1, 0]\}$ .

**Theorem C.** *Let  $\sigma$  be a holomorphic trivialization of  $\det R^q f_* \Omega_{X/\mathbb{D}}^p(\log)$ . Then we have a real analytic asymptotic expansion for its  $L^2$  norm*

$$\log h_{L^2}(\sigma, \sigma) = \alpha^{p,q} \log |t|^2 + \beta^{p,q} \log \log |t|^{-1} + C + O\left(\frac{1}{\log |t|}\right),$$

with

$$\alpha^{p,q} = -\frac{1}{2\pi i} \operatorname{tr} \left( {}^\ell \log T_s \mid \operatorname{Gr}_{F_\infty}^p H^k(X_\infty) \right) \quad (k = p + q)$$

and

$$\beta^{p,q} = \sum_{r=-k}^k r \dim \operatorname{Gr}_{F_\infty}^p \operatorname{Gr}_{k+r}^W H^k(X_\infty)$$

and a constant  $C \in \mathbb{R}$ .

This result refines [Pet84, Prop. 2.2.1]. Notice that he could only show that  $\alpha^{p,q}$  is a rational number extracted from the monodromy in a non-precise manner, and similarly for  $\beta^{p,q}$ . Our

contribution thus clarifies the exact relationship to the limiting mixed Hodge structure. This result refines [Pet84, Prop. 2.2.1] in that it gives the values of the coefficients  $\alpha^{p,q}$  and  $\beta^{p,q}$  in terms of the limiting Hodge structure, without flatness assumption on the filtered piece of the Hodge filtration.

In the semi-stable case (hence unipotent monodromies) the coefficient  $\alpha^{p,q}$  is known to be zero. The determination of the coefficient  $\alpha^{p,q}$  thus reduces to the following purely algebraic geometric consideration. To compare to the semi-stable case we exhibit a diagram

$$\begin{array}{ccc} Y & \longrightarrow & X \\ g \downarrow & & \downarrow f \\ \mathbb{D} & \xrightarrow{\rho} & \mathbb{D}, \end{array}$$

where  $\rho(t) = t^\ell$  is some ramified covering and  $Y \rightarrow \mathbb{D}$  is semi-stable. The diagram is Cartesian over  $\mathbb{D}^\times$ . There is a natural inclusion of locally free sheaves

$$\rho^* R^q f_* \Omega_{X/\mathbb{D}}^p(\log) \subseteq R^q g_* \Omega_{Y/\mathbb{D}}^p(\log).$$

The quotient is a torsion sheaf supported on the origin, and can hence be written as

$$\bigoplus_{j=1}^{h^{p,q}} \frac{\mathcal{O}_{\mathbb{D},0}}{t^{a_j} \mathcal{O}_{\mathbb{D},0}},$$

for some integers  $a_j \geq 0$ . The rational numbers  $\alpha_j^{p,q} = \frac{a_j}{\ell} \in [0, 1)$  are independent of the choice of semi-stable reduction, and we call them the elementary exponents of the  $(p, q)$  Hodge bundle. Their sum is equal to the sought for  $\alpha^{p,q}$  in Theorem C. We prove the following fundamental theorem in the theory of degenerations of Hodge structures, of independent interest:

**Theorem D.** *The elementary exponents  $\alpha_j^{p,q}$  of the  $(p, q)$  Hodge bundle are such that  $\exp(-2\pi i \alpha_j^{p,q})$  constitute the eigenvalues of  $T_s$ , the semi-simple part of the monodromy acting on  $\mathrm{Gr}_{F_\infty}^p H^{p+q}(X_\infty)$  (with multiplicities).*

In fact our arguments provide more general results for Deligne extensions for degenerations of Hodge structures. These results constitute the content of Section 2 (see in particular Theorem 2.6 and Corollary 2.8). Theorem C and Theorem D generalize Theorem A of [EFiMM18] and analogous results by Halle–Nicaise [HN12] and Boucksom–Jonsson [BJ17] which consider constructions coming from canonical bundles. With respect to these references, we stress that the theorems of this section do not assume any Calabi–Yau hypothesis, and apply to general graded pieces of the Hodge filtration. As an anonymous referee pointed out, Theorem D could also be formulated within the general framework of mixed Hodge modules.

**1.4. Relations with mirror symmetry.** In this section we return to the initial motivation for the introduction of the BCOV invariant, discussed at the beginning of the text. More computational endeavours and applications of the invariant are the topic of a sequel to the present article [EFiMM19], where we prove a mirror symmetry conjecture, at genus 1, for Calabi–Yau hypersurfaces in projective space and their mirror family.

Consider  $f: X \rightarrow \mathbb{D}$  to be a maximally unipotent projective degeneration of Calabi–Yau  $n$ -folds<sup>3</sup>, meaning that for the monodromy operator  $T$  on the local system corresponding to  $H^n$ ,

<sup>3</sup>This is also known as a large complex structure limit of Calabi–Yau manifolds.

we have  $(T - 1)^n \neq 0$  and  $(T - 1)^{n+1} = 0$ . Denote by  $X_\infty$  a general fiber. In this setting, mirror symmetry predicts the existence of a Calabi–Yau mirror  $X_\infty^\vee$  and an ample class  $[H]$  such that,

$$(1.4) \quad \kappa_f = \frac{(-1)^{n+1}}{12} \int_{X_\infty^\vee} c_{n-1}(X_\infty^\vee) \cap [H].$$

See for instance the introduction in Yoshikawa’s [Yos17] and the discussion by Klemm–Pandharipande [KP08, Sec. 4].

Generally, even when potential mirrors are known, such as in Batyrev’s framework using toric Fano varieties [Bat94], this seems to be out of reach. However, in the special case when  $X_\infty$  is an abelian variety of dimension  $n \geq 2$  or a hyperkähler variety, we can confirm that both sides of (1.4) are zero. The right hand side vanishes, because  $X_\infty^\vee$  is also an abelian or hyperkähler variety and so  $c_{n-1}(X_\infty^\vee) = 0$ . The vanishing of the left hand side is due to the constancy of the BCOV invariant for such families, cf. Proposition 5.12. Besides, in the direction of the conjecture we can prove the following corollary, which is a consequence of the general form of Theorem A (Theorem 6.5 *infra*):

**Corollary.** *Suppose that  $f: X \rightarrow \mathbb{D}$  is a projective degeneration of Calabi–Yau varieties, with unipotent monodromies. Then*

$$12\kappa_f \in \mathbb{Z}.$$

Consider momentarily a Calabi–Yau 3-fold  $Z$  and  $[H]$  an ample cohomology class in  $H^2(Z)$ . Then by the known semi-stability of  $T_Z$  and the Bogomolov–Gieseker inequality, with  $c_1 = 0$ , we have

$$\int_Z c_2(Z) \cap [H] \geq 0.$$

Taking into account (1.4), this leads us to make the following conjecture:

**Conjecture A.** *If  $f: X \rightarrow \mathbb{D}$  is a projective degeneration of 3-dimensional Calabi–Yau varieties with maximally unipotent monodromy, then  $\kappa_f \geq 0$ .*

We remark that the conjecture is true for abelian 3-folds, since in this case our BCOV invariant is constant so that  $\kappa_f$  is 0. Most importantly, it is also known to hold for the mirror quintic family, as in this case the BCOV predictions were confirmed by Fang–Lu–Yoshikawa [FLY08]. In the general case, we expect that the explicit formulas we exhibit for  $\kappa_f$  will provide a useful tool towards the proof of the conjecture. For instance, in the semi-stable and Kulikov case, a combination of Theorem A and the classification of the possible special fibers lead to the following positive answer (cf. Proposition 7.11):

**Proposition.** *The Conjecture A is true for semi-stable and Kulikov degenerations of strict Calabi–Yau threefolds.*

More ambitiously, we could ask about the sign of the coefficient  $\kappa_f$  for an arbitrary degeneration, not necessarily have maximal unipotent monodromy. This question, for which we don’t have a conjectural answer, is closely related to the problem of determining the birational type of the moduli space of polarized Calabi–Yau 3-folds. We thank K.-I. Yoshikawa for bringing our attention to this fact.

In another direction, it is expected (cf. [Kon95, p. 137]) that birational Calabi–Yau manifolds have the same  $B$ -models. Since the BCOV invariant should only depend thereupon, this expectation should afford the following realization:

**Conjecture B.** *If  $X$  and  $X'$  are birational Calabi–Yau manifolds, then  $\tau_{BCOV}(X) = \tau_{BCOV}(X')$ .*

The conjecture extends the corresponding conjecture in [FLY08] in dimension 3, which in this generality remains open. In these lines, Maillot–Rössler [MR12] proved that if  $X$  and  $X'$  are defined over  $\mathbb{Q}$ , then a power of the quotient  $\tau_{BCOV}(X)/\tau_{BCOV}(X')$  is a rational number.

After a preliminary version of this work was circulated, Y. Zhang was lead to extend our construction and produce a BCOV type invariant for Calabi–Yau pairs  $(X, Y)$  (cf. [Zha]). These pairs consist of a compact Kähler manifold  $X$  together with a smooth divisor  $Y$  in some linear series  $|mK_X|$ . It can be expected that this construction be a useful auxiliary tool in the proof of Conjecture B.

As a final consideration, we remark that the current known constructions of mirror Calabi–Yau varieties, e.g. Batyrev’s [Bat94], do not always produce Calabi–Yau manifolds, but rather orbifolds. It would thus be desirable to extend these constructions to this context, possibly replacing Dolbeault cohomology with Chen–Ruan cohomology. In this direction we quote the extensive work of Yoshikawa [Yos04, Yos09, Yos12, Yos13a, Yos13b, Yos17], who considered equivariant analytic torsion of K3 surfaces with involution. A running theme is the relationship between equivariant torsions and Borchers products. This is remarkable since the non-equivariant BCOV torsion is trivial for K3 surfaces.

**1.5. Notations and conventions.** For the convenience of the reader, we introduce the various notations and conventions that are being used in this work.

A *Calabi–Yau* variety is, for the purposes of this article, a compact connected complex Kähler manifold  $X$  with trivial canonical bundle  $K_X \simeq \mathcal{O}_X$ . We say that a Calabi–Yau variety  $X$  of dimension  $n$  is a *strict Calabi–Yau* variety if moreover  $H^i(X, \mathcal{O}_X) = 0$  for  $0 < i < n$ . Hence, except for K3 surfaces, neither hyperkähler varieties are strict Calabi–Yau varieties, nor are abelian varieties of dimension at least 2. Smooth hypersurfaces of degree  $n + 1$  in  $\mathbb{P}^n$  are strict Calabi–Yau varieties. By the Bogomolov–Beauville decomposition theorem [Bog74, Bea83], in the algebraic category these can be realized as finite étale quotients of varieties of the form  $T \times V \times H$ , where  $T$  is a complex torus,  $V$  is a product of strict Calabi–Yau varieties and  $H$  a product of hyperkähler varieties.

A *degeneration*  $f: X \rightarrow S$  is a flat morphism of reduced and irreducible complex analytic spaces, with connected fibers and smooth general fibers. Often we will take  $S = \mathbb{D}$ , a disc centered at 0, and then we suppose that the morphism is smooth outside of the origin. In that case we denote by  $X_\infty$  a general fiber, and by  $X_0$  the fiber above the origin. The differentiable type of  $X_\infty$  is independent of the choice of a general fiber. A degeneration  $f: X \rightarrow \mathbb{D}$  is said to have *normal crossings* if  $X$  is non-singular and  $X_0$  is a not necessarily reduced simple normal crossings divisor in  $X$ . If  $X_0$  is furthermore reduced, then we say that  $f$  is semi-stable. We may equivalently talk about *normal crossings (resp. semi-stable) degenerations*.

A *projective* morphism  $f: X \rightarrow S$  is a proper morphism of analytic spaces such that, locally with respect to the base,  $f$  factors through a closed immersion  $\mathbb{P}^n \times S$  followed by the projection on the second variable. A *projective degeneration* is a morphism which is both a degeneration and projective. A *rational* Kähler structure on a complex manifold is a Kähler form such that the associated cohomology class is rational. A *Kähler morphism*  $f: X \rightarrow S$  is a proper submersion of complex manifolds, admitting a Kähler form  $\omega$  on  $X$ .<sup>4</sup>

A *germ* of any of the above types  $T$  (e.g. projective) of morphisms is the localization of a morphism of compact complex manifolds, of type  $T$ .

<sup>4</sup>It would be enough to assume the existence of a smooth closed real  $(1, 1)$  form on  $X$ , inducing a Kähler form on fibers.



## 2. LOGARITHMIC EXTENSIONS OF HODGE BUNDLES

In this section we recall Deligne's extension of a local system over the punctured unit disc  $\mathbb{D}^\times$ . We also review Steenbrink's construction of the lower extension of the local system of de Rham cohomology of a normal crossings degeneration, together with its Hodge filtration. For these questions we follow closely the presentation expounded in [PS08, Chap. 11], [Ste76, Sec. 2] and [Ste77, Sec. 2]. Finally, we study the behaviour of the relative Hodge filtration with respect to semi-stable reduction. The comparison of Hodge filtrations before and after semi-stable reduction produces some elementary divisors, and we show that they exactly correspond to the eigenvalues of the semi-simple part of the monodromy operator acting on the limiting Hodge filtration.

**2.1. Deligne's extensions.** Let  $(\mathcal{V}, \nabla)$  be a finite rank flat holomorphic vector bundle on  $\mathbb{D}^\times$ . Let  $q: \mathbb{H} \rightarrow \mathbb{D}^\times$  be the universal covering map  $q(\tau) = \exp(2\pi i\tau)$ , where  $\mathbb{H}$  is the upper half-plane. The  $\mathbb{C}$ -vector space of multi-valued flat sections of  $\mathcal{V}$  is, by definition

$$V_\infty := \Gamma(\mathbb{H}, q^*\mathcal{V})^{q^*\nabla} = \ker(q^*\nabla: \Gamma(\mathbb{H}, q^*\mathcal{V}) \rightarrow \Gamma(\mathbb{H}, q^*\mathcal{V} \otimes \Omega_{\mathbb{H}})).$$

This vector space is finite dimensional. The monodromy transformation is the automorphism  $T \in \text{End}_{\mathbb{C}}(V_\infty)$  induced by  $\tau \mapsto \tau + 1$ . We assume that  $T$  is quasi-unipotent. We may then introduce the Chevalley decomposition  $T = T_s T_u = T_u T_s$  of  $T$  into a semi-simple automorphism  $T_s$  and a unipotent automorphism  $T_u$ . The logarithm of  $T_u$  is well-defined. We denote  $N = \frac{1}{2\pi i} \log T_u$ . To define a logarithm of  $T_s$ , one chooses a branch of the logarithm on  $\mathbb{C}^\times$ . For a fixed choice of branch (still denoted  $\log$ ) we denote  $S = \frac{1}{2\pi i} \log T_s$ . We can thus define  $\frac{1}{2\pi i} \log T := S + N$ .

The vector bundle  $\mathcal{V}$  on  $\mathbb{D}^\times$  can be uniquely extended to a vector bundle  $\mathcal{V}_{\log}$  on  $\mathbb{D}$ , referred to as the *Deligne extension*, in such a way that:

- (i) there is an identification  $\mathcal{V}_{\log}|_{\mathbb{D}^\times} \xrightarrow{\sim} \mathcal{V}$ , depending only on the choice of branch of logarithm.
- (ii) the connection  $\nabla$  extends to a connection with regular singular poles

$$\nabla: \mathcal{V}_{\log} \longrightarrow \mathcal{V}_{\log} \otimes \Omega_{\mathbb{D}}(\log[0]).$$

Here we denote by  $\Omega_{\mathbb{D}}(\log[0])$  the sheaf of meromorphic differentials on  $\mathbb{D}$  with at most a simple pole at 0.

- (iii) the eigenvalues of  $-2\pi i \text{Res}_0 \nabla$  belong to  $(2\pi i\mathbb{Q}) \cap \log \mathbb{C}^\times$ , for the given branch of  $\log$ .
- (iv) the operator  $T$  induces a vector bundle endomorphism of  $\mathcal{V}_{\log}$ , whose fiber  $T_0$  is related to  $\text{Res}_0 \nabla$  by

$$T_0 = \exp(-2\pi i \text{Res}_0 \nabla).$$

Two frequent choices in (iii) are the *upper* and *lower* branches. The upper branch takes values in  $\mathbb{R} + 2\pi i(0, 1)$ , and will be denoted  ${}^u\log$ . The lower branch takes values in  $\mathbb{R} + 2\pi i(-1, 0]$ , and will be denoted  ${}^\ell\log$ . In later geometric constructions we will mostly encounter the lower branch.

Whenever the monodromy is unipotent, the extension  $\mathcal{V}_{\log}$  is called the *canonical extension* of  $\mathcal{V}$ .<sup>5</sup> Notice that, when it exists, the canonical extension “commutes” with any finite base change  $q \mapsto q^\ell$ .

An explicit description of  $\mathcal{V}_{\log}$  will be necessary. The regular sections of  $\mathcal{V}_{\log}$  are obtained from twisted flat multivalued sections as follows. Let  $\mathbf{e} \in V_\infty$ . Since  $\mathbf{e}(\tau + 1) = T\mathbf{e}(\tau)$ , the twisted section

$$(2.1) \quad \tilde{\mathbf{e}}(\tau) := \exp(-\tau \log T)\mathbf{e}(\tau)$$

<sup>5</sup>In the literature one often refers to the lower extension as the canonical extension, e.g. [PS08, Def. 11.4]. In the present article, this terminology is reserved to the unipotent case.

is invariant under  $\tau \mapsto \tau + 1$  and descends to a global holomorphic section of  $\mathcal{V}$  on  $\mathbb{D}^\times$ , denoted  $\tilde{\mathbf{e}}(q)$ . From the flatness of  $\mathbf{e}(\tau)$  and the relation  $2\pi i d\tau = dq/q$ , it is straightforward to check the equality on  $\mathbb{D}^\times$

$$\nabla \tilde{\mathbf{e}}(q) = -(\log T)\tilde{\mathbf{e}}(q)d\tau = -\frac{1}{2\pi i}(\log T)\tilde{\mathbf{e}}(q)\frac{dq}{q}.$$

The procedure  $\mathbf{e}(\tau) \mapsto \tilde{\mathbf{e}}(q)$  describes a  $\mathbb{C}$ -linear injective morphism

$$\varphi: V_\infty \longrightarrow \Gamma(\mathbb{D}^\times, \mathcal{V})$$

and induces an isomorphism, depending only on the choice of branch of logarithm

$$V_\infty \otimes \mathcal{O}_{\mathbb{D}^\times} \xrightarrow{\sim} \mathcal{V}.$$

Then, one defines

$$\mathcal{V}_{\log} := \varphi(V_\infty) \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{D}}.$$

In other words, we declare that  $\tilde{\mathbf{e}}$  as above is holomorphic at  $q = 0$ . It is straightforward to check that  $\mathcal{V}_{\log}$  satisfies the properties (i)–(iv) stated above.

A formal consequence of the construction is the isomorphism described as “first twisting and then evaluating at 0”:

$$(2.2) \quad \begin{aligned} \psi: V_\infty &\xrightarrow{\sim} \mathcal{V}_{\log}(0) \\ \mathbf{e} &\longmapsto \tilde{\mathbf{e}}(0). \end{aligned}$$

By definition, this isomorphism is equivariant with respect to the monodromy transformations, namely  $T$  acting on  $V_\infty$  and  $T_0$  acting on  $\mathcal{V}_{\log}(0)$ .

A final reminder concerns unipotent reduction. By assumption, the semi-simple endomorphism  $T_s$  has finite order. We choose  $\ell \geq 1$  with  $T_s^\ell = \text{id}$ . Let  $\rho: \mathbb{D} \rightarrow \mathbb{D}$  be the finite ramified cover  $\rho(t) = t^\ell$ . The space of multi-valued flat sections of the pull-back  $(\rho^*\mathcal{V}, \rho^*\nabla)$ , say  $U_\infty$ , is actually isomorphic to  $V_\infty$ :

$$(2.3) \quad \begin{aligned} \nu: V_\infty &\xrightarrow{\sim} \Gamma(\mathbb{H}, q^* \rho^* \mathcal{V})^{q^* \rho^* \nabla} =: U_\infty \\ \mathbf{e}(\tau) &\longmapsto \mathbf{e}(\ell\tau). \end{aligned}$$

The monodromy transformation on  $U_\infty$  is unipotent and identifies to  $T^\ell = T_u^\ell$ , with logarithm  $\ell N$ . The flat vector bundle  $(\rho^*\mathcal{V}, \rho^*\nabla)$  thus affords a canonical extension, that we denote by  $\mathcal{U} = (\rho^*\mathcal{V})_{\log}$ . The procedure of twisting and evaluating at 0 for  $\mathcal{V}_{\log}$  and  $\mathcal{U}$  provides a commutative diagram of isomorphisms

$$(2.4) \quad \begin{array}{ccccc} \mathbf{e}(\tau) & & V_\infty & \xrightarrow{\sim} & \mathcal{V}_{\log}(0) & & \tilde{\mathbf{e}}(0) \\ & & \downarrow \nu & & \downarrow \sim & & \downarrow \\ \mathbf{e}(\ell\tau) & & U_\infty & \xrightarrow{\sim} & \mathcal{U}(0) & & {}^\rho \tilde{\mathbf{e}}(0). \end{array}$$

The action of  $T_s$  on  $V_\infty$  induces an action of  $T_s$  on  $U_\infty$  and  $\mathcal{U}(0)$  through the isomorphisms above. On  $U_\infty$ , this automorphism is induced by the translation  $\tau \mapsto \tau + 1/\ell$ .

With these notations, it is clear that given linearly independent elements of the form  $\tilde{\mathbf{e}}_1(0), \dots, \tilde{\mathbf{e}}_r(0)$ , the corresponding  ${}^\rho \tilde{\mathbf{e}}_1(0), \dots, {}^\rho \tilde{\mathbf{e}}_r(0)$  in  $\mathcal{U}(0)$  are linearly independent as well, and reciprocally. For future usage we record the following lemma. For simplicity of exposition we restrict ourselves to lower extensions, but a similar statement holds for other extensions. Notice that for the lower extension, it follows from the construction that there is a natural inclusion  $\rho^* \mathcal{V}_{\log} \subseteq (\rho^*\mathcal{V})_{\log} = \mathcal{U}$ .

**Lemma 2.1.** *Let  $\mathcal{V}_{\log}$  be the lower extension of  $(\mathcal{V}, \nabla)$ , and let  $\sigma \in \Gamma(\mathbb{D}, \mathcal{V}_{\log})$  be such that  $\sigma(0) \neq 0$ . Define  $k \geq 0$  as the largest integer such that  ${}^\rho\sigma := t^{-k} \rho^* \sigma \in \mathcal{U}$ . Then  $k \leq \ell - 1$  and  ${}^\rho\sigma(0) \in \mathcal{U}(0)$  is an eigenvector of  $T_s$  of eigenvalue  $\exp(-2\pi i k / \ell)$ .*

*Proof.* We choose  $\mathbf{e}_1, \dots, \mathbf{e}_r \in V_\infty$  a basis of eigenvectors of  $T_s$ , whose eigenvalues we write  $\exp(-2\pi i k_1 / \ell), \dots, \exp(-2\pi i k_r / \ell)$ , where  $0 \leq k_j \leq \ell - 1$ . By construction, it follows that  ${}^\rho\tilde{\mathbf{e}}_j(0) \in \mathcal{U}(0)$  is an eigenvector of  $T_s$  of eigenvalue  $\exp(-2\pi i k_j / \ell)$ . We write

$$\sigma = \sum_j f_j(q) \tilde{\mathbf{e}}_j(q),$$

where the  $f_j(q)$  are holomorphic functions on  $\mathbb{D}$ . Because  $\sigma(0) \neq 0$ , at least one of the functions  $f_j$  does not vanish at 0. Observe that

$$\begin{aligned} (\rho^* \tilde{\mathbf{e}}_j)(t) &= \tilde{\mathbf{e}}_j(t^\ell) = \tilde{\mathbf{e}}_j(\ell\tau) \\ &= \exp(-2\pi i(S+N)\ell\tau) \mathbf{e}_j(\ell\tau) \\ &= \exp(-2\pi i(-k_j/\ell)\ell\tau) \exp(-2\pi i N\ell\tau) \mathbf{e}_j(\ell\tau) \\ &= t^{k_j} \exp(-2\pi i N\ell\tau) \mathbf{e}_j(\ell\tau) \end{aligned}$$

where  $\exp(-2\pi i \ell N\tau) \mathbf{e}_j(\ell\tau) =: {}^\rho\tilde{\mathbf{e}}_j(t)$  belongs to  $\mathcal{U}$ . Hence, the pull-back  $\rho^* \sigma$  can be written

$${}^\rho\sigma = \sum_j t^{k_j} f_j(0) {}^\rho\tilde{\mathbf{e}}_j(t) + O(t^\ell).$$

Define  $k$  as the smallest  $k_j$  such that  $f_j(0) \neq 0$ . In particular  $k \leq \ell - 1$ . Then

$$t^{-k} {}^\rho\sigma = \sum_{k_j=k} f_j(0) {}^\rho\tilde{\mathbf{e}}_j(0) + \sum_{k_j>k} t^{k_j-k} f_j(0) {}^\rho\tilde{\mathbf{e}}_j(0) + O(t^{\ell-k})$$

and thus

$$(t^{-k} {}^\rho\sigma)(0) = \sum_{j: k_j=k} f_j(0) {}^\rho\tilde{\mathbf{e}}_j(0) \neq 0,$$

which is an eigenvector of  $T_s$  of eigenvalue  $\exp(-2\pi i k / \ell)$ . This concludes the proof.  $\square$

**Definition 2.2.** *With notations as in Lemma 2.1, we define the elementary exponent of  $\sigma$  to be the rational number  $\kappa(\sigma) = k/\ell \in \mathbb{Q} \cap ]0, 1[$ . Hence,  ${}^\rho\sigma(0)$  is an eigenvector of  $T_s$  of eigenvalue  $\exp(-2\pi i \kappa(\sigma))$ .*

**2.2. Steenbrink's construction.** Let  $X$  be a complex manifold of dimension  $m$  with a normal crossings divisor  $D$ , such that  $D_{\text{red}}$  is locally given by an equation  $z_1 \dots z_k = 0$  in suitable holomorphic coordinates  $z_1, \dots, z_m$ . Recall that the sheaf of logarithmic differentials  $\Omega_X(\log D)$  is the  $\mathcal{O}_X$ -module locally generated by  $\frac{dz_1}{z_1}, \dots, \frac{dz_k}{z_k}, dz_{k+1}, \dots, dz_m$ .

Let  $f: X \rightarrow \mathbb{D}$  be a projective normal crossings degeneration, with fibres of dimension  $n$ . We denote its restriction to  $\mathbb{D}^\times$  by  $f^\times$ . Locally, the special fiber is given by an equation  $z_1^{m_1} \dots z_k^{m_k} = 0$ . Pull-back of differential forms induces an injection  $f^* \Omega_{\mathbb{D}}(\log[0]) \rightarrow \Omega_X(\log X_0)$ . The sheaf of relative logarithmic forms is then defined by

$$\Omega_{X/\mathbb{D}}(\log) = \Omega_X(\log X_0) / f^* \Omega_{\mathbb{D}}(\log[0]).$$

This is a locally free sheaf, and we define  $\Omega_{X/\mathbb{D}}^p(\log) = \wedge^p \Omega_{X/\mathbb{D}}(\log)$ . The exterior differential makes  $\Omega_{X/\mathbb{D}}^\bullet(\log)$  a complex, named the logarithmic de Rham complex of  $f$ . After Steenbrink [Ste76, Sec. 2], its  $k$ -th hypercohomology sheaf  $R^k f_* \Omega_{X/\mathbb{D}}^\bullet(\log)$  defines an extension of the flat bundle  $\mathcal{V} := R^k f_*^\times \mathbb{C} \otimes \mathcal{O}_{\mathbb{D}^\times}$ , for which the Gauss-Manin connection has logarithmic singularities,

whose residue has eigenvalues in  $[0, 1) \cap \mathbb{Q}$ . It is hence the lower Deligne extension  $\mathcal{V}_{\log}$  of  $\mathcal{V}$ . With notations as in the previous subsection, we let  $V_\infty$  be the space of multi-valued flat sections of  $\mathcal{V}$  on  $\mathbb{D}^\times$ . By parallel transport, the space  $V_\infty$  is canonically isomorphic to the  $k$ -th cohomology of a general fiber of  $f$ , also denoted  $H^k(X_\infty, \mathbb{C})$  or simply  $H^k(X_\infty)$ . In this setting, the isomorphism (2.2) gives an isomorphism

$$(2.5) \quad \psi: H^k(X_\infty) \xrightarrow{\sim} R^k f_* \Omega_{X/\mathbb{D}}^\bullet(\log)(0).$$

It is the composition of the isomorphisms stated in [Ste76, Prop. 2.16 and Thm. 2.18]. Notice that this isomorphism depends on the fixed choice of holomorphic parameter on  $\mathbb{D}$ .

One defines the Hodge filtration bundles  $\mathcal{F}^p \mathcal{V}_{\log}$  of  $\mathcal{V}_{\log} = R^k f_* \Omega_{X/\mathbb{D}}^\bullet(\log)$ , or simply  $\mathcal{F}_{\log}^p$ , by considering the sections of the Hodge filtration on the smooth part  $\mathcal{F}^p \mathcal{V}$  that extend to  $\mathcal{V}_{\log}$ . It is equivalently the filtration induced by the bête filtration  $\Omega_{X/\mathbb{D}}^{\geq p}(\log)$  of the complex  $\Omega_{X/\mathbb{D}}^\bullet(\log)$ . If  $j: \mathbb{D}^\times \hookrightarrow \mathbb{D}$  is the inclusion, we may thus write

$$(2.6) \quad \mathcal{F}_{\log}^p = j_*(\mathcal{F}^p \mathcal{V}) \cap \mathcal{V}_{\log} \subseteq j_* \mathcal{V}.$$

In [Ste77, Thm 2.11] Steenbrink shows that the sheaves  $R^q f_* \Omega_{X/\mathbb{D}}^p(\log)$  are locally free. From the  $E_1$  degeneration of the Hodge-de Rham spectral sequence in the smooth case, it then follows that for  $R^k f_* \Omega_{X/\mathbb{D}}^\bullet(\log)$  we have

$$(2.7) \quad \mathcal{F}_{\log}^p / \mathcal{F}_{\log}^{p+1} = R^{k-p} f_* \Omega_{X/\mathbb{D}}^p(\log).$$

These are called  $(p, k-p)$  Hodge bundles, or sometimes referred to as simply Hodge bundles.

Let us now consider a semi-stable reduction diagram

$$(2.8) \quad \begin{array}{ccc} Y & \xrightarrow{r} & X \\ g \downarrow & & \downarrow f \\ \mathbb{D} & \xrightarrow{\rho} & \mathbb{D}, \end{array}$$

where  $\rho(t) = t^\ell$  and  $r$  is generically finite. The previous discussion applied to  $R^k g_* \mathbb{C} \otimes \mathcal{O}_{\mathbb{D}^\times}$  produces the canonical extension  $\mathcal{U} = R^k g_* \Omega_{Y/\mathbb{D}}(\log)$  with its Hodge filtration by vector subbundles that we may denote  ${}^\rho \mathcal{F}^p$ . Hence on  $R^k g_* \Omega_{Y/\mathbb{D}}(\log)$  we have

$${}^\rho \mathcal{F}^p / {}^\rho \mathcal{F}^{p+1} = R^{k-p} g_* \Omega_{Y/\mathbb{D}}^p(\log).$$

Observe that the  $k$ -th cohomology of a general fiber of  $g$ ,  $H^k(Y_\infty)$ , is naturally identified with  $H^k(X_\infty)$  (see the isomorphism (2.3)). Hence we now have an isomorphism (cf. (2.4))

$${}^\rho \psi: H^k(X_\infty) = H^k(Y_\infty) \xrightarrow{\sim} \mathcal{U}(0) = R^k g_* \Omega_{Y/\mathbb{D}}^\bullet(\log)(0).$$

The isomorphisms  $\psi$  and  ${}^\rho \psi$  are to be compared with [Ste77, (2.12)]. Taking the fiber at 0 of the Hodge filtration  ${}^\rho \mathcal{F}^\bullet$  and transporting it to  $H^k(X_\infty)$  through  ${}^\rho \psi$ , we obtain Steenbrink's limiting Hodge filtration on  $H^k(X_\infty)$ , that we denote by  $F^p H^k(X_\infty)$  or simply  $F_\infty^p$ . To sum up, we have

$$(2.9) \quad \begin{array}{ccc} \mathcal{U}(0) & \cong & {}^\rho \mathcal{F}^p(0) \\ \sim \uparrow {}^\rho \psi & & \sim \uparrow {}^\rho \psi \\ H^k(X_\infty) & \cong & F_\infty^p. \end{array}$$

The limiting Hodge filtration depends on the choice of holomorphic parameter on  $\mathbb{D}$ . The canonical object is the “nilpotent orbit”  $\{\exp(\lambda N) F_\infty^\bullet, \lambda \in \mathbb{C}\}$  in a suitable Grassmannian.

*Remark 2.3.* In [Ste77] Steenbrink allows  $Y$  in (2.8) to be the  $V$ -manifold, and  $g$  to be semi-stable in the sense of  $V$ -manifolds. Such a situation naturally arises as follows. We write  $f^{-1}(0) = \sum_{i=1}^r m_i D_i$ , where the divisors  $D_i$  are smooth, intersect transversally, and the  $m_i$  are their multiplicities in the schematic fiber  $f^{-1}(0)$ . Let  $\ell = \text{lcm}(m_1, \dots, m_r)$ ,  $\rho(t) = t^\ell$  and perform the base change  $X \times_\rho \mathbb{D}$ , of  $f$  by  $\rho$ . Then define  $Y$  as the normalization of  $X \times_\rho \mathbb{D}$ . Steenbrink shows that  $Y$  is a  $V$ -manifold and the structure morphism  $g: Y \rightarrow \mathbb{D}$  is semi-stable in the sense of  $V$ -manifolds. Our discussion equally applies to this setting.

To conclude this subsection, we compare Hodge filtrations and Hodge bundles before and after semi-stable reduction. Notice that there are inclusions, induced by pull-back of differential forms from  $X$  to  $Y$ ,

$$\begin{aligned} \rho^* R^k f_* \Omega_{X/\mathbb{D}}^\bullet(\log) &\subseteq R^k g_* \Omega_{Y/\mathbb{D}}^\bullet(\log), \\ \rho^* \mathcal{F}_{\log}^p &\subseteq {}^\rho \mathcal{F}^p, \quad \rho^* R^q f_* \Omega_{X/\mathbb{D}}^p(\log) \subseteq R^q g_* \Omega_{Y/\mathbb{D}}^p(\log). \end{aligned}$$

The respective quotients of these inclusions are torsion sheaves supported at the point 0. We may thus write

$$\frac{\rho \mathcal{F}^p}{\rho^* \mathcal{F}_{\log}^p} \simeq \bigoplus_{j=1}^{h^p} \frac{\mathcal{O}_{\mathbb{D},0}}{t^{a_j^p} \mathcal{O}_{\mathbb{D},0}},$$

for some integers  $a_j^p \geq 0$ , and similarly for the Hodge bundles

$$\frac{R^q g_* \Omega_{Y/\mathbb{D}}^p(\log)}{\rho^* R^q f_* \Omega_{X/\mathbb{D}}^p(\log)} \simeq \bigoplus_{j=1}^{h^{p,q}} \frac{\mathcal{O}_{\mathbb{D},0}}{t^{b_j^{p,q}} \mathcal{O}_{\mathbb{D},0}},$$

for some integers  $b_j^{p,q} \geq 0$ . We indicated by  $h^p$  the rank of  $\mathcal{F}_{\log}^p$  and  $h^{p,q}$  the rank of  $R^q f_* \Omega_{X/\mathbb{D}}^p(\log)$ .

**Lemma 2.4.** *The integers  $a_j^p$  and  $b_r^{p,q}$  satisfy*

$$0 \leq a_j^p \leq \ell - 1, \quad 0 \leq b_r^{p,q} \leq \ell - 1,$$

where  $\ell = \deg \rho$ . Moreover, the rational numbers  $a_j^p / \ell$  and  $b_r^{p,q} / \ell$  are independent of the choice of semi-stable reduction.

*Proof.* The first statement is a direct consequence of Lemma 2.1. The second assertion easily follows from two facts: first, in the unipotent case, formation of canonical extensions commutes with pull-back by  $t \mapsto t^m$  on  $\mathbb{D}$ ; second, any two semi-stable reduction diagrams are dominated by a third one. This is combined with (2.6)–(2.7), applied in the unipotent case.  $\square$

The second claim of the lemma can be reformulated by saying that the rational numbers  $a_j^p / \ell$  and  $b_j^{p,q} / \ell$  only depend on the local system  $R^k f_*^\times \mathbb{C}$  and its associated variation of Hodge structures. We give them a name:

**Definition 2.5** (Elementary exponents of Hodge bundles). *The rational numbers  $\alpha_j^p := a_j^p / \ell$  are called the elementary exponents of  $p$ -th level of the Hodge filtration for  $R^k f_*^\times \mathbb{C}$ . The rational numbers  $\alpha_j^{p,q} := b_j^{p,q} / \ell$  are called the elementary exponents of the  $(p, q)$  Hodge bundle for  $R^k f_*^\times \mathbb{C}$  ( $k = p + q$ ). We also denote by  $\alpha^{p,q} = \sum_{j=1}^{h^{p,q}} \alpha_j^{p,q}$ , i.e. the sum of all the elementary exponents of the  $(p, q)$  Hodge bundle.*

In the rest of this section we relate the elementary exponents to the eigenvalues of  $T_s$  acting on the limiting Hodge filtration.

**2.3. Elementary exponents and eigenvalues of monodromy.** We maintain the setting and notations of the previous subsections. Hence  $f: X \rightarrow \mathbb{D}$  is a normal crossings projective degeneration over the unit disc. Also,  $g: Y \rightarrow \mathbb{D}$  is a semi-stable reduction of  $f$  as in (2.8), existing after some finite base change  $\rho(t) = t^\ell$ . We compared the Hodge filtrations  $\mathcal{F}_{\log}^\bullet$  on  $\mathcal{V}_{\log} := R^k f_* \Omega_{X/\mathbb{D}}^\bullet(\log)$  and  ${}^\rho \mathcal{F}^\bullet$  on  $\mathcal{U} := R^k g_* \Omega_{Y/\mathbb{D}}^\bullet(\log)$ , producing the elementary exponents of Definition 2.5. The limiting Hodge filtration  $F_\infty^\bullet$  on  $H^k(X_\infty) = H^k(Y_\infty)$  was obtained after identifying the latter with  $\mathcal{U}(0)$  via the isomorphism  ${}^\rho \psi$ , and then transporting the filtration  ${}^\rho \mathcal{F}^\bullet(0)$  through this identification (cf. (2.9)). Recall that the semi-simple part of the monodromy operator  $T_s$  acts on  $\mathcal{U}(0)$ .

**Theorem 2.6.** *Let  $\sigma_1, \dots, \sigma_h$  be a basis of the  $\mathcal{O}_{\mathbb{D}}$ -module  $\mathcal{F}_{\log}^p$ . Then there exists another basis  $\theta_1, \dots, \theta_h$  with the following properties:*

(a) *for every  $m = 1, \dots, h$ , we have*

$$\bigoplus_{j=1}^m \mathcal{O}_{\mathbb{D}} \sigma_j = \bigoplus_{j=1}^m \mathcal{O}_{\mathbb{D}} \theta_j.$$

(b) *there exist integers  $0 \leq a_j \leq \ell - 1$  such that the elements  ${}^\rho \theta_j := t^{-a_j} \rho^* \theta_j$  define a  $\mathcal{O}_{\mathbb{D}}$ -basis of  ${}^\rho \mathcal{F}^p$ .*

(c) *the elementary exponent  $\kappa(\theta_j)$  equals  $a_j/\ell$ . Hence the fiber at 0 element  ${}^\rho \theta_j(0) \in \mathcal{U}(0)$  is an eigenvector of  $T_s$  of eigenvalue  $\exp(-2\pi i a_j/\ell)$ .*

*In particular, the operator  $T_s$  preserves the limiting Hodge filtration  $F_\infty^\bullet$  and the rational numbers  $a_j/\ell$  are, modulo reordering, the elementary exponents of  $p$ -th level of the Hodge filtration for  $R^k f_* \mathbb{C}$ .*

*Proof.* Fix a basis of  $V_\infty$ , say  $\mathbf{e}_1, \dots, \mathbf{e}_r$ , made of eigenvectors of  $T_s$ . We write their eigenvalues as  $\exp(-2\pi i k_1/\ell), \dots, \exp(-2\pi i k_r/\ell)$ , with  $0 \leq k_j \leq \ell - 1$  ordered increasingly. Recall the notations  $\bar{\mathbf{e}}_j$  and  ${}^\rho \bar{\mathbf{e}}_j$  for the associated  $\mathcal{O}_{\mathbb{D}}$ -module bases of  $\mathcal{V}_{\log}$  and  $\mathcal{U}$ . We construct the basis  $\theta_1, \dots, \theta_h$  inductively.

We start by applying Lemma 2.1 to  $\sigma_1$ . Hence there exists an integer  $a_1$ , with  $0 \leq a_1 \leq \ell - 1$ , such that  ${}^\rho \sigma_1 := t^{-a_1} \rho^* \sigma_1 \in \mathcal{U}$  and  ${}^\rho \sigma_1(0)$  is an eigenvector of  $T_s$  (in particular non-zero). Hence (a) is satisfied for  $k = 1$  with the choice  $\theta_1 = \sigma_1$ . Also (c) is satisfied.

Suppose we already constructed  $\theta_1, \dots, \theta_m$  as in the statement, for some  $m < h$ . More precisely, this means:

- condition (a) holds restricted to the range  $1, \dots, m$ ;
- for every  $j \leq m$  there exists an integer  $0 \leq a_j \leq \ell - 1$  such that  ${}^\rho \theta_j = t^{-a_j} \rho^* \theta_j$  belongs to  ${}^\rho \mathcal{F}^p$ , and the vectors  ${}^\rho \theta_1(0), \dots, {}^\rho \theta_m(0)$  are linearly independent;
- $\kappa(\theta_j) = a_j/\ell$  and  ${}^\rho \theta_j(0)$  is an eigenvector of  $T_s$  of eigenvalue  $\exp(-2\pi i a_j/\ell)$ .

In particular, the vector  $\sigma_{m+1}(0)$  is linearly independent with  $\theta_1(0), \dots, \theta_m(0)$ . We apply Lemma 2.1 with  $\sigma = \sigma_{m+1}$ . We write the  $T_s$ -eigenvalue of  ${}^\rho \sigma(0)$  as  $\exp(-2\pi i a/\ell)$ , for some  $0 \leq a \leq \ell - 1$ . Two cases can occur. In the first case,  ${}^\rho \sigma(0)$  is already linearly independent with  ${}^\rho \theta_1(0), \dots, {}^\rho \theta_m(0)$ . Then we take  $\theta_{m+1} = \sigma$  and  $a_{m+1} = a$ . In the second case, there is a non-trivial linear relation:

$$(2.10) \quad {}^\rho \sigma(0) = \sum_{j=1}^m \mu_j {}^\rho \theta_j(0).$$

We infer for the coefficients  $\mu_j$  that, either  $\mu_j = 0$ , or  $\mu_j \neq 0$  and then  ${}^\rho \theta_j(0)$  has eigenvalue  $\exp(-2\pi i a/\ell)$ . Hence, if  $\mu_j \neq 0$  then  $a_j = a$  and  ${}^\rho \theta_j = t^{-a} \rho^* \theta_j$ . We denote by  $J(\sigma)$  the non-empty

subset of  $\{1, \dots, m\}$  for which  $a_j = a$ . Define now the sections

$$\sigma' := \sigma - \sum_{j \in J(\sigma)} \mu_j \theta_j \in \Gamma(\mathbb{D}, \mathcal{F}_{\log}^{\rho})$$

and

$$\hat{\sigma}' = t^{-a} \rho^* \sigma' = {}^{\rho} \sigma - \sum_{j \in J(\sigma)} \mu_j {}^{\rho} \theta_j \in \Gamma(\mathbb{D}, {}^{\rho} \mathcal{F}^{\rho}).$$

Due to (2.10), this section is necessarily of the form

$$\hat{\sigma}' = t^{b-a} \sum_{k_j=b} \gamma_j {}^{\rho} \tilde{\mathbf{e}}_j(t) + \sum_{k_j>b} t^{k_j-a} \gamma_j {}^{\rho} \tilde{\mathbf{e}}_j(t) + t^{-a} R(t^{\ell}),$$

for some constants  $\gamma_j$ , some minimal  $b > a$  and  $R(t^{\ell}) = \rho^* R(s)$ , where  $R(s)$  a regular section of  $\mathcal{Y}_{\log}$  vanishing at  $s = 0$ . Notice the first sum is non-trivial. Otherwise, by the minimality of  $b$ , it would necessarily be  $\sigma' = R(s)$ , which vanishes at 0 and hence entails

$$\sigma(0) = \sum_j \mu_j \theta_j(0),$$

contradicting the linear independence of  $\sigma(0) = \sigma_{m+1}(0)$  with  $\theta_1(0), \dots, \theta_m(0)$  observed before. Now

$$t^{-(b-a)} \hat{\sigma}' = t^{-b} \rho^* \sigma' = {}^{\rho} \sigma'$$

is a regular section of  ${}^{\rho} \mathcal{F}^{\rho}$  and its value at 0 is

$$\sum_{k_j=b} \gamma_j {}^{\rho} \tilde{\mathbf{e}}_j(0),$$

which is non-zero and an eigenvector of  $T_s$  of eigenvalue  $\exp(-2\pi i b/\ell)$ . Hence  $\kappa(\sigma') = b/\ell$ . Furthermore,  ${}^{\rho} \sigma'$  is independent with the sections  ${}^{\rho} \theta_j$  with  $j \in J(\sigma)$ . Indeed, the eigenvalue of  ${}^{\rho} \sigma'(0)$  is  $\exp(-2\pi i b/\ell)$  and for  $j \in J(\sigma)$  the eigenvalue of  ${}^{\rho} \theta_j(0)$  is  $\exp(-2\pi i a/\ell)$ , and  $b > a$ . Still, it could be that  ${}^{\rho} \sigma'(0)$  is linearly dependent with the  ${}^{\rho} \theta_j(0)$ , with  $j \notin J(\sigma)$ . If such is the case, then we repeat the argument, starting with a non-trivial linear relation

$${}^{\rho} \sigma'(0) = \sum_{j \in \{1, \dots, m\} \setminus J(\sigma)} \mu'_j {}^{\rho} \theta_j(0).$$

In particular, this produces a new subset of indices  $J(\sigma') \subset \{1, \dots, m\}$  disjoint with  $J(\sigma)$ . This procedure clearly comes to an end after a finite number of steps, since  $J(\sigma') \cap J(\sigma) = \emptyset$  and  $\kappa(\sigma') = b/\ell > a/\ell = \kappa(\sigma)$ . The outcome is a section  $\theta_{m+1}$  with the expected properties. Namely, the sections  $\theta_1, \dots, \theta_{m+1}$  are independent and satisfy

$$\bigoplus_{j=1}^{m+1} \mathcal{O}_{\mathbb{D}} \sigma_j = \bigoplus_{j=1}^{m+1} \mathcal{O}_{\mathbb{D}} \theta_j.$$

Furthermore, for some  $0 \leq a_{m+1} \leq \ell - 1$ ,  ${}^{\rho} \theta_{m+1} = t^{-a_{m+1}} \rho^* \theta_{m+1}$  is a regular section of  ${}^{\rho} \mathcal{F}^{\rho}$ , and its value at 0 is an eigenvector of  $T_s$  of eigenvalue  $\exp(-2\pi i a_{m+1}/\ell)$ .  $\square$

*Remark 2.7.* (i) Recall that the limiting Hodge structure  $F_{\infty}^{\bullet}$  depends on the choice of holomorphic coordinate on  $\mathbb{D}$ . However, the nilpotent orbit  $\{\exp(\lambda N) F_{\infty}^{\bullet}, \lambda \in \mathbb{C}\}$  is canonical. The operators  $T_s$  and  $\exp(N\lambda)$  commute, and therefore  $T_s$  preserves  $\exp(\lambda N) F_{\infty}^{\bullet}$  as well. Also, the eigenvalues of  $T_s$  on  $F_{\infty}^{\rho}$  and  $\exp(\lambda N) F_{\infty}^{\rho}$  are the same, and hence they only depend on the nilpotent orbit. This is consistent with the fact that the elementary exponents do not depend on the choice of coordinate on  $\mathbb{D}$ . It follows that, for the purpose of comparing the

elementary exponents with the monodromy eigenvalues,  $\mathbb{D}$  can be taken to have any radius, as opposed to the radius 1 assumption we made so far.

- (ii) For one-variable variations of *polarized* Hodge structures, the invariance of the limiting Hodge filtration under the action of  $T_s$  is already observed by Schmid [Sch73, (4.9)]. A different argument for the invariance in the geometric case is provided by Steenbrink [Ste77, Thm. 2.13].
- (iii) Theorem 2.6 can be stated in the more general context of variations of Hodge structures. We restricted to the geometric case for the sake of conciseness.

As a consequence, we derive the corresponding statement for the elementary exponents of the Hodge bundles:

**Corollary 2.8.** *The elementary exponents  $\alpha_j^{p,q}$  of the  $(p, q)$  Hodge bundle for  $R^k f_*^\times \mathbb{C}$  ( $k = p + q$ ) are such that the  $\exp(-2\pi i \alpha_j^{p,q})$  constitute the eigenvalues of  $T_s$  acting on  $\mathrm{Gr}_{F_\infty}^p H^k(X_\infty)$  (with multiplicities). In particular, we have*

$$\alpha^{p,q} = -\frac{1}{2\pi i} \mathrm{tr} \left( {}^\ell \log T_s \mid \mathrm{Gr}_{F_\infty}^p H^{p+q}(X_\infty) \right).$$

*Proof.* Recall that the graded quotients  $\mathcal{F}_{\log}^p / \mathcal{F}_{\log}^{p+1}$  are locally free and isomorphic to  $R^{k-p} f_* \Omega_{X/\mathbb{D}}^p(\log)$ . Similarly after semi-stable reduction. The result easily follows from Theorem 2.6, by choosing a basis  $\sigma_1, \dots, \sigma_h$  of  $\mathcal{F}_{\log}^{p+1}$  and completing it into a basis  $\sigma_1, \dots, \sigma_r$  of  $\mathcal{F}_{\log}^p$ , in such a way that the  $\sigma_{h+1}, \dots, \sigma_r$  project onto a basis of  $R^{k-p} f_* \Omega_{X/\mathbb{D}}^p(\log)$ .  $\square$

*Remark 2.9.* (i) Steenbrink's construction of the limiting Hodge filtration via  $V$ -manifolds, recalled in Remark 2.3, shows that we may take  $\ell = \mathrm{lcm}(m_1, \dots, m_r)$ , where the  $m_i$  are the multiplicities of the irreducible components of the schematic fiber  $f^{-1}(0)$ .  
(ii) In the polarized setting and for primitive cohomology groups, one has analogous statements to Theorem 2.6 and Corollary 2.8, formally with the same proof. The formulation and details are left to the reader.

For other extensions, the same argument gives an analogous result of Theorem 2.6. However, it is not clear that the successive quotients of the (extended) Hodge filtration are locally free. Nonetheless, for the upper extension, the situation is much better. Precisely, let  ${}^u\mathcal{V}$  be the upper extension of the local system  $\mathcal{V} = R^k f_*^\times \mathbb{C} \otimes \mathcal{O}_{\mathbb{D}^\times}$  (see §2.1), and extend the Hodge filtration analogously to (2.6). Then, by [Kol86, Prop. 2.9 & Lemma 2.11]<sup>6</sup>, the extended Hodge filtration  ${}^u\mathcal{F}^\bullet$  has locally free successive quotients  ${}^u\mathcal{F}^p / {}^u\mathcal{F}^{p+1}$ . By *loc. cit.*, if  $k = n + i$  the last piece of the filtration has the form

$${}^u\mathcal{F}^n = R^i f_* \omega_{X/\mathbb{D}},$$

where  $n$  is as usual the fiber dimension of the family. We thus have a counterpart of Corollary 2.8 which generalizes our previous [EFiMM18, Theorem A] for the direct image of the relative canonical sheaf of a degenerating family of Calabi–Yau varieties:

**Proposition 2.10.** *Let  $f : X \rightarrow \mathbb{D}$  be a projective degeneration between complex manifolds. If  $g : Y \rightarrow \mathbb{D}$  denotes a semi-stable reduction as in diagram (2.8), there is a natural inclusion of locally free sheaves*

$$R^i g_* \omega_{Y/\mathbb{D}} \rightarrow \rho^* R^i f_* \omega_{X/\mathbb{D}}.$$

<sup>6</sup>The discussion in *loc. cit.* and the references used therein are valid over a one-dimensional disc.



The quotient is of the form  $\bigoplus_{j=1}^{h^{n,i}} \frac{\mathcal{O}_{\mathbb{D},0}}{t^{a_j} \mathcal{O}_{\mathbb{D},0}}$ . Then the rational numbers  $a_j / \deg \rho \in [0, 1) \cap \mathbb{Q}$  are independent of the choice of semi-stable reduction, and  $\exp(2\pi i a_j / \deg \rho)$  are the eigenvalues of  $T_s$  acting on  $F_\infty^n H^{n+i}(X_\infty)$ .

### 3. KÄHLER EXTENSIONS OF DETERMINANTS OF HODGE BUNDLES

In the theory of the BCOV line bundle, it is frequent to work with the Kähler extensions of the sheaves of differentials, rather than their logarithmic counterparts (see *e.g.* [FLY08, EFIMM18]). In this section we compare the determinant of cohomology of these two kinds of extensions. While in the comparison properties addressed in the previous section we found eigenvalues of the monodromy operator, in this section we encounter other invariants such as the total dimension of vanishing cycles.

**3.1. Kähler extensions.** Let  $f: X \rightarrow S$  be a projective degeneration between complex manifolds, of relative dimension  $n$ . We assume that  $S$  is a connected complex curve. Let  $S^\times \subseteq S$  be the (non-empty and Zariski open) locus of regular values of  $f$ . Write  $X^\times = f^{-1}(S^\times)$ . Then  $\Omega_{X^\times/S^\times}^p$  is a locally free sheaf and its determinant of cohomology  $\lambda(\Omega_{X^\times/S^\times}^p) = \det Rf_*^\times(\Omega_{X^\times/S^\times}^p)$  is defined (cf. [KM76] for the notion of determinants of perfect complexes. It generalizes the concept of maximal exterior powers of vector bundles).

**Definition 3.1.** For each  $0 \leq p \leq n$ , consider the complex of locally free sheaves

$$\tilde{\Omega}_{X/S}^p: (f^* \Omega_S)^{\otimes p} \rightarrow (f^* \Omega_S)^{\otimes p-1} \otimes \Omega_X \rightarrow \cdots \rightarrow (f^* \Omega_S) \otimes \Omega_X^{p-1} \rightarrow \Omega_X^p,$$

quasi-isomorphic on  $X^\times$  to  $\Omega_{X^\times/S^\times}^p$ . We define the Kähler extension of  $\lambda(\Omega_{X^\times/S^\times}^p)$ , as

$$(3.1) \quad \lambda(\tilde{\Omega}_{X/S}^p) = \bigotimes_{k=0}^p \lambda\left((f^* \Omega_S)^{\otimes k} \otimes \Omega_X^{p-k}\right)^{(-1)^k}.$$

*Remark 3.2.* In [Sai04, Example 4.2], the complex  $\tilde{\Omega}_{X/S}^p$  is referred to as the derived exterior power complex  $L\Lambda^q K$ , applying a functor  $L\Lambda^q$  to  $K = [f^* \Omega_S \rightarrow \Omega_X]$ . We refer the reader to *loc. cit.* for further discussions of this notion.

**3.2. Compatibility with Serre duality.** We study the compatibility of the formation of  $\lambda(\tilde{\Omega}_{X/S}^p)$  with Serre duality. For simplicity, we restrict to the local case  $S = \mathbb{D}$  and  $f$  is a submersion on  $\mathbb{D}^\times$ . As  $X$  is smooth, the relative dualizing sheaf  $\omega_{X/\mathbb{D}}$  of  $f: X \rightarrow \mathbb{D}$  is isomorphic to  $\det \Omega_{X/\mathbb{D}} \simeq K_X \otimes K_{\mathbb{D}}^{-1}$ . Then there is a canonical pairing

$$\tilde{\Omega}_{X/\mathbb{D}}^p \times \tilde{\Omega}_{X/\mathbb{D}}^{n-p} \longrightarrow \tilde{\Omega}_{X/\mathbb{D}}^n \longrightarrow \omega_{X/\mathbb{D}}.$$

The pairing is described in [Sai04, p. 420]<sup>7</sup> and extends the natural pairing on relative Kähler differentials. In particular, on the smooth locus, it is a perfect pairing. We conclude there is a morphism of complexes

$$\rho_p: \tilde{\Omega}_{X/\mathbb{D}}^p \longrightarrow R\mathrm{Hom}(\tilde{\Omega}_{X/\mathbb{D}}^{n-p}, \omega_{X/\mathbb{D}}).$$

The cone of  $\rho_p$  is acyclic on the smooth locus and its homology is supported on the singular locus. By Grothendieck-Serre duality, we have a quasi-isomorphism

$$Rf_* R\mathrm{Hom}(\tilde{\Omega}_{X/\mathbb{D}}^{n-p}, \omega_{X/\mathbb{D}}[n]) \simeq R\mathrm{Hom}(Rf_* \tilde{\Omega}_{X/\mathbb{D}}^{n-p}, \mathcal{O}_{\mathbb{D}}).$$

<sup>7</sup>To be strictly conform with *loc. cit.*, we should work with the projective morphism of regular schemes over  $\mathrm{Spec} \mathbb{C}\{t\}$  induced by  $f$ . The reader will get easily convinced that this abuse of notation is justified, for the results of this section can be checked after localization at the origin.

Taking determinants we obtain an isomorphism

$$\det(R\mathrm{Hom}(\tilde{\Omega}_{X/\mathbb{D}}^{n-p}, \omega_{X/\mathbb{D}})) \simeq \det Rf_*(\tilde{\Omega}_{X/\mathbb{D}}^{n-p})^{(-1)^{n+1}}.$$

We conclude that there is an isomorphism, where we write the tensor products of line bundles additively for lighter notations,

$$\lambda(\tilde{\Omega}_{X/\mathbb{D}}^p) \simeq (-1)^{n+1} \lambda(\tilde{\Omega}_{X/\mathbb{D}}^{n-p}) - \delta_p \cdot \mathcal{O}([0])$$

where  $\delta_p \in \mathbb{Z}$ . Outside of the origin, it extends the (canonical) Serre duality isomorphism. By dévissage we find that

$$\delta_p = \sum (-1)^k \ell_{\mathcal{O}_{\mathbb{D},0}} \left( R^k f_* \mathrm{cone}(\rho_p) \right).$$

Here  $\ell_R$  denotes the length of an  $R$ -module. It follows from [Sai04, Cor. 4.9],

$$\delta_p = (-1)^{n-p} \deg c_{n+1}^{X_0}(\Omega_{X/\mathbb{D}}).$$

Since we suppose that the total space  $X$  is smooth, we have the formula (cf. [Ful98, Ex. 14.1.5])

$$(3.2) \quad \deg c_{n+1}^{X_0}(\Omega_{X/\mathbb{D}}) = (-1)^n (\chi(X_\infty) - \chi(X_0))$$

where  $\chi(X_\infty)$  (resp.  $\chi(X_0)$ ) denotes the topological Euler characteristic of a general (resp. special) fiber of  $f: X \rightarrow \mathbb{D}$ .

We summarize the above observations in the following proposition:

**Proposition 3.3.** *Outside the origin, Serre duality induces an isomorphism*

$$\lambda(\Omega_{X^\times/\mathbb{D}^\times}^p) \simeq (-1)^{n+1} \lambda(\Omega_{X^\times/\mathbb{D}^\times}^{n-p}).$$

*It extends to an isomorphism of Kähler extensions*

$$\lambda(\tilde{\Omega}_{X/\mathbb{D}}^p) \simeq (-1)^{n+1} \lambda(\tilde{\Omega}_{X/\mathbb{D}}^{n-p}) + (-1)^{n+1-p} c \cdot \mathcal{O}([0]).$$

Here  $c = \deg c_{n+1}^{X_0}(\Omega_{X/\mathbb{D}})$  is the degree of the localized top Chern class, which is computed by (3.2).

**Corollary 3.4.** *In even relative dimension  $n = 2m$ , there is an isomorphism*

$$2\lambda(\tilde{\Omega}_{X/\mathbb{D}}^m) \simeq (-1)^{m+1} c \cdot \mathcal{O}([0]).$$

The above results adapt to a general one dimensional base  $S$  and the case of several singular fibers, and integrating over a compact Riemann surface we obtain the following corollary:

**Corollary 3.5.** *Suppose that  $S$  is a compact Riemann surface and  $f: X \rightarrow S$  is a degeneration with at most ordinary double point singularities, of even relative dimension. Then the number of singularities is even.*

*Proof.* For ordinary double point singularities, it follows from (3.2) that the localized Chern classes compute the number of singular points in the fibers (cf. [Ful98, Ex. 14.1.5 (d)]). We conclude by Corollary 3.4.  $\square$

*Remark 3.6.* In the special case of a Lefschetz pencil of degree  $d$  hypersurfaces in  $\mathbb{P}^{2m+1}$ , the above corollary indicates that there are an even number of singular fibers. This is compatible with the fact that the degree of the discriminant variety is  $(2m+2)(d-1)^{2m+1}$  by Boole's formula [GKZ08, Chap. 1].

**3.3. Comparison with the logarithmic extensions.** Let  $f: X \rightarrow \mathbb{D}$  be as before. If the special fiber  $X_0$  has not-necessarily reduced normal crossings, then  $\lambda(\Omega_{X^\times/\mathbb{D}^\times}^p)$  affords two natural extensions: the Kähler extension  $\lambda(\tilde{\Omega}_{X/\mathbb{D}}^p)$  and the logarithmic extension  $\lambda(\Omega_{X/\mathbb{D}}^p(\log))$ . Both can be compared:

$$(3.3) \quad \lambda(\tilde{\Omega}_{X/\mathbb{D}}^p) = \lambda(\Omega_{X/\mathbb{D}}^p(\log)) + \mu_p \cdot \mathcal{O}([0]),$$

for some integer  $\mu_p$ . We now describe general expressions for  $\mu_p$  in terms of the geometry of  $X_0$ . More generally, without any assumption on the special fiber, we can reduce to the normal crossings case by an embedded resolution of singularities. One then needs to keep track of the change of the determinant of cohomology of the Kähler differentials under the blowing-up process. This setting will be partially treated in a second step, after the easier normal crossings case.

Suppose that the special fiber of  $f: X \rightarrow \mathbb{D}$  is a normal crossings divisor of the form  $X_0 = \sum_{i=1}^r m_i D_i$ , with smooth  $D_i$ . Define for  $I \subseteq \{1, \dots, r\}$ ,  $D_I = \bigcap_{i \in I} D_i$ . For  $k \geq 0$  an integer, also define the codimension  $k$  stratum in  $X$

$$D(k) = \bigsqcup_{\substack{I \subseteq \{1, \dots, r\} \\ |I|=k}} D_I$$

and denote by  $a_k: D(k) \rightarrow X_0$  the natural map. Finally, given a cohomological complex  $\mathcal{C}^\bullet$ , denote by  $\mathcal{C}^{\bullet \leq k}$  the bête truncation of the complex up to degree  $k$ , namely  $\dots \rightarrow \mathcal{C}^{k-2} \rightarrow \mathcal{C}^{k-1} \rightarrow \mathcal{C}^k$ . This notation will be applied below for holomorphic de Rham complexes.

**Proposition 3.7.** *Assume that  $f: X \rightarrow \mathbb{D}$  is a projective normal crossings degeneration, and write  $X_0 = \sum_i m_i D_i$ , where the  $D_i$  are the reduced irreducible components, assumed to be smooth. Then*

$$\mu_p = (-1)^{p-1} \chi(\Omega_{X_\infty}^{\bullet \leq p-1}) - \sum_{k=1}^p (-1)^{p-k} \chi(\Omega_{D(k)}^{\bullet \leq p-k}).$$

*In particular,*

$$\mu_1 = \chi(\mathcal{O}_{X_\infty}) - \sum \chi(\mathcal{O}_{D_i}).$$

*If moreover  $f$  is semi-stable, i.e.  $X_0$  is in addition reduced, then*

$$\mu_1 = - \sum_{k \geq 2} (-1)^k \chi(\mathcal{O}_{D(k)}).$$

*Proof.* Applying the projection formula to the definition (3.1) we find

$$\lambda(\tilde{\Omega}_{X/\mathbb{D}}^p) = \sum_{k=0}^p (-1)^k \det(\Omega_{\mathbb{D}}^{\otimes k} \otimes Rf_* \Omega_X^{p-k}).$$

Here  $Rf_* \Omega_X^{p-k}$  is the whole right derived image of  $\Omega_X^{p-k}$ , which is a perfect complex on  $\mathbb{D}$ . We now want to use that for a product of two perfect complexes  $A, B$ , we have  $\det(A \otimes B) = (\text{rk } B)(\det A) + (\text{rk } A)(\det B)$ . The rank of  $Rf_* \Omega_X^{p-k}$  can be computed on the generic fiber  $X_\infty$ , and hence equals to  $\chi(\Omega_X^{p-k}|_{X_\infty})$ . From the cotangent sequence of  $f$ , one can derive the relation

$$\chi(\Omega_X^{p-k}|_{X_\infty}) = \chi(\Omega_{X_\infty}^{p-k}) + \chi(\Omega_{X_\infty}^{p-k-1}),$$

with the convention  $\chi(\Omega_{X_\infty}^{-1}) = 0$ . We find that the determinant Kähler extension can be written

(3.4)

$$\begin{aligned} \lambda(\tilde{\Omega}_{X/\mathbb{D}}^p) &= \sum_{k=0}^p (-1)^k k \left( \chi(\Omega_{X_\infty}^{p-k}) + \chi(\Omega_{X_\infty}^{p-k-1}) \right) \Omega_{\mathbb{D}} + \sum_{k=0}^p (-1)^k \lambda(\Omega_X^{p-k}) \\ &= \sum_{k=0}^{p-1} (-1)^{p-k} \chi(\Omega_{X_\infty}^k) \Omega_{\mathbb{D}} + \sum_{k=0}^p (-1)^k \lambda(\Omega_X^{p-k}) = (-1)^p \chi(\Omega_{X_\infty}^{\bullet \leq p-1}) \Omega_{\mathbb{D}} + \sum_{k=0}^p (-1)^k \lambda(\Omega_X^{p-k}). \end{aligned}$$

On the other hand, the sheaf  $\Omega_{X/\mathbb{D}}(\log)$  is already locally free, so in fact the  $p$ -th derived exterior power is a resolution of  $\Omega_{X/\mathbb{D}}^p(\log)$ . Then in a similar way we find

$$(3.5) \quad \lambda(\Omega_{X/\mathbb{D}}^p(\log)) = (-1)^p \chi(\Omega_{X_\infty}^{\bullet \leq p-1}) \Omega_{\mathbb{D}}(\log[0]) + \sum_{k=0}^p (-1)^k \lambda(\Omega_X^{p-k}(\log X_0)).$$

Comparing (3.4) and (3.5), we find

$$(3.6) \quad \lambda(\Omega_{X/\mathbb{D}}^p(\log)) - \lambda(\tilde{\Omega}_{X/\mathbb{D}}^p) = (-1)^p \chi(\Omega_{X_\infty}^{\bullet \leq p-1}) \mathcal{O}([0]) + \sum_{k=0}^p (-1)^k \left( \lambda(\Omega_X^{p-k}(\log X_0)) - \lambda(\Omega_X^{p-k}) \right).$$

For the second term on the right hand side of (3.6), we introduce the filtration by the order of poles on  $\Omega_X^k(\log)$ :

$$W_m \Omega_X^k(\log X_0) = \Omega_X^{k-m} \wedge \Omega_X^m(\log X_0), \quad m \leq k \quad \text{with} \quad \text{Gr}_m^W \Omega_X^k(\log X_0) \simeq (a_m)_* \Omega_{D(m)}^{k-m}.$$

We find

$$\lambda(\Omega_X^k(\log X_0)) - \lambda(\Omega_X^k) = \sum_{m=1}^k \lambda((a_m)_* \Omega_{D(m)}^{k-m}) = \sum_{m=1}^k \chi(\Omega_{D(m)}^{k-m}) \mathcal{O}([0]),$$

and then

$$(3.7) \quad \sum_{k=0}^p (-1)^k \left( \lambda(\Omega_X^{p-k}(\log X_0)) - \lambda(\Omega_X^{p-k}) \right) = \sum_{k=1}^p (-1)^{p-k} \chi(\Omega_{D(k)}^{\bullet \leq p-k}) \mathcal{O}([0]).$$

To sum up, equations (3.6)–(3.7) combine to

$$\lambda(\Omega_{X/\mathbb{D}}^p(\log)) - \lambda(\tilde{\Omega}_{X/\mathbb{D}}^p) = \left( (-1)^p \chi(\Omega_{X_\infty}^{\bullet \leq p-1}) + \sum_{k=1}^p (-1)^{p-k} \chi(\Omega_{D(k)}^{\bullet \leq p-k}) \right) \mathcal{O}([0]),$$

hence

$$\mu_p = (-1)^{p-1} \chi(\Omega_{X_\infty}^{\bullet \leq p-1}) - \sum_{k=1}^p (-1)^{p-k} \chi(\Omega_{D(k)}^{\bullet \leq p-k}),$$

as was to be shown. This completes the proof of the first part of the statement. The first claimed expression for  $\mu_1$  is readily checked. In the semi-stable case, we consider the natural exact sequence

$$(3.8) \quad 0 \rightarrow \mathcal{O}_{X_0} \rightarrow \mathcal{O}_{D(1)} \rightarrow \mathcal{O}_{D(2)} \rightarrow \mathcal{O}_{D(3)} \rightarrow \dots \rightarrow \mathcal{O}_{D(n)} \rightarrow 0.$$

Taking Euler characteristics, we derive

$$\chi(\mathcal{O}_{X_0}) = - \sum_{k=1}^n (-1)^k \chi(\mathcal{O}_{D(k)}).$$

By flatness of  $f: X \rightarrow \mathbb{D}$  we have  $\chi(\mathcal{O}_{X_\infty}) = \chi(\mathcal{O}_{X_0})$ . The second claimed expression for  $\mu_1$  follows.  $\square$

We now discuss a variant of Proposition 3.7 for morphisms which are not necessarily in normal crossings form. Before the statement, we need a preliminary observation. If  $f: X \rightarrow \mathbb{D}$  is a projective normal crossings degeneration, with  $X_0 = \sum m_i D_i$  as above, we have a commutative diagram of exact sequences

(3.9)

$$\begin{array}{ccccccc}
& & & & & & 0 \\
& & & & & & \downarrow \\
& & & & & & P \\
& & & & & & \downarrow \\
& & & & & & \mathcal{O}_{X_0} \\
0 & \longrightarrow & f^* \Omega_{\mathbb{D}} & \longrightarrow & f^* \Omega_{\mathbb{D}}(\log[0]) & \xrightarrow{\text{res}} & \mathcal{O}_{X_0} \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \Omega_X & \longrightarrow & \Omega_X(\log X_0) & \xrightarrow{\text{res}} & \mathcal{O}_{D(1)} \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & P & \longrightarrow & \Omega_{X/\mathbb{D}} & \longrightarrow & \Omega_{X/\mathbb{D}}(\log) \longrightarrow 0 \\
& & & & \downarrow & & \downarrow \\
& & & & 0 & & 0 \\
& & & & & & 0
\end{array}$$

We derive the existence of a quasi-isomorphism of complexes

$$\text{cone}[\Omega_{X/\mathbb{D}} \rightarrow \Omega_{X/\mathbb{D}}(\log)] \xrightarrow{\sim} \text{cone}[\mathcal{O}_{X_0} \rightarrow \mathcal{O}_{D(1)}].$$

**Proposition 3.8.** *Suppose  $f: X \rightarrow \mathbb{D}$  is a projective degeneration between complex manifolds. Write  $X_{0, \text{red}} = \sum D_i$ , and  $\tilde{D}_i$  for a desingularization of the irreducible components  $D_i$ . Denote by  $\pi: \tilde{X} \rightarrow X$  a simple normal crossings model. Then there is a canonical isomorphism*

$$\lambda(\Omega_{X/\mathbb{D}}) \xrightarrow{\sim} \lambda(\Omega_{\tilde{X}/\mathbb{D}}(\log)) + \left( \chi(\mathcal{O}_{X_\infty}) - \sum \chi(\mathcal{O}_{\tilde{D}_i}) \right) \mathcal{O}([0])$$

which induces the identity on the smooth locus.

*Proof.* The bundle  $\lambda(\Omega_{\tilde{X}/\mathbb{D}}(\log))$  is independent of the specific normal crossings model  $\tilde{X}$ , since it is built up from lower extensions of Hodge bundles. We construct one model by applying embedded resolution of singularities to  $X_0 \hookrightarrow X$ . This means that  $\tilde{X}$  is obtained by a sequence of blow ups in the special fibers, of  $X_i \rightarrow \mathbb{D}$  say, along regular centers  $Z_i$ . Denote by  $\nu: X_{i+1} \rightarrow X_i$  the blowup. In that case  $Z_i$  is necessarily regularly embedded in  $X_i$  and the exceptional divisor  $E_i$  is a projective bundle over  $Z_i$ . We moreover have the exact sequence

$$(3.10) \quad 0 \longrightarrow \nu^* \Omega_{X_i/\mathbb{D}} \longrightarrow \Omega_{X_{i+1}/\mathbb{D}} \longrightarrow \Omega_{E_i/Z_i} \longrightarrow 0,$$

where the exactness on the left can be justified by a direct computation.

Recalling that  $R\nu_* \mathcal{O}_{X_{i+1}} \simeq \mathcal{O}_{X_i}$ , and taking the determinant of the cohomology, we find that

$$\lambda(\Omega_{X_{i+1}/\mathbb{D}}) \xrightarrow{\sim} \lambda(\Omega_{X_i/\mathbb{D}}) + \chi(\Omega_{E_i/Z_i}) \cdot \mathcal{O}([0]).$$

We notice that  $\chi(\Omega_{E_i/Z_i}) = -\chi(\mathcal{O}_{E_i})$ . Indeed, the Euler exact sequence for the cotangent sheaf of a projective bundle readily implies  $R^p \nu_* \mathcal{O}_{E_i} \simeq R^{p+1} \nu_* \Omega_{E_i/Z_i}$  for all  $p$ . Hence, for corresponding

Leray spectral sequences we have

$$\begin{array}{ccc} H^q(Z_i, R^p v_* \mathcal{O}_{E_i}) & \Longrightarrow & H^{q+p}(E_i, \mathcal{O}_{E_i}) \\ \parallel & & \\ H^q(Z_i, R^{p+1} v_* \Omega_{E_i/Z_i}) & \Longrightarrow & H^{q+p+1}(E_i, \Omega_{E_i/Z_i}), \end{array}$$

and we conclude by taking Euler characteristics. At the end of the sequence of blow ups, we thus find

$$\lambda(\Omega_{\tilde{X}/\mathbb{D}}) \xrightarrow{\sim} \lambda(\Omega_{X/\mathbb{D}}) - \left( \sum \chi(\mathcal{O}_{E_i}) \right) \cdot \mathcal{O}([0]).$$

On the other hand, from (3.9) there is a quasi-isomorphism

$$(3.11) \quad \text{cone}[\Omega_{\tilde{X}/\mathbb{D}} \rightarrow \Omega_{\tilde{X}/\mathbb{D}}(\log)] \xrightarrow{\sim} \text{cone}[\mathcal{O}_{\tilde{X}_0} \rightarrow \bigoplus \mathcal{O}_{D'_i} \oplus \bigoplus \mathcal{O}_{E'_i}],$$

where the  $D'_i$  (resp.  $E'_i$ ) are the strict transforms of the  $D_i$  (res.  $E_i$ ). Recall that the Euler characteristics of the form  $\chi(\mathcal{O}_D)$  are birational invariants of complex manifolds, so that  $\chi(\mathcal{O}_{D'_i}) = \chi(\mathcal{O}_{\tilde{D}_i})$  and  $\chi(\mathcal{O}_{E'_i}) = \chi(\mathcal{O}_{E_i})$ . We derive

$$\lambda(\Omega_{\tilde{X}/\mathbb{D}}(\log)) \simeq \lambda(\Omega_{\tilde{X}/\mathbb{D}}) + \left( \sum \chi(\mathcal{O}_{\tilde{D}_i}) + \sum \chi(\mathcal{O}_{E_i}) - \chi(\mathcal{O}_{\tilde{X}_0}) \right) \cdot \mathcal{O}([0]).$$

Since the family  $\tilde{X} \rightarrow \mathbb{D}$  is flat, we have  $\chi(\mathcal{O}_{\tilde{X}_0}) = \chi(\mathcal{O}_{\tilde{X}_\infty}) = \chi(\mathcal{O}_{X_\infty})$ . The result follows by composing the isomorphisms deduced from (3.10) and (3.11).  $\square$

**Corollary 3.9.** *Under the hypothesis of Proposition 3.8, if  $X_0$  has at most rational singularities, then*

$$\lambda(\Omega_{X/\mathbb{D}}) \xrightarrow{\sim} \lambda(\Omega_{\tilde{X}/\mathbb{D}}(\log)).$$

*This in particular holds whenever  $n \geq 2$  and  $X_0$  admits at most ordinary double point singularities.*

*Proof.* Since  $X_0$  is connected with at most rational singularities, it is in particular normal irreducible. Hence, with notations as in Proposition 3.8, we have  $X_0 = D$ . We then observe  $\chi(\mathcal{O}_{\tilde{D}}) = \chi(\mathcal{O}_{X_0})$  by the rational singularities assumption and  $\chi(\mathcal{O}_{X_0}) = \chi(\mathcal{O}_{X_\infty})$  by flatness.  $\square$

**3.4. Families with at most ordinary double point singularities.** In this subsection we study the Hodge bundles of morphisms whose singular fibers have at most ordinary double point singularities.

Let  $f: X \rightarrow \mathbb{D}$  be a projective degeneration between complex manifolds, with fibers of dimension  $n$ . We assume that the special fiber  $X_0$  has at most ordinary double point singularities. Hence, on a neighborhood (in  $X$ ) of a singular point of  $f^{-1}(0)$ , there exist holomorphic coordinates  $(z_0, \dots, z_n, s)$  such that  $X$  is locally given by  $s = z_0^2 + \dots + z_n^2$  and  $f$  is the projection to  $s$ . Let  $p_1, \dots, p_r$  be the set of singular points. Let  $v: \tilde{X} \rightarrow X$  be the blow-up in  $X$  of  $p_1, \dots, p_r$  and  $g: \tilde{X} \rightarrow \mathbb{D}$  the natural morphism. Then the special fiber of  $g$  has normal crossings of the form  $\tilde{X}_0 = Z + 2 \sum_{i=1}^r E_i$ , where the  $E_i = v^{-1}(p_i) \simeq \mathbb{P}_{\mathbb{C}}^n$  are disjoint and  $Z$  is the strict transform of  $X_0$ . Moreover, the intersection  $W_i := Z \cap E_i \subset E_i$  is isomorphic to a smooth quadric in  $\mathbb{P}_{\mathbb{C}}^n$ . As an application of Corollary 2.8, we compare the Hodge bundles of such families before and after semi-stable reduction.

**Proposition 3.10.** *Let  $\rho: \mathbb{D} \rightarrow \mathbb{D}$ ,  $\rho(t) = t^\ell$ , denote a base change realizing a semi-stable reduction  $h: Y \rightarrow \mathbb{D}$  of  $f: X \rightarrow \mathbb{D}$ . Then there is a natural morphism*

$$\rho^* R^q g_* \Omega_{\tilde{X}/\mathbb{D}}^p(\log) \longrightarrow R^q h_* \Omega_{Y/\mathbb{D}}^p(\log).$$

If  $n$  is odd or  $(p, q) \neq (n/2, n/2)$ , this is an isomorphism. In the case  $(p, q) = (n/2, n/2)$ , the cokernel is isomorphic to

$$\left( \frac{\mathcal{O}_{\mathbb{D},0}}{t\mathcal{O}_{\mathbb{D},0}} \right)^{\oplus \#\text{sing}(X_0)}$$

*Proof.* The statement can be derived from Corollary 2.8, by studying the monodromy operator on the limiting mixed Hodge structure on  $H^n(X_\infty)$ . It is described by the Picard-Lefschetz theorem. For simplicity, assume that there is a single ordinary double point. If the relative dimension  $n$  is odd, then the monodromy acting on any  $H^k(X_\infty)$  is unipotent, and hence  $T_s$  is trivial on it. We now focus on the case when  $n$  is even. Since the monodromy is non-trivial at most on  $H^n(X_\infty)$ , we only need to consider  $p + q = n$ . We have to show that  $T_s$  acts trivially on  $\text{Gr}_{F_\infty}^k H^n(X_\infty)$  for  $k \neq n$ . We use the fact that  $T_s$  is an endomorphism of mixed Hodge structures on  $H^n(X_\infty)$ . We have

$$\text{Gr}_k^W H^n(X_\infty) = 0, \quad k \neq n$$

(cf. for instance [Wan97, Sec. 3]). Hence, it is enough to analyze the action of  $T_s$  on  $\text{Gr}_{F_\infty}^\bullet \text{Gr}_n^W H^n(X_\infty)$ . But  $F_\infty^\bullet$  induces a pure Hodge structure of weight  $n$  on  $\text{Gr}_n^W H^n(X_\infty)$ . If  $T_s$  acts non-trivially on  $\text{Gr}_{F_\infty}^p \text{Gr}_n^W H^n(X_\infty)$ , then it acts non-trivially on  $\text{Gr}_F^q \text{Gr}_n^W H^n(X_\infty)$ ,  $p + q = n$ . Since the only non-trivial eigenvalue of  $T_s$  on  $H^n(X_\infty)$  is  $-1$ , with multiplicity one, then necessarily  $p = q = n/2$ . This concludes the proof.  $\square$

#### 4. $L^2$ METRICS

This section revolves around the  $L^2$  metrics and their asymptotics for one-parameter degenerations of projective varieties. We provide precise expressions for the dominant and subdominant terms of the asymptotics of the  $L^2$  metrics on determinants of Hodge bundles. For this, we rely on the general statements in Section 2 and Schmid's metric characterisation of the limiting monodromy weight filtration [Sch73]. Throughout, we will freely exploit the compatibility of Schmid's limiting mixed Hodge structures with Steenbrink's geometric approach [Ste76, Ste77, GNA90].

**4.1. Generalities on  $L^2$  metrics.** We recall known facts about  $L^2$  metrics, mainly to fix notations and normalizations.

Let  $X$  be a compact Kähler manifold of dimension  $n$ . Let  $h_X$  be a Kähler metric on the holomorphic vector bundle  $T_X$ , and  $\omega$  the associated Kähler form. The expression for  $\omega$  in local holomorphic coordinates is

$$(4.1) \quad \omega = i \sum_{j,k} h_X \left( \frac{\partial}{\partial z_j}, \frac{\partial}{\partial z_k} \right) dz_j \wedge d\bar{z}_k.$$

Most of the time, we will only refer to the Kähler form, since it determines  $h_X$  by the rule (4.1).

On the De Rham cohomology  $H^k(X, \mathbb{C})$  there is a natural  $L^2$  metric: given classes  $\alpha, \beta$  and harmonic representatives  $\tilde{\alpha}, \tilde{\beta}$ , we put

$$(4.2) \quad \langle \alpha, \beta \rangle_{L^2} = \int_X \tilde{\alpha} \wedge \star \tilde{\beta} = \int_X \langle \tilde{\alpha}, \tilde{\beta} \rangle \frac{\omega^n}{n!}.$$

Here  $\star$  denotes the Hodge star operator and the inner product in the second integral is the induced pairing on differential forms, coming from the Kähler metric  $h_X$ . On Dolbeault cohomology  $H^{p,q}(X) = H^q(X, \Omega_X^p)$  we may similarly define an  $L^2$  metric. The Hodge decomposition

$$H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X)$$

is then an orthogonal sum decomposition for the  $L^2$  metrics.

Recall that, for any integer  $k \leq n$ , we define the primitive cohomology subspace of  $H^k(X, \mathbb{C})$  by

$$H_{\text{prim}}^k(X, \mathbb{C}) = \ker \left( L^{n-k+1} : H^k(X, \mathbb{C}) \rightarrow H^{2n-k+2}(X, \mathbb{C}) \right),$$

where  $L = \omega \wedge \bullet$  is the Lefschetz operator. Then, for any  $k \geq 0$  we have the Lefschetz decomposition

$$H^k(X, \mathbb{C}) = \bigoplus_{\max\{0, 2(k-n)\} \leq 2r \leq k} L^r H_{\text{prim}}^{k-2r}(X, \mathbb{C}).$$

This decomposition is orthogonal for the  $L^2$  metric. On each piece of the decomposition, the Hodge star operator is given by the rule

$$(4.3) \quad \star L^r \alpha = (-1)^{(k-2r)(k-2r+1)/2} \frac{r!}{(n-k+r)!} L^{n-k+r} C \alpha,$$

where  $C$  is the Weil operator acting as multiplication by  $i^{p-q}$  on  $H^{p,q}(X)$ . Therefore, the  $L^2$  metric on  $L^r H_{\text{prim}}^{k-2r}(X, \mathbb{C})$  reads

$$(4.4) \quad \langle L^r \alpha, L^r \beta \rangle_{L^2} = (-1)^{(k-2r)(k-2r+1)/2} \frac{r!}{(n-k+r)!} \int_X \alpha \wedge C \bar{\beta} \wedge \omega^{n-k+2r}.$$

One also defines  $H_{\text{prim}}^{p,q}(X) = H_{\text{prim}}^{p+q}(X, \mathbb{C}) \cap H^{p,q}(X)$ . Primitive cohomology groups can be put in families, in the setting of a Kähler morphism  $f : X \rightarrow S$ . The construction produces holomorphic vector bundles for which we employ similar notations, for instance  $(R^k f_* \mathbb{C})_{\text{prim}}$  or  $(R^q f_* \Omega_{X/S}^p)_{\text{prim}}$ . The analogue of the Lefschetz decomposition holds in this generality, as a decomposition of holomorphic vector bundles. For this we notice that the Lefschetz operator induced by the Kähler structure is horizontal with respect to the Gauss–Manin connection, and hence holomorphic.

At a later point we will exploit the integral structure of the cohomology groups. For a  $\mathbb{Z}$ -module of finite type  $\Lambda$ , we denote by  $\Lambda_{\text{nt}}$  the maximal torsion free quotient of  $\Lambda$ .

**Definition 4.1.** *Let  $X$  be a compact Kähler manifold, with Kähler form  $\omega$ . We define  $\text{vol}_{L^2}(H^k(X, \mathbb{Z}), \omega)$  as the square of the covolume of the lattice  $H^k(X, \mathbb{Z})_{\text{nt}} \subset H^k(X, \mathbb{R})$ , with respect to the Euclidean structure induced by the  $L^2$  metric. We adopt the convention  $\text{vol}_{L^2}(H^k(X, \mathbb{Z}), \omega) = 1$  if  $b_k(X) = 0$ .*

**Proposition 4.2.** *The following statements hold:*

- (1) *Let  $X$  be a compact Kähler manifold, and  $\omega$  a Kähler form on  $X$ .*
  - (a)  *$\text{vol}_{L^2}(H^k(X, \mathbb{Z}), \omega)$  is independent of the complex structure and only depends on the cohomology class  $[\omega]$ .*
  - (b)  *$\text{vol}_{L^2}(H^k(X, \mathbb{Z}), \omega)$  is a rational number, if  $[\omega]$  is rational.*
- (2) *Let  $f : X \rightarrow S$  be a Kähler morphism, and  $\omega$  a Kähler form on  $X$ . Then  $s \mapsto \text{vol}_{L^2}(H^k(X_s, \mathbb{Z}), \omega|_{X_s})$  is locally constant on  $S$ .*

*Proof.* The  $L^2$  metric is determined by the Hodge star operator, which is computed for images under the Lefschetz operator of primitive classes by (4.3). Since the Lefschetz decomposition is orthogonal for the  $L^2$  metric, we can restrict to these cases. Except for the Weil operator, all operators are independent of the complex structure. Passing to determinants, the Weil operator disappears. Indeed,

$$(4.5) \quad \det C = \prod_{p,q} i^{(p-q)h_{\text{prim}}^{p,q}} = 1,$$

since  $h_{\text{prim}}^{p,q} = h_{\text{prim}}^{q,p}$ . This proves (1a). It implies the statement for Kähler morphisms, since such a family is a locally trivial  $\mathcal{C}^\infty$  fibration, and the Kähler form defines a flat section of  $R^2 f_* \mathbb{R}$ .



To prove that the volume is a rational number, it is enough to prove that for a given basis  $e_1, \dots, e_{b_k}$  of  $H^k(X, \mathbb{Q})$ , the expression  $\det \langle e_i, e_j \rangle_{L^2}$  is rational. To this end, notice that the Lefschetz decomposition can be defined over  $\mathbb{Q}$ , since the Kähler form is rational:

$$H^k(X, \mathbb{Q}) = \bigoplus_r L^r H_{\text{prim}}^{k-2r}(X, \mathbb{Q}), \quad H_{\text{prim}}^k(X, \mathbb{Q}) = \ker \left( L^{n-k+1} : H^k(X, \mathbb{Q}) \rightarrow H^{2n-k+2}(X, \mathbb{Q}) \right).$$

As remarked before, this decomposition is orthogonal for the  $L^2$  metric and we can hence restrict ourselves to considering  $\det \langle v_i, v_j \rangle_{L^2}$  for a basis  $v_1, \dots, v_d$  of  $L^r H_{\text{prim}}^{k-2r}(X, \mathbb{Q})$ . Recalling now (4.3), we see that  $\star L^r$  acting on  $H_{\text{prim}}^{k-2r}(X, \mathbb{Q})$  can be decomposed as

$$(4.6) \quad H_{\text{prim}}^{k-2r}(X, \mathbb{C}) \xrightarrow{C} H_{\text{prim}}^{k-2r}(X, \mathbb{C}) \xrightarrow{L^{n-k+r}} H^{2n-k+r}(X, \mathbb{C}) \xrightarrow{q} H^{2n-k+r}(X, \mathbb{C}),$$

where  $q$  is multiplication by a rational number. Furthermore, the determinant of the Weil operator  $C$  is 1 as remarked in (4.5). Hence, since the other operators in (4.6) preserve the rational structure, we have in fact

$$\det \star L^r H_{\text{prim}}^{k-2r}(X, \mathbb{Q}) = \det L^{n-k+r} H_{\text{prim}}^{k-2r}(X, \mathbb{Q}).$$

The pairing  $\det \langle v_i, v_j \rangle_{L^2}$  hence reduces to the determinant of a matrix of integrals of top degree rational cohomology classes, *i.e.* the determinant of a matrix with rational entries. It is therefore a rational number.  $\square$

**4.2. The singularities of  $L^2$  metrics.** Let  $f: X \rightarrow \mathbb{D}$  be a projective normal crossings degeneration, with  $n$ -dimensional fibers. We fix a Kähler form on  $X^\times$  which is rational on fibers, *e.g.* induced by a projective embedding. The Hodge bundles  $R^q f_* \Omega_{X/\mathbb{D}}^p(\log)$  are then endowed with singular  $L^2$  metrics, smooth over  $\mathbb{D}^\times$ . The determinant bundle  $\det R^q f_* \Omega_{X/\mathbb{D}}^p(\log)$  inherits a singular  $L^2$  metric. In [Zuc82, Pet84, CKS86], building on the work of Schmid [Sch73], the asymptotic expansion of the  $L^2$  metric is considered, including first and second order derivatives. The coefficients of the asymptotic expressions provided therein are not neatly formulated in terms of the limiting mixed Hodge structure. Our aim is to address this point for the dominant and subdominant terms.

To formulate our results, consider the limiting mixed Hodge structure  $(F_\infty^\bullet, W_\bullet)$  on  $H^k(X_\infty)$  ( $k = p + q$ ), where  $F_\infty^\bullet$  is the limiting Hodge structure and  $W_\bullet$  is the monodromy weight filtration (cf. Schmid [Sch73]). We introduce the following invariants:

$$(4.7) \quad \alpha^{p,q} = -\frac{1}{2\pi i} \operatorname{tr} \left( \ell \log T_s \mid \operatorname{Gr}_{F_\infty}^p H^k(X_\infty) \right)$$

and

$$(4.8) \quad \beta^{p,q} = \sum_{r=-k}^k r \dim \operatorname{Gr}_{F_\infty}^p \operatorname{Gr}_{k+r}^W H^k(X_\infty).$$

In the course of the proof of Theorem 4.4 below, we need the following lemma.

**Lemma 4.3.** *The invariants  $\beta^{p,q}$  satisfy the following two identities:*

- (1)  $\beta^{p,q} = -\beta^{q,p}$ .
- (2)  $\beta^{p,q} = \beta^{n-q, n-p}$ .

*In particular,  $\beta^{p,q} = -\beta^{n-p, n-q}$ .*

*Proof.* Since  $\mathrm{Gr}_{k+r}^W H^k(X_\infty)$  is a pure Hodge structure of weight  $k+r$  we have

$$\dim \mathrm{Gr}_{F_\infty}^p \mathrm{Gr}_{k+r}^W H^k(X_\infty) = \dim \mathrm{Gr}_{F_\infty}^{k+r-p} \mathrm{Gr}_{k+r}^W H^k(X_\infty).$$

Moreover, the limiting mixed Hodge structure is equipped with a nilpotent operator  $N$  on  $H^k(X_\infty)$  which induces an isomorphism

$$N^r : \mathrm{Gr}_{F_\infty}^p \mathrm{Gr}_{k+r}^W H^k(X_\infty) \xrightarrow{\sim} \mathrm{Gr}_{F_\infty}^{p-r} \mathrm{Gr}_{k-r}^W H^k(X_\infty).$$

A direct combination of these observations proves the first point.

The second point follows from the Lefschetz isomorphism, which induces isomorphisms (cf. Schmid [Sch73, Theorem 6.16])

$$L^{n-k} : \mathrm{Gr}_{F_\infty}^p \mathrm{Gr}_{k+r}^W H^k(X_\infty) \xrightarrow{\sim} \mathrm{Gr}_{F_\infty}^{p+n-k} \mathrm{Gr}_{r+2n-k}^W H^{2n-k}(X_\infty).$$

□

The statement of the following theorem refers to Mumford good hermitian metrics on holomorphic vector bundles [Mum77]. In rank one, this notion can be formally extended to  $\mathbb{Q}$ -line bundles. For instance, if a  $\mathbb{Q}$ -line bundle is represented by a couple  $(L, m)$  formed by a line bundle  $L$  together with an integer  $m$ , then a good hermitian metric on  $(L, m)$  will just be a good hermitian metric on  $L$ .

**Theorem 4.4.** (1) *Let  $\sigma$  be a holomorphic trivialization of  $\det R^q f_* \Omega_{X/\mathbb{D}}^p(\log)$ . Then there is an asymptotic expansion*

$$\log h_{L^2}(\sigma(t), \sigma(t)) = \alpha^{p,q} \log |t|^2 + \beta^{p,q} \log \log |t|^{-1} + C + O\left(\frac{1}{\log |t|}\right)$$

for some real constant  $C$ .

(2) *On the  $\mathbb{Q}$ -line bundle  $\det R^q f_* \Omega_{X/\mathbb{D}}^p(\log) \otimes \mathcal{O}(\alpha^{p,q}[0])$ , the  $L^2$  metric induces a good hermitian metric, in the sense of Mumford.*

*Proof of Theorem 4.4.* First of all, we reduce both (1) and (2) to the semi-stable case: by Corollary 2.8, the change of the  $L^2$  metric under semi-stable reduction is accounted for by the term  $|t|^{2\alpha^{p,q}}$ . For the rest of the argument, we can hence assume that  $f : X \rightarrow \mathbb{D}$  is semi-stable.

In the semi-stable case, the proof of (2) was first outlined by Zucker [Zuc82, Sec. 1]. Further details were later worked out by Peters [Pet84, Sec. 2 & 3]. A complete argument is provided by Cattani–Kaplan–Schmid [CKS86, Thm. 5.21 & Prop. 5.22]. These authors deal with degenerations of Hodge structures in several variables. In the particular case of one-parameter degenerations, their results imply the Mumford goodness of the  $L^2$  metric on  $\det R^q f_* \Omega_{X/\mathbb{D}}^p(\log)$ .

For the proof of (1) in the semi-stable case, Schmid’s nilpotent orbit theorem [Sch73] implies a general expansion of the form

$$(4.9) \quad \log h_{L^2}(\sigma(t), \sigma(t)) = \beta \log \log |t|^{-1} + C + O\left(\frac{1}{\log |t|}\right),$$

for some integer  $\beta$  and some constant  $C$ . This is a weak form of [Pet84, Sec. 2.2], whose proof follows the same lines. The rest of the argument will be devoted to showing that  $\beta = \beta^{p,q}$ .

We claim it is enough to establish the weaker inequality  $\beta \leq \beta^{p,q}$ , namely

$$(4.10) \quad h_{L^2}(\sigma(t), \sigma(t)) = O\left((\log |t|)^{\beta^{p,q}}\right).$$

Indeed, suppose this estimate is satisfied for all  $p, q$ . Let  $\sigma'$  be a trivializing section of  $\det R^{n-q} f_* \Omega_{X/\mathbb{D}}^{n-p}(\log)$ . Then we have the estimate

$$(4.11) \quad h_{L^2}(\sigma'(t), \sigma'(t)) = O\left((\log|t|)^{\beta^{n-p, n-q}}\right) = O\left((\log|t|)^{-\beta^{p,q}}\right),$$

where we applied Lemma 4.3. These norms can be compared, since Serre duality induces an  $L^2$  isometry

$$(4.12) \quad \det R^q f_* \Omega_{X/\mathbb{D}}^p(\log) \simeq \det R^{n-q} f_* \Omega_{X/\mathbb{D}}^{n-p}(\log)^\vee.$$

Here we implicitly used that  $\det \Omega_{X/\mathbb{D}}(\log) = \omega_{X/\mathbb{D}}$ , since we are in the semi-stable setting. Combining (4.11) and (4.12) we find the reverse inequality  $\beta_{p,q} \leq \beta$ , and we thus conclude by (4.10).

We now proceed to prove (4.10). For the discussion, we rely on Section 2. First of all, we fix a subset  $E = \{\mathbf{e}_1, \dots, \mathbf{e}_h\}$  of  $F_\infty^p H^k(X_\infty)$  which projects to a basis in  $\mathrm{Gr}_{F_\infty}^p H^k(X_\infty)$  and which is adapted to the weight type filtration

$$W_\ell \mathrm{Gr}_{F_\infty}^p H^k(X_\infty) = W_\ell \cap F_\infty^p / W_\ell \cap F_\infty^{p+1}.$$

In other words, for each  $\ell$ , there is a subset  $E_\ell$  of  $E$  such that the elements of  $E_\ell$  are in  $W_\ell \cap F_\infty^p \setminus W_{\ell-1} \cap F_\infty^p$  and project to a basis of  $\mathrm{Gr}_\ell^W \mathrm{Gr}_{F_\infty}^p H^k(X_\infty)$ .

Secondly, lift the elements  $E$  to holomorphic sections  $\{\tilde{\sigma}_j\}$  of  $\mathcal{F}_{\log}^p$ . They project to a local holomorphic frame  $\{\sigma_j\}$  of  $\mathcal{F}_{\log}^p / \mathcal{F}_{\log}^{p+1} = R^q f_* \Omega_{X/\mathbb{D}}^p(\log)$ , by Nakayama's lemma. Also, introduce the twisted sections  $\tilde{\mathbf{e}}_j(t) = e^{-2\pi i N \tau} \mathbf{e}_j(\tau)$ , for  $\tau \in \mathbb{H}$  (cf. (2.1)). These can be identified with holomorphic sections of  $R^k f_* \Omega_{X/\mathbb{D}}^\bullet(\log)$ . Under this identification, we have the equality  $\tilde{\sigma}_j(0) = \tilde{\mathbf{e}}_j(0)$  (cf. (2.2), (2.5) and (2.9)). Therefore,

$$\tilde{\sigma}_j(t) - \tilde{\mathbf{e}}_j(t) \in t \cdot \Gamma(\mathbb{D}, R^k f_* \Omega_{X/\mathbb{D}}^\bullet(\log)).$$

Together with [Gri84, Chap. II, Prop. 25], we derive for the  $L^2$  norm

$$\|\tilde{\sigma}_j(t) - \tilde{\mathbf{e}}_j(t)\|_{L^2}^2 = O\left(|t|(\log|t|)^b\right),$$

for some integer  $b$ . By Schmid's theorem [Sch73, Thm. 6.6], adapted to the present setting by Zucker [Zuc79, Prop. 3.9], we derive

$$(4.13) \quad \|\tilde{\sigma}_j(t)\|_{L^2}^2 = O\left((\log|t|)^{\ell-k}\right),$$

where  $\ell$  is such that  $\mathbf{e}_j \in E_\ell$ . Since the  $L^2$  norm on  $\mathcal{F}_{\log}^p / \mathcal{F}_{\log}^{p+1}$  is the quotient norm of the  $L^2$  norm on  $\mathcal{F}_{\log}^p$ , we have:

$$(4.14) \quad \|\sigma_1 \wedge \dots \wedge \sigma_h\|_{L^2}^2(t) \leq \prod_j \|\sigma_j\|_{L^2}^2(t) \leq \prod_j \|\tilde{\sigma}_j\|_{L^2}^2(t).$$

Combining (4.13) and (4.14), together with  $\mathrm{Gr}_\ell^W \mathrm{Gr}_{F_\infty}^p H^k(X_\infty) \simeq \mathrm{Gr}_{F_\infty}^p \mathrm{Gr}_\ell^W H^k(X_\infty)$ , we conclude the claimed estimate (4.10)  $\square$

*Remark 4.5.* (1) Under a flatness assumption, Peters gives an expression for the leading term of the asymptotics of  $\log h_{L^2}$  on  $\det \mathcal{F}_{\log}^p$  [Pet84, Prop. 2.2.1]. This is compatible with our result, up to a sign mistake in *loc. cit.*, due to a confusion between lower and upper extensions.

- (2) As for Theorem 2.6 and Corollary 2.8, there is a counterpart of Theorem 4.4 for more general degenerations of Hodge structures, and in particular for the upper extension  ${}^u\mathcal{F}^\bullet$  (and their graded quotients) in the geometric case. In this situation, only the exponent  $\alpha^{p,q}$  needs to be changed to the corresponding elementary exponent for the upper extensions. The rest of the asymptotic expansion remains the same, as it is determined after semi-stable reduction.

## 5. BCOV METRICS AND INVARIANT FOR CALABI–YAU VARIETIES

In this section we recall the construction of the BCOV bundle following Fang–Lu–Yoshikawa [FLY08]. It is named after Bershadsky–Cecotti–Ooguri–Vafa [BCOV94], who developed a mostly conjectural technique for computing “higher loop string amplitudes”. For a Kähler family of Calabi–Yau manifolds, the bundle can be endowed with a Quillen type metric, independent of the particular choice of Kähler structure. In relative dimension 3, Fang–Lu–Yoshikawa could extract from this Quillen metric an important invariant of Calabi–Yau 3-folds, called the BCOV invariant. This is a suitable normalization of a combination of holomorphic analytic torsions. It is actually this invariant, rather than the original quantity in [BCOV94], that is expected to fulfill the predictions in *loc. cit.* in connection with mirror symmetry. The case of the mirror quintic family was successfully solved in [FLY08]. The analogous of this conjectural program in dimension 4 has been proposed by Klemm–Pandharipande [KP08], and further studied in dimension 5 by Pandharipande–Zinger [PZ10]. However a right counterpart of the BCOV invariant in arbitrary dimension, independent of the Kähler structure, was still missing. Filling this gap is the ultimate purpose of this section.

In this section, we wish to distinguish the dualizing (or canonical) sheaf, from Kähler forms. We herein adopt the notation  $K_X$  for the canonical sheaf of a complex manifold  $X$ , and similarly for the relative setting.

**5.1. Background on Quillen metrics.** We summarize the work of Bismut–Gillet–Soulé on Quillen metrics on holomorphic determinant bundles, and notably the curvature formula. This is the basis for the later construction of the Quillen-BCOV metric. The discussion is mostly based on [BGS88c], for which [BGS88a, BGS88b] develop necessary foundations.

Let  $f: X \rightarrow S$  be a Kähler morphism of complex manifolds of relative dimension  $n$ . Let  $h_X$  be a Kähler metric on  $T_X$ , with Kähler form  $\omega$ . Let  $(E, h_E)$  be a hermitian holomorphic vector bundle on  $X$ . The determinant line bundle  $\lambda(E)$  is a holomorphic line bundle on  $S$  with fibers

$$\lambda(E)_s = \bigotimes_q \det H_{\bar{\partial}}^q(X_s, E|_{X_s})^{(-1)^q}, \quad s \in S.$$

Here  $H_{\bar{\partial}}^q(X_s, E|_{X_s})$  is the  $q$ -th Dolbeault cohomology group of the holomorphic vector bundle  $E|_{X_s}$ . The construction is compatible with the Knudsen–Mumford determinant of cohomology [KM76] (see [BGS88c, Sec. 2 & Sec. 3] for two different proofs). The  $\mathcal{C}^\infty$  complex line bundle underlying  $\lambda(E)$  is a particular instance of the general construction of Bismut–Freed [BF86a, BF86b], generalizing ideas of Quillen [Kvi85]. Their method simultaneously produces a hermitian metric (and a unitary connection), the so-called Quillen metric, that we now explain.

Introduce  $L^2$  hermitian inner products on the constituents of the Dolbeault complex  $A^{0,q}(X_s, E|_{X_s})$  of  $(0, q)$ -forms with values in  $E|_{X_s}$ . Given  $u, v \in A^{0,q}(X_s, E|_{X_s})$ , we put

$$(5.1) \quad \langle u, v \rangle_{L^2, s} = \frac{1}{(2\pi)^n} \int_{X_s} \langle u, v \rangle \frac{\omega^n}{n!}.$$

On the right hand side,  $\langle \cdot, \cdot \rangle$  is the pointwise hermitian inner product on  $E|_{X_s} \otimes \Omega_{X_s}^{(0,q)}$ , depending on the metrics  $h_E$  on  $E$  and  $h_X$  on  $T_X$ . Next, we take the formal adjoint  $\bar{\partial}^*$  of the Dolbeault operator  $\bar{\partial}$ , characterized by the equation  $\langle \bar{\partial}^* u, v \rangle_{L^2, s} = \langle u, \bar{\partial} v \rangle_{L^2, s}$ . The Dolbeault Laplacian is then defined as  $\Delta_{\bar{\partial}, s}^{(0,q)} = (\bar{\partial} + \bar{\partial}^*)^2$ . This is a positive elliptic partial differential operator of second order acting on  $A^{0,q}(X_s, E|_{X_s})$ . It has a discrete spectrum with finite multiplicities. The kernel of  $\Delta_{\bar{\partial}, s}^{(0,q)}$  is by definition the space of harmonic forms in  $A^{0,q}(X_s, E|_{X_s})$ . Hodge theory provides a canonical isomorphism

$$H_{\bar{\partial}}^q(X_s, E|_{X_s}) \xrightarrow{\sim} \ker \Delta_{\bar{\partial}, s}^{(0,q)} \subset A^{0,q}(X_s, E|_{X_s}).$$

Through this isomorphism, the restriction of the  $L^2$  metric to harmonic forms induces a hermitian product on  $H_{\bar{\partial}}^q(X_s, E|_{X_s})$ . By taking exterior products and duals, the determinant of cohomology  $\lambda(E)_s$  inherits a metric, denoted  $h_{L^2, s}$  and called  $L^2$  metric. We remark that the pointwise  $L^2$  metric on  $\lambda(E)$  thus defined is in general not smooth in  $s \in S$ . It is nevertheless smooth when all the higher direct images  $R^q f_* E$  are locally free, which is for instance the case of Hodge bundles. In any event, the possible lack of smoothness can be corrected by taking into account the rest of the spectrum of  $\Delta_{\bar{\partial}, s}^{(0,q)}$ , as follows. Let  $\{\lambda_k\}_{k \geq 1}$  be the strictly positive eigenvalues of  $\Delta_{\bar{\partial}, s}^{(0,q)}$ , ordered increasingly and repeated according to multiplicities. Introduce the spectral zeta function

$$\zeta_{\bar{\partial}}^{(0,q)}(u) = \sum_{k \geq 1} \frac{1}{\lambda_k^u}, \quad \operatorname{Re}(u) \gg 0.$$

This is absolutely convergent and holomorphic for  $\operatorname{Re}(u) \gg 0$ , and meromorphically continues to the whole complex plane  $\mathbb{C}$ . It is holomorphic at  $u = 0$ , and one defines the determinant of the Laplacian (with the zero eigenvalue removed) by

$$\det \Delta_{\bar{\partial}, s}^{(0,q)} = \exp \left( - \frac{d}{du} \zeta_{\bar{\partial}}^{(0,q)}(u) \right) \Big|_{u=0}.$$

The *holomorphic analytic torsion* of  $(E|_{X_s}, h_E)$ , computed with respect to  $\omega$ , is the real number

$$T(s) = \sum_q (-1)^q q \log \det \Delta_{\bar{\partial}, s}^{(0,q)}.$$

Sometimes it is convenient to spell out the hermitian data involved, and we may then write  $T(E|_{X_s}, h_E, \omega)$  or other self-explanatory variants of this notation. We notice that the function  $s \mapsto T(s)$  is in general not smooth. Finally, the *Quillen metric* on  $\lambda(E)_s$  is

$$h_{Q, s} = \exp(T(s)) \cdot h_{L^2, s}.$$

The Quillen metric is a smooth hermitian metric on  $\lambda(E)$  [BGS88c, Thm. 1.6]. In particular, when the  $L^2$  metric is smooth, the function  $s \mapsto T(s)$  is smooth as well. For the purpose of this article, a relevant such example to bear in mind is  $\lambda(\Omega_{X/S}^p)$ , where  $\Omega_{X/S}^p$  is endowed with the metric deduced from  $h_X$ .

The curvature theorem of Bismut–Gillet–Soulé [BGS88c, Thm. 1.27] computes the first Chern–Weil form of  $(\lambda(E), h_Q)$ . That is, if  $\nabla_{\lambda(E), h_Q}$  is the Chern connection of  $h_Q$ , then

$$c_1(\lambda(E), h_Q) = \frac{i}{2\pi} \nabla_{\lambda(E), h_Q}^2 \in A^{1,1}(S).$$

The curvature formula lifts the codimension one part of the Grothendieck–Riemann–Roch relationship to an equality of differential forms on  $S$ :

$$(5.2) \quad c_1(\lambda(E), h_Q) = f_* (\text{ch}(E, h_E) \text{td}(T_X, h_X))^{(1,1)}.$$

Here  $\text{ch}(E, h_E)$  and  $\text{td}(T_X, h_X)$  are the Chern–Weil representatives of  $\text{ch}(E)$  and  $\text{td}(T_X)$ , and  $f_*$  is the fiber integral operation on differential forms.

An important property of Quillen metrics, central to the proof of the curvature formula, is the anomaly equation [BGS88c, Thm. 1.23]. This describes the dependence of the Quillen metric on the metrics  $h_E$  and  $h_X$ . Since we don't explicitly apply the anomaly formula in this paper, we refer the interested reader to *loc. cit.* for details.

**5.2. The BCOV line bundle and its Quillen-BCOV metric.** Let  $f: X \rightarrow S$  be a Kähler morphism whose fibers are Calabi–Yau manifolds of dimension  $n$ . Let  $\omega$  be a Kähler form on  $X$ . We endow the relative cotangent bundle  $\Omega_{X/S}^1$  and its exterior powers  $\Omega_{X/S}^p$  with the induced smooth hermitian metrics.

**Definition 5.1.** *The BCOV bundle of the family  $f: X \rightarrow S$  is the line bundle on  $S$*

$$\lambda_{BCOV}(X/S) := \bigotimes_p \lambda(\Omega_{X/S}^p)^{(-1)^p p}.$$

From the choice of Kähler structure and the induced metrics on the  $\Omega_{X/S}^p$ , the line bundle  $\lambda_{BCOV}$  carries a  $L^2$  and a Quillen type metric, that we denote  $h_{L^2}$  and  $h_Q$ , respectively. Let us momentarily assume that  $S$  is reduced to a point. Hence we deal with a single Calabi–Yau manifold  $X$ . Following [FLY08, Sec. 4] we put

$$A(X, \omega) = \exp\left(-\frac{1}{12} \int_X (\log \varphi) c_n(T_X, \omega)\right), \quad \text{with} \quad \varphi = \frac{i^{n^2} \eta \wedge \bar{\eta}}{\|\eta\|_{L^2}^2} \frac{n!}{\omega^n}.$$

Here  $\eta$  is a holomorphic trivialization of  $\Omega_X^n$  and  $c_n(T_X, \omega)$  is the Chern–Weil representative of  $c_n(T_X)$  associated to the Kähler metric. For a Ricci-flat Kähler metric, the factor  $A(X, \omega)$  simplifies to

$$A(X, \omega) = \text{vol}(X, \omega)^{\frac{\chi(X)}{12}}, \quad \text{where} \quad \text{vol}(X, \omega) = \frac{1}{(2\pi)^n} \int_X \frac{\omega^n}{n!}.$$

In the general family setting  $f: X \rightarrow S$ , the function  $s \mapsto A(X_s, \omega|_{X_s})$  is clearly smooth, and will be denoted  $A(X/S, \omega)$ .

**Definition 5.2.** *The Quillen-BCOV metric on  $\lambda_{BCOV}$  is the smooth hermitian metric*

$$h_{Q,BCOV} := A(X/S, \omega) \cdot h_Q.$$

*We refer to the pair  $(\lambda_{BCOV}, h_{Q,BCOV})$  as the BCOV hermitian line bundle.*

Recall the characterization of the Weil–Peterson form  $\omega_{WP} = c_1(f_* K_{X/S}, h_{L^2})$  (cf. [Tia87]). The curvature of the BCOV hermitian line bundle was computed in [FLY08, Thm. 4.9], building on the curvature formula of Bismut–Gillet–Soulé (5.2):

**Proposition 5.3.** *The curvature of  $(\lambda_{BCOV}, h_{Q,BCOV})$  is given by  $\frac{\chi(X_\infty)}{12} \omega_{WP}$ . In particular the BCOV hermitian line bundle is independent of the choice of Kähler structure.*

**5.3. The  $L^2$ -BCOV metric.** In this subsection, we work with general compact Kähler manifolds, not necessarily of Calabi–Yau type. We define a renormalized  $L^2$  norm on the BCOV bundle. It has the feature that it is independent of the choice of Kähler form (cf. Proposition 5.6).

**Definition 5.4.** (1) Let  $X$  be a compact Kähler manifold of dimension  $n$ , with Kähler form  $\omega$ . We define

$$(5.3) \quad B(X, \omega) := \prod_{k=1}^{2n} \text{vol}_{L^2}(H^k(X, \mathbb{Z}), \omega)^{(-1)^{k+1}k/2}.$$

(2) Let  $f: X \rightarrow S$  be a Kähler morphism with Kähler structure form  $\omega$ , whose fibers are compact complex manifolds of dimension  $n$ . We define a function  $B(X/S, \omega)(s) := B(X_s, \omega|_{X_s})$ . It follows from Proposition 4.2 that  $B(X/S, \omega)$  is in fact locally constant.

(3) The  $L^2$ -BCOV metric, or rescaled  $L^2$  metric, on  $\lambda_{\text{BCOV}}$  is the  $\mathcal{C}^\infty$  metric

$$h_{L^2, \text{BCOV}} = B(X/S, \omega) \cdot h_{L^2}.$$

*Remark 5.5.* (i) The definition drives some inspiration from Kato’s formalism of heights of motives [Kat14], see specially paragraph 1.3 in *loc. cit.*

(ii) Since Poincaré duality is a unimodular pairing, it is not difficult to prove that

$$\text{vol}_{L^2}(H^k(X, \mathbb{Z}), \omega) \text{vol}_{L^2}(H^{2n-k}(X, \mathbb{Z}), \omega) = 1.$$

Hence the product in the definition of  $B(X, \omega)$  can be written more succinctly as

$$B(X, \omega) = \prod_{k=0}^{n-1} \text{vol}_{L^2}(H^k(X, \mathbb{Z}), \omega)^{(-1)^k(n-k)}.$$

In particular for a (simply connected) Calabi–Yau 3-fold  $X$  with a Kähler form  $\omega$ , we find that

$$(5.4) \quad B(X, \omega) = \text{vol}_{L^2}(H^0(X, \mathbb{Z}), \omega)^3 \text{vol}_{L^2}(H^2(X, \mathbb{Z}), \omega).$$

**Proposition 5.6.** The  $L^2$ -BCOV metric  $h_{L^2, \text{BCOV}}$  is independent of the Kähler structure.

*Proof.* We can check the statement pointwise, and hence we may work with a single compact Kähler manifold  $(X, \omega)$ . For each  $k$ , inspired by an identity from [Kat14, 1.3] we define

$$L(H^k) = \sum_{p+q=k} p \det H^q(X, \Omega_X^p),$$

where as usual we adopt additive notations for tensor products. Consider also the complex conjugate line

$$\overline{L(H^k)} = \sum_{p+q=k} \overline{p \det H^q(X, \Omega_X^p)} = \sum_{p+q=k} (k-p) \det H^q(X, \Omega_X^p).$$

The  $L^2$  metric induces a metric on  $L(H^k)$ , as well as  $\overline{L(H^k)}$ . Since complex conjugation  $L(H^k) \rightarrow \overline{L(H^k)}$  is an isometry (as real vector spaces), we have an  $L^2$  isometry

$$2L(H^k) = L(H^k) + \overline{L(H^k)} = \sum_{p+q=k} k \det H^p(X, \Omega^q) = k \det H^k(X, \mathbb{C}).$$

Up to sign, the line  $\det H^k(X, \mathbb{C})$  has a natural element determined by the integral structure, namely  $e_1 \wedge \dots \wedge e_{b_k}$  for a basis  $e_1, \dots, e_{b_k}$  of  $H^k(X, \mathbb{Z})_{\text{nt}}$ . Dividing by the norm of this section we find that the right hand side, and hence the left hand side, don’t depend on the Kähler structure.

More precisely the  $L^2$  metric on  $L(H^k)$  multiplied by  $\text{vol}_{L^2}(H^k(X, \mathbb{Z}), \omega)^{-k/2}$  is independent of the choice of Kähler form. Since the BCOV bundle is clearly given by

$$\sum_{0 \leq k \leq 2n} (-1)^k L(H^k),$$

the proposition follows.  $\square$

**5.4. The BCOV invariant for Calabi–Yau  $n$ -folds.** Let now  $X$  be a compact Calabi–Yau  $n$ -fold, with Kähler form  $\omega$ . The vector bundles  $\Omega_X^p$  inherit smooth hermitian metrics. The “virtual” vector bundle

$$\bigoplus_p (-1)^p p \Omega_X^p$$

has a well-defined holomorphic analytic torsion depending on  $\omega$  and written  $\mathcal{T}_{BCOV}(X, \omega)$ . It also carries the metrics  $h_{Q,BCOV}$  and  $h_{L^2,BCOV}$ .

**Definition 5.7.** *Let  $X$  be a Calabi–Yau  $n$ -fold. The BCOV invariant of  $X$  is the real number given by*

$$\tau_{BCOV}(X) = h_{Q,BCOV} / h_{L^2,BCOV}$$

*In other words, for any auxiliary Kähler form  $\omega$ ,*

$$\tau_{BCOV}(X) = \frac{A(X, \omega)}{B(X, \omega)} \mathcal{T}_{BCOV}(X, \omega),$$

*where  $A(X, \omega)$  and  $B(X, \omega)$  are as in definitions 5.2 and 5.4.*

The terminology invariant is justified by the following proposition.

**Proposition 5.8.** *The BCOV invariant  $\tau_{BCOV}$  depends only on the complex structure of  $X$ .*

*Proof.* This is the combination of Proposition 5.3 and Proposition 5.6.  $\square$

*Remark 5.9.* By Remark 5.5 (ii), it is easily seen that our BCOV invariant generalizes the BCOV invariant defined by Fang–Lu–Yoshikawa for Calabi–Yau 3-folds [FLY08, Definition 4.13], up to an explicit power of  $2\pi$  due to different normalizations.

Let us now discuss the differential equation satisfied by the BCOV invariant for families. Let  $f: X \rightarrow S$  be a Kähler morphism between complex connected manifolds, whose fibers are  $n$ -dimensional Calabi–Yau varieties. Then  $s \mapsto \log \tau_{BCOV}(X_s)$  yields a smooth function on the space of parameters. Let now  $\omega$  be an auxiliary choice of Kähler form on  $X$ . Endow the Hodge bundles of  $f$  with the associated  $L^2$  metrics. After Fang–Lu [FL05], we define the  $k$ -th Hodge form on  $S$  as the following combination of Chern–Weil forms:

$$(5.5) \quad \omega_{H^k} = \sum_{p=0}^k c_1(\mathcal{F}^p R^k f_* \Omega_{X/S}^\bullet, h_{L^2}).$$

By Griffiths’ computation of the curvature of Hodge bundles, this is known to be a semi-positive (1,1) form on  $S$ , cf. *loc. cit.*. It can equivalently be written

$$(5.6) \quad \omega_{H^k} = \sum_{p=0}^k p c_1(R^{k-p} f_* \Omega_{X/S}^p, h_{L^2}) \in c_1(L(H^k)).$$

Since  $B(X/S, \omega)$  is constant, from the curvature formula for the Quillen-BCOV metric [FLY08, Thm. 4.9], recalled in Proposition 5.3, and the very definition of  $\tau_{BCOV}$ , we obtain the following proposition:



**Proposition 5.10.** *Let  $f: X \rightarrow S$  be a Kähler morphism between connected complex manifolds, whose fibers are Calabi–Yau manifolds of dimension  $n$ . Then there is an equality of differential forms on  $S$*

$$(5.7) \quad dd^c \log \tau_{BCOV} = \sum_{k=0}^{2n} (-1)^k \omega_{H^k} - \frac{\chi}{12} \omega_{WP}$$

where  $\chi$  is the topological Euler characteristic of any fiber of  $f$ .

*Remark 5.11.* (i) Proposition 5.10 actually holds for the slightly more general Kähler fibrations considered by Bismut–Gillet–Soulé [BGS88b].

(ii) In the physics literature, equation (5.7) is referred to as the *holomorphic anomaly equation of  $F_1$* , see [BCOV94, Eq. (3.10)]. In relative dimension 3, it is part of an infinite system of differential equations relating some partition functions  $F_g$ , where  $g \geq 0$  runs over all the possible genera of compact Riemann surfaces.

**5.5. Triviality of the BCOV invariant for special geometries.** Recall the Beauville–Bomogolov classification, that any Calabi–Yau variety is an étale quotient of a product  $T \times V \times H$ , where  $T$  is an complex torus,  $V$  is a product of strict Calabi–Yau varieties, and  $H$  is a product of hyperkähler varieties. Recall that a hyperkähler variety is a Kähler manifold whose  $H^{2,0}$  is spanned by a holomorphic symplectic form. The facts we need regarding hyperkähler manifolds can be found in [Hit92].

We are grateful to Ken–Ichi Yoshikawa for sharing with us his argument for the triviality of the BCOV torsion for hyperkähler variety equipped with a Ricci flat metric.

**Proposition 5.12.** *If  $f: X \rightarrow S$  is a projective morphism of complex analytic spaces, whose fibers are either abelian varieties of dimension at least 2 or hyperkähler varieties, then the function  $s \mapsto \tau_{BCOV}(X_s)$  is locally constant on  $S$ .*

*Proof.* Since the statement is local over the base and only depends on  $S_{\text{red}}$ , we can first desingularize  $S$  and further assume that  $S$  is a polydisc. By Proposition 4.2 it suffices to show that the BCOV torsion  $s \mapsto \mathcal{T}_{BCOV}(X_s, \omega|_{X_s})$  is constant if we compute it with respect to a Ricci flat metric  $\omega$  in the Kähler class providing a projective embedding. Even more, we claim that  $\mathcal{T}_{BCOV}(X_s, \omega|_{X_s}) = 1$ . We may thus assume that  $S$  is reduced to a point, and work with a single variety  $X$ . The case of abelian varieties is actually well-known, and we refer for instance to the remark in [Ber03, p. 154].

Now for the hyperkähler case. Let  $n$  be the dimension of  $X$ . Recall that  $n$  is necessarily even. The BCOV torsion decomposes as a product,

$$\mathcal{T}_{BCOV}(X, \omega) = \prod_{p=1}^n T(\Omega_X^p, \omega)^{(-1)^p p},$$

where  $T(\Omega_X^p, \omega)$  is the holomorphic analytic torsion of  $\Omega_X^p$  endowed with the hermitian metric induced by  $\omega$ , and computed with respect to the Kähler structure  $\omega$ . The holomorphic analytic torsion of a hermitian vector bundle  $E$  satisfies (see for instance [GS91, Thm. 1.4])

$$(5.8) \quad T(E, \omega) = T(E^\vee \otimes K_X, \omega)^{(-1)^{n+1}},$$

where  $K_X$  is equipped with the metric coming from the Ricci flat metric on  $T_X$ . For the Ricci-flat metric, there is an isometry  $K_X \simeq \mathcal{O}_X$ , where the latter is equipped with the trivial metric. Indeed, if  $\eta$  is a holomorphic symplectic 2 form, then  $\eta^{n/2}$  is a holomorphic trivialization of  $K_X$ . Furthermore, Ricci flatness implies that the pointwise norm of  $\eta^{n/2}$  is constant. A suitable rescaling of  $\eta$  thus provides the claimed isometry. From (5.8) we then infer  $T(E, \omega) = T(E^\vee, \omega)^{-1}$ .

Therefore, to conclude, it will be enough to prove that the symplectic holomorphic form induces a holomorphic isometry  $\Omega_X^p \simeq \wedge^p T_X \simeq (\Omega_X^p)^\vee$ . This is the content of the following lemma, probably well-known to the specialists, which we state separately.  $\square$

**Lemma 5.13.** *Let  $X$  be a hyperkähler manifold, with non-degenerate holomorphic symplectic form  $\eta$ . Then the induced isomorphism  $\wedge^p T_X \simeq \Omega_X^p$  is, up to a constant, an isometry.*

*Proof.* Since  $X$  is hyperkähler, the Ricci flat metric defines a Kähler form with respect to three orthogonal parallel complex structures,  $I, J, K$ , satisfying  $I^2 = J^2 = K^2 = IJK = -1$ . After renormalization by a scalar, it is possible to write

$$(5.9) \quad \eta(\bullet, \bullet) = \frac{g(J\bullet, \bullet) + ig(K\bullet, \bullet)}{2}.$$

We prove that contracting  $\eta$  by a holomorphic tangent vector provides an isometry  $T_X \simeq \Omega_X$ , where the holomorphic tangent and cotangent bundles are given by the complex structure  $I$ . The general statement follows. If  $v \in T_X$ , then  $Iv = iv$ , and so  $v = \xi - iI\xi$  where  $\xi = \operatorname{Re} v$ . A direct computation shows that  $\|v\|^2 = 2\|\xi\|^2$ . On the other hand, by (5.9) one can conclude  $\eta(v, \bullet) = g(J\xi + iK\xi, \bullet)$ . By definition,  $\|g(x, \bullet)\|^2 = \langle g(x, \bullet), \overline{g(x, \bullet)} \rangle = \langle x, \bar{x} \rangle$ , and hence

$$\|\eta(v)\|^2 = \langle J\xi + iK\xi, J\xi - iK\xi \rangle = \langle J\xi, J\xi \rangle + \langle K\xi, K\xi \rangle = 2\|\xi\|^2 = \|v\|^2.$$

Here we have used that  $J$  and  $K$  preserve the metric. This concludes the proof.  $\square$

## 6. GENERAL ASYMPTOTICS OF THE BCOV INVARIANT

In this section we investigate the singular behaviour of the BCOV invariant along one-parameter degenerations of Calabi–Yau manifolds.

After initially comparing, in the first subsection, the various extensions of the BCOV bundle, we go on to establish the logarithmic behavior of the BCOV invariant along one-parameter degenerations. There we provide a closed formula for general normal crossings projective degenerations of Calabi–Yau varieties (cf. Theorem 6.5). This can be recast as providing the boundary conditions of the holomorphic anomaly equation (cf. Proposition 5.10), and it has proven to be key to the proof of the known cases of the BCOV conjecture, cf. [FLY08]. We then proceed to determine the subdominant term of the BCOV invariant in terms of limiting mixed Hodge structures (cf. Proposition 6.8). These statements will be the point of departure for the computations for special geometries in Section 7.

**6.1. Kähler and logarithmic extensions of the BCOV bundle.** Let  $f: X \rightarrow \mathbb{D}$  be a projective degeneration between complex manifolds. The BCOV line bundle  $\lambda_{BCOV}(X^\times/\mathbb{D}^\times)$  affords a natural extension to  $\mathbb{D}$ , the so-called Kähler extension:

**Definition 6.1.** *The Kähler extension of the BCOV bundle  $\lambda(X^\times/\mathbb{D}^\times)$  is defined as*

$$\tilde{\lambda}_{BCOV} = \bigotimes_p \lambda(\tilde{\Omega}_{X/\mathbb{D}}^p)^{(-1)^p p},$$

where we recall that  $\lambda(\tilde{\Omega}_{X/\mathbb{D}}^p)$  is the Kähler extension of  $\lambda(\Omega_{X^\times/\mathbb{D}^\times}^p)$ , cf. Definition 3.1.

If the singular fiber  $X_0$  has normal crossings, then there is another natural extension: the logarithmic extension.

**Definition 6.2.** *Let  $f: X \rightarrow \mathbb{D}$  be a projective normal crossings degeneration. The logarithmic extension of the BCOV bundle is defined as*

$$\lambda_{BCOV}(\log) = \bigotimes_p \lambda(\Omega_{X/\mathbb{D}}^p(\log))^{(-1)^p p}.$$

In our previous work [EFiMM18], the singularities of the Quillen-BCOV metric were formulated in terms of the Kähler extension of the BCOV bundle. In contrast, the degeneration of the  $L^2$ -BCOV metric is well understood for the logarithmic extension, thanks to Theorem 4.4. A comparison of both extensions is needed in order to extract the singularities of the BCOV invariant. We define  $\mu_{BCOV}$  as the integer realizing this comparison, assuming that  $f: X \rightarrow \mathbb{D}$  is a normal crossings projective degeneration:

$$\tilde{\lambda}_{BCOV} = \lambda_{BCOV}(\log) + \mu_{BCOV} \mathcal{O}([0]).$$

Hence, recalling the definition (3.3) of the integers  $\mu_p$ , we have

$$\mu_{BCOV} = \sum_{p=0}^n p(-1)^p \mu_p.$$

We now exploit the explicit computation of  $\mu_p$  provided by Proposition 3.7, in order to give an expression for  $\mu_{BCOV}$ . We make use of the notation introduced in section 3.3.

**Proposition 6.3.** *Let  $f: X \rightarrow \mathbb{D}$  be a projective normal crossings degeneration, and write  $X_0 = \sum m_i D_i$ . Assume that the smooth fibers are Calabi–Yau  $n$ -folds. Let  $d(k) = \dim D(k) = n - k + 1$ . Then*

$$\begin{aligned} \mu_{BCOV} = & -\frac{(9n+5)n}{24} \chi(X_\infty) - \sum_{k=1}^n (-1)^k \frac{(9n+3k+2)d(k)}{24} \chi(D(k)) \\ & - \frac{(-1)^n}{12} \sum_{k=1}^n \int_{D(k)} c_1(\Omega_{D(k)}) c_{n-k}(\Omega_{D(k)}). \end{aligned}$$

*Proof.* Recall the expression we obtained for  $\mu_p$  in Proposition 3.7:

$$\mu_p = (-1)^{p-1} \chi(\Omega_{X_\infty}^{\bullet \leq p-1}) - \sum_{k=1}^p (-1)^{p-k} \chi(\Omega_{D(k)}^{\bullet \leq p-k}).$$

Therefore

$$\begin{aligned} \mu_{BCOV} = & - \sum_{p=1}^n p \chi(\Omega_{X_\infty}^{\bullet \leq p-1}) - \sum_{p=1}^n \sum_{k=1}^p (-1)^k p \chi(\Omega_{D(k)}^{\bullet \leq p-k}) \\ (6.1) \quad & = - \sum_{p=1}^n p \chi(\Omega_{X_\infty}^{\bullet \leq p-1}) - \sum_{k=1}^n (-1)^k \sum_{p=k}^n p \chi(\Omega_{D(k)}^{\bullet \leq p-k}). \end{aligned}$$

For an integer  $d \geq 0$ , define  $S_d = 0 + \dots + d$ . Then

$$\begin{aligned} \sum_{p=1}^n p \chi(\Omega_{X_\infty}^{\bullet \leq p-1}) = & \sum_{j=0}^n (-1)^j (S_n - S_j) \chi(\Omega_{X_\infty}^j) = S_n \chi(X_\infty) - \sum_{j=0}^n (-1)^j S_j \chi(\Omega_{X_\infty}^j) \\ (6.2) \quad & = \frac{n^2 + n}{2} \chi(X_\infty) - \sum_{j=0}^n (-1)^j \frac{j^2 + j}{2} \chi(\Omega_{X_\infty}^j). \end{aligned}$$

There are formulas for such sums. We will invoke them below. Let's first tackle the remaining terms in (6.1). They have a similar structure:

$$\begin{aligned}
(6.3) \quad \sum_{k=1}^n (-1)^k \sum_{p=k}^n p \chi(\Omega_{D(k)}^{\bullet \leq p-k}) &= \sum_{k=1}^n (-1)^k \sum_{p=0}^{n-k} (p+k) \chi(\Omega_{D(k)}^{\bullet \leq p}) \\
&= \sum_{k=1}^n (-1)^k \sum_{p=0}^{n-k} p \chi(\Omega_{D(k)}^{\bullet \leq p}) + \sum_{k=1}^n (-1)^k k \sum_{p=0}^{n-k} \chi(\Omega_{D(k)}^{\bullet \leq p}) \\
&= \sum_{k=1}^n (-1)^k \sum_{p=1}^{d(k)} p \chi(\Omega_{D(k)}^{\bullet \leq p-1}) + \sum_{k=1}^n (-1)^k (k-1) \sum_{p=0}^{n-k} \chi(\Omega_{D(k)}^{\bullet \leq p}),
\end{aligned}$$

where we used that  $d(k) = \dim D(k) = n - k + 1$ . The first term in (6.3) is computed as in (6.2):

$$(6.4) \quad \sum_{k=1}^n (-1)^k \sum_{p=1}^{d(k)} p \chi(\Omega_{D(k)}^{\bullet \leq p-1}) = \sum_{k=1}^n (-1)^k \left( \frac{d(k)^2 + d(k)}{2} \chi(D(k)) - \sum_{j=0}^{d(k)} (-1)^j \frac{j^2 + j}{2} \chi(\Omega_{D(k)}^j) \right).$$

For the second term in (6.3), we observe  $\sum_{p=0}^{n-k} \chi(\Omega_{D(k)}^{\bullet \leq p}) = d(k) \chi(D(k)) - \sum_{p=1}^{d(k)} (-1)^p p \chi(\Omega_{D(k)}^p)$ , hence

$$\begin{aligned}
(6.5) \quad \sum_{k=1}^n (-1)^k (k-1) \sum_{p=0}^{n-k} \chi(\Omega_{D(k)}^{\bullet \leq p}) &= \sum_{k=1}^n (-1)^k (k-1)(n-k+1) \chi(D(k)) \\
&\quad - \sum_{k=1}^n (-1)^k (k-1) \sum_{p=1}^{d(k)} (-1)^p p \chi(\Omega_{D(k)}^p).
\end{aligned}$$

To simplify (6.2)–(6.5), we infer from [EFiMM18, Lemma 4.6]:

$$(6.6) \quad \sum_{j=1}^{d(k)} (-1)^j j \chi(\Omega_{D(k)}^j) = (-1)^{d(k)} \frac{d(k)}{2} \int_{D(k)} c_{d(k)}(\Omega_{D(k)}) = \frac{d(k)}{2} \chi(D(k)),$$

$$\begin{aligned}
(6.7) \quad \sum_{j=1}^{d(k)} (-1)^j \frac{j^2 + j}{2} \chi(\Omega_{D(k)}^j) &= \frac{d(k)(3d(k) + 7)}{24} \chi(D(k)) \\
&\quad + \frac{(-1)^{d(k)}}{12} \int_{D(k)} c_1(\Omega_{D(k)}) c_{d(k)-1}(\Omega_{D(k)}).
\end{aligned}$$

A similar expression holds for  $X_\infty$  instead of  $D(k)$ . Taking into account that  $c_1(\Omega_{X_\infty}) = 0$ , it reads

$$(6.8) \quad \sum_{j=1}^n (-1)^j \frac{j^2 + j}{2} \chi(\Omega_{X_\infty}^j) = \frac{n(3n+7)}{24} \chi(X_\infty).$$

Also, thanks to (6.6), equation (6.4) simplifies to

$$(6.9) \quad \sum_{k=1}^n (-1)^k \sum_{p=1}^{d(k)} p \chi(\Omega_{D(k)}^{\bullet \leq p-1}) = \sum_{k=1}^n (-1)^k \frac{(k-1)(n-k+1)}{2} \chi(D(k))$$

To conclude, it suffices to plug (6.6)–(6.9) into (6.2)–(6.5) and then add up.  $\square$

**6.2. Logarithmic behaviour of the BCOV invariant.** Recall from the introduction that there is a strong motivation for finding the asymptotic behaviour of the invariant. For the sake of motivation, consider the special case of a projective degeneration of Calabi–Yau 3-folds  $f : X \rightarrow \mathbb{D}$  admitting a single ordinary double point over the origin, Fang–Lu–Yoshikawa [FLY08, Theorem 8.2] proved that

$$\log \tau_{BCOV}(X_t) = \frac{1}{6} \log |t|^2 + O(\log \log |t|^{-1}).$$

This was used in *loc. cit.* in order to prove an instance of genus 1 mirror symmetry for the quintic 3-folds.

More recently, in [LX19], Liu and Xia study systematically the logarithmic behavior of the BCOV invariant of [FLY08]. More precisely they study the limits

$$\kappa_f := \lim_{t \rightarrow 0} \frac{\log \tau_{BCOV}(X_t)}{\log |t|^2}$$

where  $t$  is a local parameter around  $0 \in \mathbb{D}$  for a projective degeneration  $X \rightarrow \mathbb{D}$  of Calabi–Yau 3-folds. It follows from [FLY08, Theorem 9.1] that the limit exists and is a real number. In [Yos15, Thm. 0.1], Yoshikawa proved it is a rational number. In [LX19] formulas are obtained for sums of local contributions, for a compact one-dimensional base  $S$  of a generically smooth family of Calabi–Yau 3-folds  $X \rightarrow S$ . They conjecture that these formulas localize in a precise sense under some specific conditions. In this subsection we address the corresponding matters in arbitrary dimension.

Before we proceed with the statement of the first theorem, we need to recall a definition from [EFiMM18]. Given a projective degeneration  $f : X \rightarrow \mathbb{D}$  between complex manifolds, whose smooth fibers are Calabi–Yau manifolds, we have an injective morphism of line bundles

$$\mathbf{ev} : f^* f_* K_{X/\mathbb{D}} \hookrightarrow K_{X/\mathbb{D}}.$$

We may equivalently see  $\mathbf{ev}$  as a global section of  $K_{X/\mathbb{D}} \otimes (f^* f_* K_{X/\mathbb{D}})^{-1}$ .

**Definition 6.4.** We denote  $B = \text{div } \mathbf{ev}$ . We say that  $f : X \rightarrow \mathbb{D}$  is a Kulikov family if  $B = 0$ .

The divisor  $B$  is effective. Because the smooth fibers have trivial canonical bundle,  $B$  is supported on the special fiber  $X_0$ . To sum up, we have

$$K_{X/\mathbb{D}} = f^* f_* K_{X/\mathbb{D}} \otimes \mathcal{O}(B), \quad |B| \subseteq X_0.$$

**Theorem 6.5.** Let  $f : X \rightarrow \mathbb{D}$  be a germ of a projective degeneration between complex algebraic varieties, whose smooth fibers are Calabi–Yau manifolds. Then the limit

$$(6.10) \quad \kappa_f := \lim_{t \rightarrow 0} \frac{\log \tau_{BCOV}(X_t)}{\log |t|^2}$$

exists and is a rational number. Moreover, if  $X$  is non-singular and the special fiber  $X_0 = \sum m_i D_i$  has normal crossings, then

$$(6.11) \quad \begin{aligned} \kappa_f = & \frac{3n+1}{12} (\chi(X_\infty) - \chi(X_0)) + \sum_{k=1}^{n+1} (-1)^k \frac{(k-1)(3k+6n+2)}{24} \chi(D(k)) \\ & - \frac{(-1)^n}{12} \int_B c_n(\Omega_X) - \sum_{k=1}^n \frac{(-1)^n}{12} \int_{D(k)} c_1(\Omega_{D(k)}) c_{n-k}(\Omega_{D(k)}) \\ & - \frac{\alpha}{12} \chi(X_\infty) - \sum_{0 \leq p, q \leq n} (-1)^{p+q} p \alpha^{p,q}, \end{aligned}$$

where  $\alpha = \frac{1}{2\pi i} \operatorname{tr} \left( {}^u \log T_s \mid \operatorname{Gr}_{F_\infty}^n H^n(X_\infty) \right)$ .

*Remark 6.6.* (i) The restriction to *germs* of projective morphisms of algebraic varieties stems from the application of our previous work [EFiMM18]. In turn *loc. cit.* relies on Yoshikawa's theorem on the singularities of Quillen metrics [Yos07], where the compactness of the base of the fibration is needed. The germ assumption is a technical one, satisfied in most applications. It should be possible to remove this assumption from a functorial lifting of the Grothendieck–Riemann–Roch theorem to the level of line bundles. An example of this in the case of curves is provided by [Eri13], where the first author applies Deligne's Riemann–Roch isomorphism to obtain the singularities of the Quillen metric.

- (ii) The limit  $\kappa_f$  depends only on the restriction of  $f$  to  $\mathbb{D}^\times$ , although the existence of a model over  $\mathbb{D}$  is needed.
- (iii) Notice that the first sum in the expression of  $\kappa_f$  runs from  $k = 1$  to  $n + 1$ , in contrast to the first sum in the expression for  $\mu_{BCOV}$  (Proposition 6.3), which runs from  $k = 1$  to  $n$ . Observe that  $D(n + 1)$  is zero dimensional, and that  $\chi(D(n + 1)) = \#D(n + 1)$  could be non-trivial.
- (iv) The integral  $\int_B c_n(\Omega_X)$  can be worked out explicitly in terms of the geometry/topology of the components of the special fiber  $X_0$ , and their incidence relations. Instead of giving a cumbersome general expression, we refer the reader to the proof of Theorem 7.3 for an example of use.

*Proof of Theorem 6.5.* Because  $\kappa_f$  only depends on the restriction of  $f$  to  $\mathbb{D}^\times$ , it is enough to prove the second assertion. By resolution of singularities we can suppose that  $X$  is smooth and  $X_0 = \sum n_i D_i$  is a divisor with normal crossings. Also we may take an auxiliary integral Kähler structure  $\omega$  induced by a projective embedding.

Let  $\tilde{\lambda}_{BCOV}$  be the Kähler extension of the BCOV bundle. Let  $\sigma$  be a trivializing holomorphic section. In [EFiMM18, Cor. 4.9] we showed that

$$(6.12) \quad \log h_{Q,BCOV}(\sigma, \sigma) = \left\{ \frac{9n^2 + 11n + 2}{24} (\chi(X_\infty) - \chi(X_0)) - \frac{\alpha}{12} \chi(X_\infty) + \frac{(-1)^{n+1}}{12} \int_B c_n(\Omega_X) \right\} \log |t|^2 + o(\log |t|).$$

Notice that in *loc. cit.* the last term in the asymptotics was written as  $\int_B c_n(\Omega_{X/\mathbb{D}})$ , which equals  $\int_B c_n(\Omega_X)$  as stated above. Recall now the relation  $\tilde{\lambda}_{BCOV} = \lambda_{BCOV}(\log) + \mu_{BCOV} \mathcal{O}([0])$ . This means that if we are given trivializing sections  $\sigma$  as above, and  $\sigma'$  of  $\lambda_{BCOV}(\log)$ , then the relation between the respective Quillen-BCOV square norms is

$$\log h_{Q,BCOV}(\sigma, \sigma) = \log h_{Q,BCOV}(\sigma', \sigma') - \mu_{BCOV} \log |t|^2 + \text{continuous}.$$

We can take  $\sigma'$  of the form

$$\sigma' = \bigotimes_{p,q} (\sigma^{(p,q)})^{(-1)^{p+q} p},$$

where  $\sigma^{(p,q)}$  trivializes  $\det R^q f_* \Omega_{X/\mathbb{D}}^p(\log)$ . Then, by Theorem 4.4 and because the  $L^2$  volumes  $\operatorname{vol}_{L^2}(H^k(X_t, \mathbb{Z}), \omega|_{X_t})$  stay constant by Proposition 4.2, we find

$$(6.13) \quad \log h_{L^2,BCOV}(\sigma', \sigma') = \left( \sum_{p,q} (-1)^{p+q} p \alpha^{p,q} \right) \log |t|^2 + O(\log \log |t|^{-1}).$$

But  $\log \tau_{BCOV}(X_t) = \log h_{Q,BCOV}(\sigma', \sigma') - \log h_{L^2,BCOV}(\sigma', \sigma')$ . Therefore the conclusion is achieved by combining equations (6.12)–(6.13) with Proposition 6.3, together with the relation

$$(6.14) \quad \chi(X_0) = - \sum_{k=1}^{n+1} (-1)^k \chi(D(k)) = - \sum_{k=1}^n (-1)^k \chi(D(k)) + (-1)^n \chi(D(n+1)).$$

□

**Corollary 6.7.** *Let  $f: X \rightarrow \mathbb{D}$  be a germ of a normal crossings projective degeneration of algebraic varieties, whose smooth fibers are Calabi–Yau manifolds. Write  $X_0 = \sum_{i=1}^r m_i D_i$  and define  $M = \text{lcm}(m_1, \dots, m_r)$ . Then*

$$12M\kappa_f \in \mathbb{Z}.$$

*If all the monodromies are unipotent, then  $12\kappa_f \in \mathbb{Z}$ .*

*Proof.* The first statement follows from the following observations. First,  $\text{lcm}(m_1, \dots, m_r)$  kills the semi-simple part of the monodromy endomorphisms acting on the cohomology groups  $H^k(X_\infty)$ , and hence all the  $M\alpha$  and  $M\alpha^{p,q}$  are integers. Second, the numerators  $(k-1)(3k+6n+2)$  in the second sum in (6.11) are always even integers. The second claim follows analogously, using that the  $\alpha$  and  $\alpha^{p,q}$  vanish. □

**6.3. The subdominant term in the asymptotics of the BCOV invariant.** Let  $f: X \rightarrow \mathbb{D}$  be a germ of a projective degeneration between complex algebraic varieties, whose smooth fibers are Calabi–Yau manifolds. In the previous subsection, we provided a general expression for the leading term of the asymptotic behaviour of the function  $t \mapsto \log \tau_{BCOV}(X_t)$ . The following statement describes the subdominant term.

**Proposition 6.8.** *The assumptions being as above, we have an asymptotic expansion*

$$\log \tau_{BCOV}(X_t) = \kappa_f \log |t|^2 + \varrho_f \log \log |t|^{-1} + \text{continuous},$$

*where  $\varrho_f$  is given in terms of the limiting Hodge structures by*

$$\varrho_f = \frac{\chi(X_\infty)}{12} \beta^{n,0} - \sum_{p,q} (-1)^{p+q} p \beta^{p,q}.$$

*Proof.* It is enough to combine the definition of the BCOV invariant (Definition 5.7), together with [EFiMM18, Prop. 4.2], Theorem 6.5 and Theorem 4.4. □

An immediate consequence of the proposition is the existence of the following limit:

$$(6.15) \quad \tau_{BCOV, \text{lim}} = \lim_{t \rightarrow 0} \frac{\tau_{BCOV}(X_t)}{|t|^{2\kappa_f} (\log |t|^{-1})^{\varrho_f}} \in \mathbb{R}_{>0}.$$

This limit actually depends on the choice of coordinate  $t$  on  $\mathbb{D}$ . Under a change of coordinate  $t \mapsto \lambda(t)$ , the limit changes to  $|\lambda'(0)|^{2\kappa_f} \cdot \tau_{BCOV, \text{lim}}$ . This can be restated by saying that  $\tau_{BCOV, \text{lim}}$  defines a hermitian metric on the  $\mathbb{Q}$ -complex line  $(\omega_{\mathbb{D},0})^{\otimes \kappa_f}$ . By construction, this metric only depends on the restriction of  $f: X \rightarrow \mathbb{D}$  to  $\mathbb{D}^\times$ , but not on the special fiber  $X_0$ .

**Remark 6.9.** (1) As an application of Lemma 4.3, the expression for  $\varrho_f$  can equivalently be written as

$$\varrho_f = \frac{\chi(X_\infty)}{12} \beta^{n,0} + 2 \sum_{\substack{p+q < n \\ q < p}} (-1)^{p+q} (q-p) \beta^{p,q} + (-1)^n \sum_{\substack{p+q=n \\ q < p}} (q-p) \beta^{p,q}.$$

- (2) For maximally unipotent degenerations of Calabi–Yau varieties as above, the mirror map provides a (quasi-)canonical coordinate on  $\mathbb{D}$ . This choice of coordinate induces a trivialization of  $\omega_{\mathbb{D},0}$ , so that  $\tau_{BCOV, \lim}$  becomes an unambiguously defined quantity.
- (3) For open Calabi–Yau manifolds with cylindrical ends, Conlon–Mazzeo–Rochon [CMR15] have defined an avatar of the Quillen-BCOV metric. It would be interesting to explore the connection between their work and our limiting invariant  $\tau_{BCOV, \lim}$ .

**6.4. Extensions of the holomorphic anomaly equation.** Let  $f: X \rightarrow S$  be a flat projective morphism of compact connected complex manifolds, with  $\dim S = 1$ . We let  $S^\times \subseteq S$  be the locus of regular values of  $f$ , which is necessarily Zariski open and non-empty. Write  $\{P_1, \dots, P_r\}$  for the complement  $S \setminus S^\times$ . If the smooth fibers of  $f$  are Calabi–Yau manifolds, then they have a well-defined BCOV invariant. By Theorem 6.5 the smooth function  $s \mapsto \log \tau_{BCOV}(X_s)$  on  $S^\times$  extends to a locally integrable function on  $S$ . For every point  $P_i$ , the singularity of  $\log \tau_{BCOV}(X_s)$  is of logarithmic type, with an attached coefficient  $\kappa_f(P_i)$  defined as in (6.10). In general, for a locally integrable differential form  $\theta$  on  $S$ , we denote by  $[\theta]$  the current of integration against  $\theta$ .

**Lemma 6.10.** *The differential form  $dd^c \log \tau_{BCOV}$  on  $S^\times$  is locally integrable on the whole of  $S$ . Moreover, we have the equality of currents*

$$(6.16) \quad dd^c[\log \tau_{BCOV}] - [dd^c \log \tau_{BCOV}] = \sum_{i=1}^r \kappa_f(P_i) \cdot \delta_{P_i}.$$

*Proof.* For the proof one needs a complement to the analysis of the subdominant term in Proposition 6.8, in order to include its  $d$  and  $dd^c$  derivatives. On the one hand, we need a control on the remainder of the asymptotics of the Quillen-BCOV metrics (6.12). This was addressed in [EFiMM18, Prop. 4.2]: the remainder, together with its  $d$  and  $dd^c$  derivatives, is modeled on  $\log \log |t|^{-1}$  and its  $d$  and  $dd^c$  derivatives. On the other hand, we have a similar property for the  $L^2$  metrics on Hodge bundles for a choice of fiberwise rational Kähler structure, by Theorem 4.4 (2). One easily concludes from these facts that  $dd^c \log \tau_{BCOV}$  has at worst Poincaré growth, and is in particular locally integrable. Also (6.16) follows from the indicated behaviour of the remainder term and by Theorem 6.5, by a standard evaluation of the current  $dd^c[\log \tau_{BCOV}] - [dd^c \log \tau_{BCOV}]$  on test functions.  $\square$

Let us now choose a Kähler structure on  $f$ , fiberwise rational on the smooth locus. We let  $\omega_{H^k}$  be the corresponding Hodge forms, defined on  $S^\times$  by (5.5)–(5.6). Denote as before  $\omega_{WP}$  the Weil–Petersson form on  $S^\times$ . By Theorem 4.4, the differential forms  $\omega_{H^k}$  and  $\omega_{WP}$  are locally integrable and have at most Poincaré growth singularities on  $S$ . Hence they define currents  $[\omega_{H^k}]$  and  $[\omega_{WP}]$  by integration. These are all closed semi-positive currents of type  $(1, 1)$ , and in particular they have well-defined (current) cohomology classes on  $S$ , denoted  $\{\omega_{H^k}\}$  and  $\{\omega_{WP}\}$ . The cohomology classes of  $\delta_{P_i}$  are identified to the cycle cohomology classes of the points  $P_i$ , denoted  $\{P_i\}$ .

**Proposition 6.11.** *We have an equality in  $H^{1,1}(S)$*

$$\frac{\chi}{12} \{\omega_{WP}\} - \sum_{k=1}^{2n} (-1)^k \{\omega_{H^k}\} = \sum_i \kappa_f(P_i) \{P_i\},$$

where  $\chi$  is the topological Euler characteristic of a general smooth fiber of  $f$ . Consequently, there is a relation

$$\frac{\chi}{12} \int_S \omega_{WP} - \sum_{k=1}^{2n} (-1)^k \int_S \omega_{H^k} = \sum_{i=1}^r \kappa_f(P_i).$$



*Proof.* From (6.16) we infer that the extension from  $S^\times$  to the level of currents by integration of  $dd^c \log \tau_{BCOV}$  is cohomologous to the sum of cycle cohomology classes  $\{P_i\}$ . The first equality then follows from the differential equation in Proposition 5.10, applied over  $S^\times$  and extended to the level of currents by integration. The second claim is obtained by integration over  $S$ .  $\square$

## 7. ASYMPTOTICS FOR SPECIAL GEOMETRIES

The general discussion in the previous section can be bolstered if further assumptions on the degeneration  $: X \rightarrow S$  are imposed. We initially treat the cases of semi-stable minimal (or Kulikov) degenerations (cf. Proposition 7.1) as well as for ordinary double point singularities (cf. Theorem 7.3). For the small dimensions 3 and 4, we can give refined statements for general degenerations (cf. Theorem 7.6 and Theorem 7.13). As a corollary we will then deduce the conjecture of Liu-Xia of [LX19, Conj. 0.5]. We conclude the article with some algebraic geometric applications of our results, summarized here as constraints to the existence of particular degenerations of Calabi–Yau varieties.

**7.1. Kulikov degenerations.** In this subsection we consider a semi-stable germ of a projective degeneration  $f : X \rightarrow \mathbb{D}$  of smooth algebraic varieties. Furthermore, we suppose that  $f$  is Kulikov, *i.e.*  $B = 0$ . We write  $X_0 = \sum_i D_i$ .

**Proposition 7.1.** *With the above assumptions and notations, we have*

$$(7.1) \quad \kappa_f = \sum_{k=1}^{n+1} (-1)^k \frac{k(k-1)}{24} \chi(D(k)).$$

*Proof.* First, from Theorem 6.5 easily follows the intermediate expression

$$(7.2) \quad \kappa_f = \sum_{k=1}^{n+1} (-1)^k \frac{k(k-1)}{8} \chi(D(k)) - \sum_{k=1}^n \frac{(-1)^n}{12} \int_{D(k)} c_1(\Omega_{D(k)}) c_{n-k}(\Omega_{D(k)}).$$

Indeed, in the semi-stable case, the monodromy is unipotent, and therefore  $\alpha = \alpha^{p,q} = 0$  for all  $p, q$ . A standard computation, see *e.g.* [Sai04, Lemma 1.4], shows that if  $D_i^\circ = D_i \setminus \bigcup_{j \neq i} D_{ij}$ , then

$$(7.3) \quad \chi(X_\infty) = \sum \chi(D_i^\circ) = \sum_{k=1}^{n+1} (-1)^{k+1} k \chi(D(k)).$$

Combining with equation (6.14), we derive

$$\chi(X_\infty) - \chi(X_0) = \sum_{k=1}^{n+1} (-1)^{k+1} (k-1) \chi(D(k)).$$

Now it is enough to plug this relation in the general expression for  $\kappa_f$  (6.11), together with the vanishing of  $B$  and  $\alpha$ ,  $\alpha^{p,q}$ , to conclude with (7.2).

We next proceed to simplify the contribution of the integrals in (7.2). For this, we establish a recursion relating the integral over  $D(k)$  to the integral over  $D(k+1)$ . Enumerate the components of  $X_0$  as  $D_1, \dots, D_r$ . Let  $I \subseteq \{1, \dots, r\}$  be a multi-index subset of order  $k$ . Accordingly, define  $D_I = \bigcap_{i \in I} D_i$  and  $B_I = D_I \cap (\bigcup_{j \notin I} D_j)$ . The Kulikov assumption, together with the triviality (as a Cartier divisor) of  $X_0 = \sum_i D_i$  and the adjunction formula guarantee that the pairs  $(D_I, B_I)$  are log-Calabi–Yau, *i.e.*  $K_{D_I} + B_I = 0$ . Combining this with the conormal exact sequence for the

inclusions  $D_{I \cup \{j\}} \hookrightarrow D_I$  ( $j \notin I$ ), we obtain the equalities

$$\begin{aligned} \int_{D(k)} c_1(\Omega_{D(k)}) c_{n-k}(\Omega_{D(k)}) &= - \sum_{|I|=k} \int_{D_I} [B_I] \cap c_{n-k}(\Omega_{D(k)}) \\ &= -(k+1) \int_{D(k+1)} c_{n-k}(\Omega_{D(k+1)}) + \sum_{\substack{|I|=k \\ j \notin I}} \int_{D_{I \cup \{j\}}} [D_j]|_{D_{I \cup \{j\}}} c_{n-k-1}(\Omega_{D_{I \cup \{j\}}}) \\ &= (-1)^{n-k+1} (k+1) \chi(D(k+1)) + \int_{D(k+1)} c_1(\Omega_{D(k+1)}) c_{n-k-1}(\Omega_{D(k+1)}). \end{aligned}$$

In the last equality we used that  $\int_{D_I} c_{n-k}(\Omega_{D_I}) = (-1)^{n-k} \chi(D_I)$  and  $K_{D_I} = -B_I = \sum_{i \in I} [D_i]|_{D_I}$ . The result follows by applying this recursion to (7.2).  $\square$

*Remark 7.2.* In the case of K3 surfaces, an application of adjunction shows that the right hand side of (7.1) vanishes. This is in agreement with the constancy of the BCOV invariant established in Proposition 5.12, which also implies the vanishing of  $\kappa_f$ .

**7.2. Ordinary double point singularities.** In this subsection, let  $f: X \rightarrow \mathbb{D}$  be a germ of a projective degeneration between smooth algebraic varieties, with general Calabi–Yau fibers of dimension  $n$ . Suppose that  $X_0$  admits at most ordinary double point singularities.

**Theorem 7.3.** *With the above assumptions and notations, we have*

$$\kappa_f = \frac{n+1}{24} \#\text{sing}(X_0) \quad \text{and} \quad \varrho_f = \#\text{sing}(X_0) \quad \text{if } n \text{ is odd,}$$

or

$$\kappa_f = -\frac{n-2}{24} \#\text{sing}(X_0) \quad \text{and} \quad \varrho_f = 0 \quad \text{if } n \text{ is even,}$$

where  $\#\text{sing}(X_0)$  denotes the number of singular points in the fiber  $X_0$ .

For later use we record the following lemma, which follows from the conormal exact sequence for the cotangent bundle, and the Euler sequence on  $\mathbb{P}_{\mathbb{C}}^n$  restricted to  $W$ :

**Lemma 7.4.** *Let  $W$  be an irreducible degree  $d$  smooth hypersurface in  $\mathbb{P}_{\mathbb{C}}^n$ . Then*

$$\int_W c_1(\mathcal{O}_W(1)) c_{n-2}(\Omega_W) = \frac{(-1)^{n-1}}{d} \chi(W) + (-1)^n \frac{n(n+1)}{2}.$$

For the following lemma, the reader is advised to review the description of the blow up of  $X$  along the ordinary double point singularities §3.4. Also, we notice that for the morphism  $f$ , the divisor of the evaluation map  $\mathbf{ev}$  is trivial, hence  $K_X = \mathcal{O}_X$  is trivial as well:  $f$  is a Kulikov family.

**Lemma 7.5.** *Let  $\nu: \tilde{X} \rightarrow X$  be the blow up of the ordinary double points in  $X_0$ . Let  $Z$  be the strict transform of  $X_0$  in  $\tilde{X}$ . Then*

$$\int_Z c_1(\Omega_Z) c_{n-1}(\Omega_Z) = \left( (-1)^{n-1} \frac{3(n-2)}{2} \chi(Q) + (-1)^n \frac{(n-2)n(n+1)}{2} \right) \#\text{sing}(X_0),$$

where  $Q$  is any smooth quadric in  $\mathbb{P}_{\mathbb{C}}^n$ , and hence  $\chi(Q) = n + (1 + (-1)^{n+1})/2$ .

*Proof.* Let  $E = E_1 + \dots + E_r$  be the exceptional divisor, where  $r$  is the number of ordinary double point singularities. Each  $E_i$  is isomorphic to  $\mathbb{P}_{\mathbb{C}}^n$ . Since the canonical bundle  $K_X$  is trivial, we have

$$K_{\tilde{X}} = \mathcal{O}(nE).$$

Hence, because  $\tilde{X}_0 = 2E + Z$ , we find by the adjunction formula

$$K_Z = \mathcal{O}_Z((n-2)E) = \mathcal{O}_Z((n-2)W),$$

where  $W = \sum E_i \cap Z$ , and each  $E_i \cap Z$  is an irreducible smooth quadric in  $\mathbb{P}_{\mathbb{C}}^n$ . Hence

$$(7.4) \quad \int_Z c_1(\Omega_Z) c_{n-1}(\Omega_Z) = (n-2) \int_W c_{n-1}(\Omega_Z|_W).$$

From the conormal exact sequence of the immersion  $W \hookrightarrow Z$ , we derive

$$(7.5) \quad \begin{aligned} \int_W c_{n-1}(\Omega_Z|_W) &= \int_W c_{n-1}(\Omega_W) - \int_W c_1(N_{W/Z}) c_{n-2}(\Omega_W) \\ &= (-1)^{n-1} \chi(W) - \int_W c_1(N_{W/Z}) c_{n-2}(\Omega_W). \end{aligned}$$

To compute the last integral in (7.5), we notice that  $N_{W/\tilde{X}} = N_{W/Z} \oplus N_{W/E}$ , because  $Z$  and  $E$  intersect transversally. It becomes

$$c_1(N_{W/Z}) = c_1(K_W) - c_1(K_{\tilde{X}}|_W) - c_1(N_{W/E}).$$

Together with the adjunction formula for  $W \hookrightarrow E$ , we infer

$$c_1(N_{W/Z}) = c_1(K_E|_W) - c_1(K_{\tilde{X}}|_W).$$

Now recall that  $K_E = \mathcal{O}_E(-n-1)$  and  $K_{\tilde{X}}|_W = \mathcal{O}_E(nE)|_W = \mathcal{O}_E(-n)|_W$ . Therefore  $c_1(N_{W/Z}) = c_1(\mathcal{O}_E(-1)|_W)$  and

$$(7.6) \quad \int_W c_1(N_{W/Z}) c_{n-2}(\Omega_W) = - \int_W c_1(\mathcal{O}_W(1)) c_{n-2}(\Omega_W).$$

To conclude, we apply Lemma 7.4 and add up (7.4)–(7.6).  $\square$

*Proof of Theorem 7.3.* The computation of the subdominant term is a consequence of the description of the limiting Hodge structure of ordinary double points [Ste77, Example 2.15].

For the dominant term, we let  $\tilde{X} \rightarrow X$  be the blow up of the ordinary double point singularities, and we apply Theorem 6.5 to the projection  $\tilde{f}: \tilde{X} \rightarrow \mathbb{D}$ . The special fiber  $\tilde{X}_0 = 2E + Z$  is as in the previous lemma:  $Z$  is the strict transform of  $X_0$ ,  $E = \sum E_i$  is a disjoint union of  $\mathbb{P}_{\mathbb{C}}^n$ , and the intersections  $W_i = Z \cap E_i$  are irreducible smooth quadrics in these projective spaces. We put  $W = \sum W_i$  in  $Z$ . Finally, the divisor of the evaluation map  $\mathbf{ev}$  for  $\tilde{f}$  is  $nE$ . We evaluate all the contributions to  $\kappa_{\tilde{f}}$ .

For the Euler characteristic of the general fiber  $\tilde{X}_{\infty}$  we have

$$\chi(\tilde{X}_{\infty}) = 2\chi(E) + \chi(Z) - 3\chi(W).$$

To verify this, we first notice that the degree of  $c_n(\Omega_{\tilde{X}/\mathbb{D}}(\log))$  on fibers is constant. On a general fiber this is  $(-1)^n \chi(\tilde{X}_{\infty})$ . On the special fiber, we find it is equal to the degree of  $2c_n(\Omega_E(\log W)) + c_n(\Omega_Z(\log W))$ . This can be computed through the residue exact sequence (see e.g. (3.9)), and the result follows. Also,

$$\chi(\tilde{X}_0) = \chi(E) + \chi(Z) - \chi(W).$$

Using that  $\chi(E) = (n+1)\#\text{sing}(X_0)$  and  $\chi(W) = [n + (1 + (-1)^{n+1})/2]\#\text{sing}(X_0)$ , we obtain

$$(7.7) \quad \frac{3n+1}{12} (\chi(\tilde{X}_{\infty}) - \chi(\tilde{X}_0)) = \frac{(3n+1)(-n+(-1)^n)}{12} \#\text{sing}(X_0).$$

The next term in (6.11) equals

$$(7.8) \quad \frac{3n+4}{12} \chi(W) = \frac{3n+4}{12} \left( n + \frac{1+(-1)^{n+1}}{2} \right) \#\text{sing}(X_0).$$

By the conormal exact sequence of  $E \hookrightarrow \tilde{X}$ , and taking into account that  $c(\Omega_E) = (1 - c_1(\mathcal{O}_E(1)))^{n+1}$  and  $\mathcal{O}(E)|_E = \mathcal{O}_E(-1)$ , we have

$$(7.9) \quad -\frac{(-1)^n}{12} \int_{nE} c_n(\Omega_{\tilde{X}}) = \left\{ \frac{n^2(n+1)}{24} - \frac{n(n+1)}{12} \right\} \#\text{sing}(X_0).$$

For the following terms, we apply lemmas 7.4 and 7.5. Using again  $c(\Omega_E) = (1 - c_1(\mathcal{O}_E(1)))^{n+1}$  and  $K_W = \mathcal{O}_W(-n+1)$ , we find

$$(7.10) \quad -\frac{(-1)^n}{12} \left\{ \int_E c_1(\Omega_E) c_{n-1}(\Omega_E) + \int_Z c_1(\Omega_Z) c_{n-1}(\Omega_Z) + \int_W c_1(\Omega_W) c_{n-2}(\Omega_W) \right\} = \\ \left\{ -\frac{n^2(n+1)}{24} + \frac{2n-5}{24} \left( n + \frac{1+(-1)^{n+1}}{2} \right) \right\} \#\text{sing}(X_0).$$

Finally, by Proposition 3.10 we know: i) if  $n$  is odd, then  $\alpha^{p,q} = 0$  for all  $p, q$ ; ii) if  $n$  is even, then  $\alpha^{p,q} = 0$  for  $(p, q) \neq (n/2, n/2)$  and

$$\alpha^{n/2, n/2} = \frac{1}{2} \#\text{sing}(X_0).$$

Since moreover  $\alpha = 0$ , we therefore conclude

$$(7.11) \quad -\frac{\alpha}{12} \chi(X_\infty) - \sum_{p,q} (-1)^{p+q} p \alpha^{p,q} = \begin{cases} 0 & \text{if } n \text{ is odd} \\ -\frac{n}{4} \#\text{sing}(X_0) & \text{if } n \text{ is even.} \end{cases}$$

To complete the proof, one just needs to evaluate the sum (7.7) + ... + (7.11).  $\square$

**7.3. Strict Calabi–Yau varieties: dimensions 3 and 4.** In the case of degenerating families of strict Calabi–Yau 3-folds and 4-folds we can give general results on the asymptotic behaviour of the BCOV invariant, not supposing that the central fiber is a normal crossings divisor.

We suppose first that  $f: X \rightarrow \mathbb{D}$  is a germ of a projective degeneration between smooth algebraic varieties, whose smooth fibers are strict Calabi–Yau 3-folds and with special fiber  $X_0 = \sum m_i D_i$ , not necessarily of normal crossings.

**Theorem 7.6.** *With the above assumptions and notations,*

(1) *we have*

$$\kappa_f = -\frac{1}{6}(\chi(X_\infty) - \chi(X_0)) - \left( \frac{\chi(X_\infty)}{12} + 3 \right) \alpha + \alpha^{1,1} - \alpha^{1,2} - \sum \chi(\mathcal{O}_{\tilde{D}_i}) + \frac{1}{12} \int_B c_3(\Omega_X)$$

where  $\tilde{D}_i$  denotes a desingularization of each  $D_i$  and  $\alpha = \frac{1}{2\pi i} \left( {}^u \log T_s \mid \text{Gr}_{F_\infty}^3 H^3(X_\infty) \right)$ .

(2) *If  $f: X \rightarrow \mathbb{D}$  has unipotent monodromies,*

$$\kappa_f = -\frac{1}{6}(\chi(X_\infty) - \chi(X_0)) + \left( \chi(\mathcal{O}_{X_\infty}) - \sum \chi(\mathcal{O}_{\tilde{D}_i}) \right) + \frac{1}{12} \int_B c_3(\Omega_X)$$

The proof is given below. We first deduce an immediate corollary.

**Corollary 7.7.** *Suppose furthermore that  $X_0$  has at most rational singularities.*

(1) *Then*

$$\kappa_f = -\frac{1}{6}(\chi(X_\infty) - \chi(X_0)) + \alpha^{1,1} - \alpha^{1,2}.$$

(2) *If the singularities are moreover isolated, we have*

$$\kappa_f = \frac{1}{6} \mu_f - \alpha^{1,2}$$

where  $\mu_f$  denotes the Milnor number of the special fiber.

*Proof.* Since  $X_0$  is normal,  $f : X \rightarrow \mathbb{D}$  is automatically Kulikov, *i.e.*  $B = 0$ . If  $X_0$  has rational singularities one finds that  $\chi(\mathcal{O}_{\tilde{X}_0}) = \chi(\mathcal{O}_{X_0}) = \chi(\mathcal{O}_{X_\infty}) = 0$  where we used the flatness of  $f$  and the strict Calabi–Yau condition on smooth fibers. Finally, for isolated singularities the monodromy acts trivially on  $H^2$  so  $\alpha^{1,1} = 0$ , and since rational and canonical singularities are equivalent for Gorenstein complex varieties (see *e.g.* [Kol97, Corollary 11.13]) it follows from [EFiMM18, Proposition 2.8] that  $\alpha = 0$ . For isolated singularities in  $X_0$  the total dimension of the vanishing cycles  $-(\chi(X_\infty) - \chi(X_0))$  is the Milnor number  $\mu_f$ .  $\square$

*Proof of Theorem 7.6.* By the definition of the BCOV invariant, it suffices to study the asymptotic behavior of the  $L^2$ - and Quillen-BCOV metrics on the Kähler extension of the BCOV bundle. We fix a Kähler metric on the total space  $X$ , whose Kähler form is fiberwise integral on the smooth locus. The asymptotic behavior of the Quillen-BCOV metric was established in [EFiMM18, Cor 4.9] (see (6.12)) and states that for a local trivialization of  $\tilde{\lambda}_{BCOV}$

$$(7.12) \quad \log \|\sigma\|_{Q,BCOV}^2 = \left( \frac{29}{6}(\chi(X_\infty) - \chi(X_0)) - \frac{\alpha}{12}\chi(X_\infty) + \frac{1}{12} \int_B c_3(\Omega_X) \right) \log |t|^2 + o(\log |t|).$$

In fact this also holds without the assumption of having normal crossings in the special fiber.

To control the asymptotic behaviour of the  $L^2$ -BCOV metric, we first notice that the renormalizing factors are constant by Proposition 4.2, so we only need to consider the non-renormalized  $L^2$  metric on the BCOV bundle. We can then apply Serre duality (cf. Proposition 3.3) and find that there is an isometry

$$\tilde{\lambda}_{BCOV,L^2} \simeq -3\lambda(\mathcal{O}_X) + \lambda(\Omega_{X/\mathbb{D}}) + 5c \cdot \mathcal{O}([0])$$

where  $\mathcal{O}([0])$  is equipped with the trivial singular metric and  $c = -(\chi(X_\infty) - \chi(X_0))$  is a localized Chern class. We first compare with the logarithmic extension, whose  $L^2$  norm is easier to handle. By Proposition 3.8 we have  $\lambda(\Omega_{X/\mathbb{D}}) = \lambda(\Omega_{X/\mathbb{D}}(\log)) + \left( \chi(\mathcal{O}_{X_\infty}) - \sum \chi(\mathcal{O}_{\tilde{D}_i}) \right) \mathcal{O}([0])$  hence, as  $X_\infty$  is a strict Calabi–Yau threefold, we find the isometry

$$(7.13) \quad \tilde{\lambda}_{BCOV,L^2} \simeq -3\lambda(\mathcal{O}_X) + \lambda(\Omega_{X/\mathbb{D}}(\log)) + \left( 5c - \sum \chi(\mathcal{O}_{\tilde{D}_i}) \right) \cdot \mathcal{O}([0]).$$

We study individually the asymptotic behaviour of the various  $L^2$  metrics. First of all, notice that all the  $R^i f_* \mathcal{O}_X$  are locally free and commute with restriction to the fibers. To see this, if  $\tilde{X} \rightarrow X$  is a birational morphism of complex manifolds such that  $\tilde{f} : \tilde{X} \rightarrow \mathbb{D}$  is a projective normal crossings degeneration, then  $R^i f_* \mathcal{O}_X \simeq R^i \tilde{f}_* \mathcal{O}_{\tilde{X}}$ , and the latter is locally free as it is a higher direct image of a logarithmic sheaf. It easily follows, by connectedness of the fibers, that  $f_* \mathcal{O}_X = \mathcal{O}_{\mathbb{D}}$ . By Grothendieck-Serre duality, there is then an isomorphism

$$(7.14) \quad \lambda(\mathcal{O}_X) \simeq \mathcal{O}_{\mathbb{D}} - \det R^1 f_* \mathcal{O}_X + \det R^2 f_* \mathcal{O}_X - R^3 f_* \mathcal{O}_X \simeq \mathcal{O}_{\mathbb{D}} - \det R^1 f_* \mathcal{O}_X + \det R^2 f_* \mathcal{O}_X + f_* \omega_{X/\mathbb{D}}.$$

This is an isometry for the  $L^2$  norms. The squared  $L^2$  norm of 1 in the first factor is a constant multiple of the volume of the fixed Kähler form on the fiber, and hence constant. For the next two terms, by Theorem 4.4, for a local trivialization  $\sigma_{p,q}$  of  $\det R^q f_* \Omega_{X/\mathbb{D}}^p(\log)$  we have  $\log \|\sigma_{p,q}\|_{L^2}^2 = \alpha^{p,q} \log |t|^2 + o(\log |t|)$ . Also, for a local trivialization  $\eta$  of  $f_* \omega_{X/\mathbb{D}}$ , by [EFiMM18, Theorem A] we have  $\log \|\eta\|_{L^2}^2 = -\alpha \log |t|^2 + o(\log |t|)$  so that a local trivialization  $\sigma$  of  $\lambda(\mathcal{O}_X)$  satisfies

$$\log \|\sigma\|_{L^2}^2 = (-\alpha^{0,1} + \alpha^{0,2} - \alpha) \log |t|^2 + o(\log |t|).$$

For the  $L^2$  metric on  $\lambda(\Omega_{X/\mathbb{D}}(\log))$ , expanding the determinant and performing the same analysis as in (7.14), we find that the norm of a local trivialization  $\sigma$  of  $\lambda(\Omega_{X/\mathbb{D}}(\log))$  is

$$\log \|\sigma\|_{L^2}^2 = (\alpha^{1,0} - \alpha^{1,1} + \alpha^{1,2} - \alpha^{1,3}) \log |t|^2 + o(\log |t|).$$

All in all, combining (7.12), (7.13) and the subsequent computations we find a general expression for  $\kappa_f$ . The assumption that a general fiber is strict Calabi–Yau assures the vanishing of several terms ( $\alpha^{0,1} = \alpha^{0,2} = \alpha^{1,0} = \alpha^{1,3} = 0$ ), giving the first part of the theorem. The case of unipotent monodromies is a simplification of the main result under this assumption.  $\square$

For the purposes of the statements below, we use the notation  $D_{ij}, D_{ijk}$ , etc. for  $D_i \cap D_j$ ,  $D_i \cap D_j \cap D_k$  etc. We also abuse notation and write e.g.  $[D_i^2 D_j]$  for the class  $c_1(\mathcal{O}(D_i)) \cap [D_{ij}]$  and identify top degree intersection products with their degrees. We call quadruple points those points lying on four components on a normal crossings union of three dimensional varieties. With this understood, we can then record the following lemma.

**Lemma 7.8.** *Let  $f: X \rightarrow \mathbb{D}$  be a semi-stable projective degeneration, with strict Calabi–Yau 3-fold smooth fibers, and write  $X_0 = \sum D_i$ . Denote by  $Q$  the number of quadruple points on  $X_0$ . Then we have*

$$\sum_{k \neq i, k \neq j, i < j} c_1(\mathcal{O}(D_k))^2 \cap [D_{ij}] = -4Q.$$

If  $f$  is furthermore assumed to be Kulikov, then

$$\sum_{i < j} c_1(K_{D_{ij}})^2 \cap [D_{ij}] = 8Q.$$

*Proof.* First note the relation  $0 = c_1(\mathcal{O}(X_0)) = \sum c_1(\mathcal{O}(D_i))$ . Then it follows that, since  $[D_{ijk}] = c_1(\mathcal{O}(D_k)) \cap [D_{ij}]$ ,

$$0 = \sum_{\ell} c_1(\mathcal{O}(D_{\ell})) \cap [D_{ijk}] = [D_i^2 D_j D_k] + [D_i D_j^2 D_k] + [D_i D_j D_k^2] + \sum_{l \notin \{i, j, k\}} [D_{ijkl}].$$

The last sum is the number of quadruple points on  $D_{ijk}$ . As any quadruple point appears on four different components, taking sums over all possible combinations we find the first identity. If  $f$  is Kulikov, then all the components  $D_{ij}$  are log-Calabi–Yau so that  $K_{D_{ij}} = -\sum_{l \neq i, l \neq j} c_1(\mathcal{O}(D_l))$ . Then,

$$\sum_{i < j} c_1(K_{D_{ij}})^2 \cap [D_{ij}] = \sum_{i < j} \left( \sum_{l \neq i, l \neq j} c_1(\mathcal{O}(D_l)) \right)^2 \cap [D_{ij}] = -4Q + 2 \binom{4}{2} Q = 8Q.$$

$\square$

**Corollary 7.9.** *Suppose that  $f: X \rightarrow \mathbb{D}$  is a semi-stable, Kulikov germ of a projective degeneration between smooth algebraic varieties, with strict Calabi–Yau 3-fold generic fibers. Then, if  $Q$  denotes the number of quadruple points, we have*

$$\begin{aligned} 12\kappa_f &= \chi(D(2)) - 6Q \\ &= 12\chi(D(1), \mathcal{O}_{D(1)}) - 2Q. \end{aligned}$$

Notice that in particular  $\kappa_f \equiv \frac{-Q}{6} \pmod{1}$ .

*Proof.* According to Proposition 7.1 we have

$$(7.15) \quad 12\kappa_f = \chi(D(2)) - 3\chi(D(3)) + 6\chi(D(4)).$$

Since  $K_{D_{ijk}} + B_{ijk}$  is trivial (see the proof of the same Proposition 7.1), we have

$$\chi(D_{ijk}) = \#\{\text{quadruple points on } D_{ijk}\}$$

so that  $\chi(D(3)) = 4Q$ , which provides the first equality. Using the same argument and the sequence (3.8) we find as  $\chi(\mathcal{O}_{X_0}) = \chi(\mathcal{O}_{X_\infty}) = 0$

$$\chi(\mathcal{O}_{D(1)}) = \chi(\mathcal{O}_{D(2)}) - \chi(\mathcal{O}_{D(3)}) + Q = \chi(\mathcal{O}_{D(2)}) - Q.$$

This expression together with an application of the Noether formula, for the surfaces  $D_{ij}$ , and Lemma 7.8 furnishes the second formula in the corollary.  $\square$

*Remark 7.10.* The corollary implies the conjecture of Liu–Xia [LX19, Conj. 0.5], simplified with the computations as in the previous lemma,

$$\sum_{i < j} c_1(\mathcal{O}(D_i))c_1(\mathcal{O}(D_j)) \cap [D_{ij}] - \left( \sum_{i < j} c_1(\mathcal{O}(D_i))c_1(\mathcal{O}(D_j)) \right) \cap \left( \sum_{i < j} [D_{ij}] \right) = -2Q.$$

**Proposition 7.11.** *Let  $f: X \rightarrow \mathbb{D}$  be a germ of a projective degeneration of 3-dimensional strict Calabi–Yau varieties, which is moreover semi-stable and Kulikov, of maximal unipotent monodromy. Then  $\kappa_f \geq 0$ . In other words, Conjecture A in the introduction is true under these assumptions.*

*Proof.* First of all, since a quadruple point lies on 6 different double surfaces, we find that the expression for  $\kappa_f$  in Corollary 7.9 takes the form

$$(7.16) \quad 12\kappa_f = \sum_{ij;a} (\chi(D_{ij;a}) - n_{ij;a}),$$

where the sum is over the irreducible components  $D_{ij;a}$  of all double surfaces  $D_{ij}$  and  $n_{ij;a}$  denotes the number of quadruple points on each such component.

Second, the hypotheses on the morphism impose the following restrictions on the geometry of the special fiber: after [CL16, Lemma 4.1], the components  $D_{ij;a}$  are rational surfaces, and the triple curves draw a cycle  $C_{ij;a}$  of rational curves on each of them. As in the proof of Proposition 7.1,  $C_{ij;a}$  is an anti-canonical divisor of  $D_{ij;a}$ . On  $D_{ij;a}$ , the number  $n_{ij;a}$  of quadruple points equals the number of triple curves, and the dual graph of  $C_{ij;a}$  is homeomorphic to a circle. Together with (7.16), we see that the proposition follows from the lemma below.  $\square$

**Lemma 7.12.** *Suppose  $S$  is a rational surface and  $D \in |K_S^{-1}|$ . Then one of the following holds:*

- $D$  has a dual graph homeomorphic to a star.
- The number  $d$  of irreducible components of  $D$  is bounded above by  $\chi(S)$ .

*Proof.* By the birational classification of surfaces,  $S$  is a succession of  $r$  blowups  $p: S \rightarrow S_{\min}$  of a minimal rational surface  $S_{\min}$ . These are either  $\mathbb{P}^2$  or the Hirzebruch surfaces  $F_e = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-e))$ . We first provide an argument reducing to these cases.

The anti-canonical bundles of  $S$  and  $S_{\min}$  are related by

$$(7.17) \quad K_S^{-1} = p^*(K_{S_{\min}}^{-1}) \otimes \mathcal{O}(-E),$$

where  $E = \sum_{\ell=1}^r E_\ell$  is the exceptional divisor. Notice that the relationship (7.17) implies that  $D + E \in |p^*K_{S_{\min}}^{-1}| = |K_{S_{\min}}^{-1}|$ . In other words, there exists a global section  $s \in H^0(K_{S_{\min}}^{-1})$  such that  $D + E = \text{div } p^*s$ . Let  $d'$  be the number of irreducible components of  $\text{div } s$ . Then the number of irreducible components of  $\text{div } p^*s$  is bounded by  $d' + r$ . It follows that  $d \leq d' + r$ . If we can show that  $d' \leq \chi(S_{\min})$  we conclude that  $d \leq \chi(S_{\min}) + r = \chi(S)$ .

If  $S = \mathbb{P}^2$ , then  $K_S^{-1} = \mathcal{O}(3)$  and  $\chi(S) = 3$  from which the statement in the lemma easily follows.

If  $S = \mathbb{F}_e$ , then  $\chi(S) = 4$  and  $K_S^{-1} \equiv 2C_0 + (e+2)f$ , where  $C_0$  is the unique irreducible curve such that  $C_0^2 = -e$  and  $f$  is any fiber of the natural map  $\pi: S \rightarrow \mathbb{P}^1$ . Write

$$D = aC_0 + \sum_{k=1}^b f_k + D'$$

where  $f_k$  denote fibers of  $\pi$  and  $D'$  contains neither any fiber nor  $C_0$ . Since  $D'.f = 2 - a \geq 0$ , we proceed according to the possibilities of  $a$ .

If  $a = 2$ , then  $D'$  is empty. Since  $C_0$  is a section of  $\pi$ , this means that the dual graph of  $D$  is a star. Notice that this topological type does not change under blowup.

If  $a = 1$ , then  $D'.f = 1$  and hence  $D'$  is irreducible. On the one hand,  $D.C_0 = -e + 2$ . On the other hand,  $D.C_0 = -e + b + D'.C_0 \geq -e + b$  from which we infer that  $b \leq 2$ . Hence the number of components is bounded by  $4 = \chi(S)$ . The case  $a = 0$  is similar to the case  $a = 1$ .  $\square$

In the case that  $f: X \rightarrow \mathbb{D}$  is a projective degeneration of Calabi–Yau 4-folds, a similar proof as in Theorem 7.6 yields the below theorem. As in the 3-dimensional case, for simplicity, we state it only for strict Calabi–Yau 4-folds. It is a refinement of the case of normal crossings in Theorem 6.5.

**Theorem 7.13.** *Suppose that  $f: X \rightarrow \mathbb{D}$  is a germ of a projective degeneration between smooth algebraic varieties, whose smooth fibers are strict Calabi–Yau 4-folds and with special fiber  $X_0 = \sum m_i D_i$ , not necessarily of normal crossings. Then*

$$\kappa_f = -\frac{1}{12}(\chi(X_\infty) - \chi(X_0)) + \left(-\frac{\chi(X_\infty)}{12} + 4\right)\alpha + 2\alpha^{1,1} - 2\alpha^{1,2} + 2\alpha^{1,3} + 2\left(2 - \sum \chi(\mathcal{O}_{\widetilde{D}_i})\right) - \frac{1}{12} \int_B c_4(\Omega_X)$$

where  $\widetilde{D}_i$  denotes a desingularization of  $D_i$ . Here  $\alpha = \frac{1}{2\pi i} \left( u \log T_s \mid \text{Gr}_{F_\infty}^4 H^4(X_\infty) \right)$ .

The following is a straightforward corollary following the lines of Corollary 7.7.

**Corollary 7.14.** *Suppose moreover that  $X_0$  has at most rational singularities.*

- (1) Then  $\kappa_f = -\frac{1}{12}(\chi(X_\infty) - \chi(X_0)) + 2\alpha^{1,1} - 2\alpha^{1,2} + 2\alpha^{1,3}$ .
- (2) If the singularities are moreover isolated, we have

$$\kappa_f = -\frac{\mu_f}{12} + 2\alpha^{1,3}$$

where  $\mu_f$  denotes the Milnor number of the special fiber.

*Proof.* We just comment on the vanishing of  $\alpha^{1,1}$  and  $\alpha^{1,2}$  implicit in the second claim: since the singularities are isolated, the monodromy action is trivial on  $H^2(X_\infty)$  and  $H^3(X_\infty)$ .  $\square$

*Remark 7.15.* For ordinary double point singularities, the corollaries 7.7 and 7.14 are compatible with Theorem 7.3. Indeed, for ordinary double points the Milnor number equals the number of singular points, and we know the monodromy from the Picard-Lefschetz theorem.

**7.4. Bounds on ordinary double point singularities.** In [FL05], Fang–Lu used the differential equation satisfied by the BCOV invariant to prove the non-existence of complete curves in some moduli spaces of polarized Calabi–Yau varieties. In this section, we apply Theorem 7.3 to improve on their work. We also remark, in Proposition 7.19, a consequence for abelian varieties and hyperkähler varieties.

We first place ourselves in the setting of §6.4 and Propostion 6.11. Hence,  $f: X \rightarrow S$  is a flat projective morphism of compact connected complex manifolds, with  $\dim S = 1$ . We let  $S^\times \subseteq S$  be



the locus of regular values of  $f$ , and we write  $\{P_1, \dots, P_r\}$  for the complement  $S \setminus S^\times$ . We suppose that the smooth fibers have trivial canonical bundle, with topological Euler characteristic  $\chi$ . We choose a Kähler structure on  $f$ , fiberwise rational on the smooth locus, and we denote by  $\omega_{H^k}$  the corresponding Hodge forms, and  $\omega_{WP}$  the Weil–Petersson form. Recall these are smooth semi-positive  $(1, 1)$  forms on  $S^\times$ , with at worst Poincaré growth along  $S \setminus S^\times$ . Following Fang–Lu, we will say that the family  $f: X \rightarrow S$  is *primitive* if the Hodge forms  $\omega_{H^k}$  vanish for all  $k \neq n$ .<sup>8</sup> They establish in *loc. cit.*, Corollary 2.10, the following inequality:

$$(7.18) \quad \omega_{H^n} \geq 2\omega_{WP}.$$

While the authors work with strict Calabi–Yau manifolds, an examination of their proof shows the validity of this inequality for Calabi–Yau varieties in the broad sense. Combining (7.18) together with Proposition 6.11, we readily infer:

**Proposition 7.16.** *If  $f: X \rightarrow S$  is a primitive family of Calabi–Yau varieties as above, then*

$$(-1)^n \sum_{i=1}^r \kappa_f(P_i) \leq \left( -2 + (-1)^n \frac{\chi}{12} \right) \text{vol}(\omega_{WP}),$$

where  $\text{vol}(\omega_{WP}) = \int_S \omega_{WP}$ .

It follows from the proposition that if  $(-1)^{n+1}\chi > -24$  and  $f$  is non-isotrivial, then  $f$  has at least one singular fiber. This observation was already made by Fang–Lu [FL05, Cor. 1.3]. We now prove variants of their result.

**Corollary 7.17.** *In odd relative dimension  $n$ , assume that  $\chi > -24$  and  $f: X \rightarrow S$  as above is a non-isotrivial primitive degeneration of Calabi–Yau manifolds. Suppose furthermore that  $f$  has at most ordinary double points. Let  $\#\text{sing}(X/S)$  be the total number of singular points in the fibers of  $f$ . Then*

$$\#\text{sing}(X/S) \geq \frac{48 + 2\chi}{n + 1}.$$

*Proof.* The inequality results as an application of Proposition 7.16 and Theorem 7.3. We need to observe that  $\omega_{WP} = c_1(f_*K_{X/S}, h_{L^2})$  and that the  $L^2$  metric is Mumford good on  $f_*K_{X/S}$  by [EFiMM18, Thm. A & Prop. 2.8]. Alternatively, in this case  $f_*K_{X/S}$  is a lower extension and we can apply Theorem 4.4 and the fact that  $\alpha^{n,0} = 0$ . In any event, this entails

$$\text{vol}(\omega_{WP}) = \deg f_*K_{X/S} \geq 1,$$

the later inequality being due to the non-isotriviality assumption [Pet84, Thm. 5.3.1].  $\square$

**Corollary 7.18.** *In even relative dimension  $n \geq 4$ , assume that  $\chi < 24$  and  $f: X \rightarrow S$  as above is a non-isotrivial primitive degeneration of Calabi–Yau manifolds, admitting at most ordinary double point singularities. Then*

$$\#\text{sing}(X/S) \geq \left\lceil \frac{48 - 2\chi}{n - 2} \right\rceil^{\text{even}},$$

where for  $x \in \mathbb{R}$ ,  $\lceil x \rceil^{\text{even}}$  denotes the smallest even integer  $n$  with  $n \geq x$ .

*Proof.* For the inequality  $\#\text{sing}(X/S) \geq (48 - 2\chi)/(n - 2)$ , the proof goes as in Corollary 7.17, and is left as an exercise to the reader. To conclude we apply Corollary 3.5.  $\square$

<sup>8</sup>Actually Fang–Lu define the primitive Calabi–Yau manifolds as those strict Calabi–Yau manifolds whose Kuranishi deformations have this vanishing property for the Hodge forms. Also they define primitivity by vanishing for  $k < n$ . In the integrally polarized case, our notion coincides with theirs as follows from Serre duality.

For particular geometries we have a stronger non-existence result, which can be proven by other means. We include it as an illustration of our techniques:

**Proposition 7.19.** *If  $f : X \rightarrow S$  is a projective degeneration of abelian varieties of dimension at least 2 or hyperkähler varieties of dimension at least 4, then no fibers of  $f$  have only ordinary double point singularities.*

*Proof.* This follows from the constancy of the BCOV invariant for such families (cf. Proposition 5.12) and Theorem 7.3, which expresses the logarithmic term of the BCOV invariant in terms of the number of ordinary double point singularities.  $\square$

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