FAMILIES OF HYPERSURFACES OF LARGE DEGREE

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dedicated to Eckart Viehweg

ABSTRACT. Grauert and Manin showed that a non-isotrivial family of compact complex hyperbolic curves has finitely many sections. We consider a general moving enough family of high enough degree hypersurfaces in a complex projective space. We show the existence of a strict closed subset of its total space that contains the image of all its sections.

1. INTRODUCTION

Grauert [Grauert-65] and Manin [Manin-63] solved Mordell conjecture for curves over function fields. Lang generalised this statement in [Lang-86].

Conjecture (Lang's conjecture over function fields)**.** *Let $\pi : \mathcal{X} \to Y$ be a projective surjective morphism of complex algebraic manifolds, whose generic fibre is of general type. If π is not birationally isotrivial, then there is a proper subscheme of $\mathcal{X}$ that contains the image of all sections of π.*

Grauert's proof can be red as a construction of first order differential equations fulfilled by all but a finite number of sections of the family π. First order differential equations are also enough to deal with families of manifolds with ample cotangent bundles ([Noguchi-81][Moriwaki-95]).

We implement this idea in higher dimensions with higher order differential equations, in a case where the positivity assumption is made only for the canonical bundle. We consider a family of hypersurfaces of $\mathbb{P}^{n+1}$ parametrised by a curve C and given by a section of an ample line bundle $L = \lambda \boxtimes \mathcal{O}_{\mathbb{P}^{n+1}}(d)$ on $C \times \mathbb{P}^{n+1}$.

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{\iota} & C \times \mathbb{P}^{n+1} \\ \downarrow \pi & \swarrow pr_1 & \\ C & & \end{array}$$

We will assume that the genus of C and the relative dimension n are at least 2. The degree of the hypersurfaces in the family is the integer d. We will say that the family is *moving enough* if $\deg \lambda$ is large. We prove

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Main Theorem. *For a general moving enough family of high enough degree hypersurfaces of a complex projective space, there is a proper algebraic subset of the total space that contains the image of all its sections.*

"General" refers to the family being chosen outside a proper algebraic subset of the parameter space. This assumption ensures in particular that the family is not birationally isotrivial.

By Noether normalisation theorem and the primitive element theorem, every algebraic manifold $\mathcal{X}$ of dimension n defined over the field of rational functions $\mathbb{C}(Y)$ of a connected manifold Y is birational over $\mathbb{C}(Y)$ to an (usually singular) hypersurface in $\mathbb{P}^{n+1}_{\mathbb{C}(Y)}$.

We point out that Noguchi [Noguchi-85] gave a proof of Lang's conjecture when the smooth members of the family are hyperbolic under the assumption that the smooth part is hyperbolically embedded in the total space. Even though Kobayashi conjectured that a generic hypersurface of large degree of $\mathbb{P}^{n+1}$ is hyperbolic, our main result would not follow. Our proof does not rely on properties of families of hyperbolic manifolds, like normality. We benefit however from the recent works dealing with Kobayashi conjecture, especially from [Demailly-95].

The first part of our work (section 2) describes general tools for dealing with higher order jets. The second part (sections 3, 4, 5) is devoted to the proof of the

Theorem 1. *For every general moving enough family of high enough degree hypersurfaces of $\mathbb{P}^{n+1}$, there is a non-trivial differential equation of order $n+1$ fulfilled by all its sections.*

Then, adapting general techniques in universal families originating in the works of Clemens [Clemens-86], Voisin [Voisin-96] and Siu [Siu-04], we obtain the main theorem in the third part (section 6).

I thank Claire Voisin for the nice idea she gave to me for computing nef cones. I discussed the subject of this paper with many people over more than four years. I would like to thank them all, in a single sentence.

2. Jet spaces for sections

Note that Lang's conjecture would follow from a positive answer to the case when the parameter space is a curve. We consider a proper morphism of complex manifolds $\pi : \mathcal{X} \to C$, that we regard as a family of n-dimensional compact complex varieties parametrised by a connected compact complex smooth curve C. We intend to construct the jet spaces for the sections of π, allowing finite extension of the base field $\mathbb{C}(C)$ that is, finite covers of the curve C.

2.1. **Jets of order one.**

We follow the ideas of Grauert [Grauert-65].

Consider a section $s : C_\rho \to \mathcal{X}$ of the pull-back family $\pi_\rho : \rho^\star\mathcal{X} \to C_\rho$, where $\rho : C_\rho \to C$ is a finite morphism of curves.

$$\begin{array}{ccc} \rho^\star\mathcal{X} & \xrightarrow{\rho} & \mathcal{X} \\ s \uparrow \downarrow \pi_\rho & & \downarrow \pi \\ C_\rho & \xrightarrow{\rho} & C \end{array}$$

The map ${}^t ds : s^\star\Omega_{\mathcal{X}} \to \Omega_{C_\rho}$ satisfies ${}^t ds \circ s^{\star t} d\pi_\rho = Id_{\Omega_{C_\rho}}$, is hence surjective and provides a rank one quotient of $s^\star\Omega_{\mathcal{X}}$. The corresponding curve $s_1 : C_\rho \to \mathcal{X}_{\rho,1}$ inside the bundle $\pi_{0,\rho,1} : \mathcal{X}_{\rho,1} := \mathbb{P}(\rho^\star\Omega_{\mathcal{X}}) \to \rho^\star\mathcal{X}$ of rank one quotients of $\rho^\star\Omega_{\mathcal{X}}$

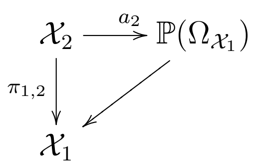
$$\begin{array}{ccccc} \mathcal{X}_{\rho,1} & \xrightarrow{\rho} & \mathcal{X}_1 & := & \mathbb{P}(\Omega_{\mathcal{X}}) \\ \downarrow \pi_{0,\rho,1} & & \downarrow \pi_{0,1} & & \\ \rho^\star\mathcal{X} & \xrightarrow{\rho} & \mathcal{X} & & \\ s \uparrow \downarrow \pi_\rho & & \downarrow \pi & & \\ C_\rho & \xrightarrow{\rho} & C & & \end{array} \qquad (s_1 : C_\rho \to \mathcal{X}_{\rho,1})$$

lifts s (i.e. $\pi_{0,\rho,1} \circ s_1 = s$), is therefore a section of $\pi_{\rho,1} : \mathcal{X}_{\rho,1} \to C_\rho$. It avoids the divisor $\mathcal{D}_1 := \mathbb{P}(\Omega_{\mathcal{X}/C})$ of vertical differentials, the divisor of the section of $\pi^\star T_C \otimes \mathcal{O}_{\Omega_{\mathcal{X}}}(1)$ given by ${}^t d\pi : \pi^\star\Omega_C \to \Omega_{\mathcal{X}}$. We have to study the positivity properties of this line bundle, which transfer into mobility properties of the forbidden divisor $\mathcal{D}_1$.

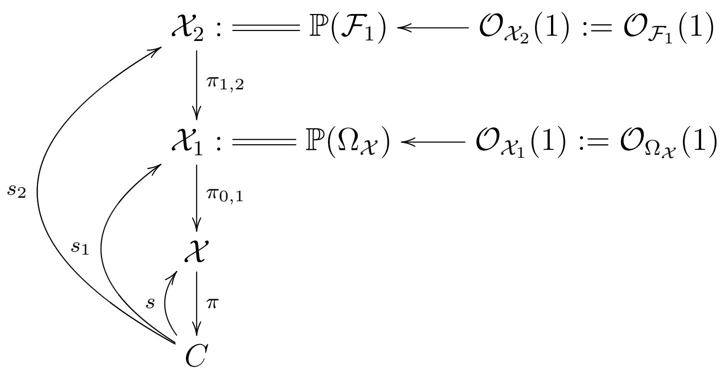
2.2. **Second order jets.** The rest of the construction, that does not depend on the map π but only on the total space $\mathcal{X}$, is Demailly-Semple construction [Demailly-95]. We will omit the cover ρ. As in the preceding section, the curve $s_1 : C \to \mathcal{X}_1$ lifts to a curve inside the bundle of rank one quotients of $\Omega_{\mathcal{X}_1}$. More precisely, the rank one quotient ${}^t ds_1 : s_1^\star\Omega_{\mathcal{X}_1} \to \Omega_C$ fulfils the relation ${}^t ds_1 \circ s_1^{\star t} d\pi_{0,1} = {}^t ds$. The map ${}^t ds_1$ at the point $[{}^t ds]$ of $\mathcal{X}_1$ vanishes on the image by ${}^t d\pi_{0,1}$ of forms in the kernel of the tautological quotient ${}^t ds$. In other words, ${}^t ds_1$ is a rank one quotient of the quotient $\mathcal{F}_1$ of $\Omega_{\mathcal{X}_1}$ defined by the following diagram on $\mathcal{X}_1$.

$$\begin{array}{ccccccc} & 0 & & 0 & & & \\ & \downarrow & & \downarrow & & & \\ & S & = & S & & & \\ & \downarrow & & \downarrow {}^t d\pi_{0,1} & & & \\ 0 \to & \pi_{0,1}^\star\Omega_{\mathcal{X}} & \xrightarrow{{}^t d\pi_{0,1}} & \Omega_{\mathcal{X}_1} & \to & \Omega_{\mathcal{X}_1/\mathcal{X}} & \to 0 \\ & \downarrow & & \downarrow q_1 & & || & \\ 0 \to & \mathcal{O}_{\mathcal{X}_1}(1) & \xrightarrow{{}^t d\pi_{0,1}} & \mathcal{F}_1 & \to & \Omega_{\mathcal{X}_1/\mathcal{X}} & \to 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ & 0 & & 0 & & 0 & \end{array}$$

Define the second order jet space to be $\pi_{1,2} : \mathcal{X}_2 := \mathbb{P}(\mathcal{F}_1) \to \mathcal{X}_1$. As in the formalism of Arrondo, Sols and Speiser [A-S-S-97], we need to keep track of the injective map $a_2 = \mathbb{P}(q_1) : \mathcal{X}_2 \to \mathbb{P}(\Omega_{\mathcal{X}_1})$ given by the quotient $q_1 : \Omega_{\mathcal{X}_1} \to \mathcal{F}_1$.

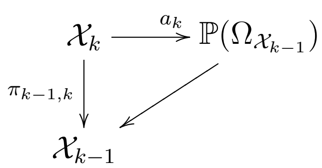


We hence get a map $s_2 : C \to \mathcal{X}_2$ defined by the quotient ${}^t ds_1 : s_1^\star \mathcal{F}_1 \to \Omega_C$. Note that $\pi_{1,2}$ is the restriction to $\mathbb{P}(\mathcal{F}_1) \subset \mathbb{P}(\Omega_{\mathcal{X}_1})$ of the map defined by the quotient $\pi_1^\star \Omega_{\mathcal{X}} \xrightarrow{{}^t d\pi_{01}} \Omega_{\mathcal{X}_1}$ so that the relation ${}^t ds_1 \circ s_1^{\star t} d\pi_{0,1} = {}^t ds$ is rephrased saying that the map s_2 is a lifting of s_1 (i.e. $\pi_{1,2} \circ s_2 = s_1$).



The map ${}^t d\pi_{01} : \mathcal{O}_{\mathcal{X}_1}(1) \to \mathcal{F}_1$ gives rise to a section of $\pi_1^\star \mathcal{O}_{\mathcal{X}_1}(-1) \otimes \mathcal{O}_{\mathcal{X}_2}(1)$ whose divisor $\mathcal{D}_2 := \mathbb{P}(\Omega_{\mathcal{X}_1/\mathcal{X}}) \subset \mathcal{X}_2$ is not hit by the curve s_2 associated with the quotient ${}^t ds_1$, for $\Omega_C \simeq s_1^\star \mathcal{O}_{\mathcal{X}_1}(1) \to^{s_1^{\star t} d\pi_{01}} s_1^\star \mathcal{F}_1 \to^{{}^t ds_1} \Omega_C$ vanishes nowhere.

2.3. **Higher order jets.** This scheme inductively leads to the construction of the k^{th}-order jet space $\pi_{k-1,k} : \mathcal{X}_k \to \mathcal{X}_{k-1}$, together with a map $a_k : \mathcal{X}_k \to \mathbb{P}(\Omega_{\mathcal{X}_{k-1}})$ that completes the commutative diagram.



Note that $a_k^\star \mathcal{O}_{\Omega_{\mathcal{X}_{k-1}}}(1) = \mathcal{O}_{\mathcal{X}_k}(1)$. The bundle $\mathcal{F}_k$ on $\mathcal{X}_k$ is the quotient of $\Omega_{\mathcal{X}_k}$ defined by

$$
\begin{array}{ccccccc}
 & 0 & & 0 & & & \\
 & \downarrow & & \downarrow & & & \\
 & S_k & = & S_k & & & \\
 & \downarrow & & \downarrow {}^t d\pi_{k-1,k} & & & \\
0\to & \pi^\star_{k-1,k}\Omega_{\mathcal{X}_{k-1}} & \xrightarrow{{}^t d\pi_{k-1,k}} & \Omega_{\mathcal{X}_k} & \to & \Omega_{\mathcal{X}_k/\mathcal{X}_{k-1}} & \to 0\\
 & \downarrow & & \downarrow q_k & & || & \\
0\to & \mathcal{O}_{\mathcal{X}_k}(1) & \longrightarrow & \mathcal{F}_k & \to & \Omega_{\mathcal{X}_k/\mathcal{X}_{k-1}} & \to 0\\
 & \downarrow & & \downarrow & & \downarrow & \\
 & 0 & & 0 & & 0 &
\end{array}
$$

The $(k+1)^{th}$-order jet space is $\pi_{k,k+1} \ : \ \mathcal{X}_{k+1} := \mathbb{P}(\mathcal{F}_k) \to \mathcal{X}_k$ and the map $a_{k+1} \ : \ \mathcal{X}_{k+1} \to \mathbb{P}(\Omega_{\mathcal{X}_k})$ is the injective map associated with the quotient $q_k \ : \ \Omega_{\mathcal{X}_k} \to \mathcal{F}_k$. Note that the relative dimension of $\pi_{k+1,k}$ is equal to that of $\pi_{k-1,k}$ that is n. Therefore

$$\dim \mathcal{X}_k = (k+1)n + 1.$$

Now, given a section $s \ : \ C \to \mathcal{X}$ of the family $\pi \ : \ \mathcal{X} \to C$, assuming that we have constructed the lifts $s_i \ : \ C \to \mathcal{X}_i$ up to the level k, we get the $(k+1)^{th}$-order jet $s_{k+1} \ : \ C \to \mathcal{X}_{k+1}$ by considering the surjective map ${}^t ds_k \ : \ s_k^\star \mathcal{F}_k \to \Omega_C$ built from the relation ${}^t ds_k \circ s_k^{\star t} d\pi_{k-1,k} = {}^t ds_{k-1}$. Recall that the tautological quotient bundle $\mathcal{O}_{\mathcal{X}_{k+1}}(1)$ pulls-back to C via s_{k+1} into the considered quotient Ω_C :

$$s_{k+1}^\star \mathcal{O}_{\mathcal{X}_{k+1}}(1) = \Omega_C.$$

The map ${}^t d\pi_{k-1,k} \ : \ \mathcal{O}_{\mathcal{X}_k}(1) \to \mathcal{F}_k$ gives rise to a divisor $\mathcal{D}_{k+1} = \mathbb{P}(\Omega_{\mathcal{X}_k/\mathcal{X}_{k-1}})$ in $\mathcal{X}_{k+1} = \mathbb{P}(\mathcal{F}_k)$ in the linear system $|\pi_{k+1,k}^\star \mathcal{O}_{\mathcal{X}_k}(-1) \otimes \mathcal{O}_{\mathcal{X}_{k+1}}(1)|$ that the curve s_{k+1} avoids.

2.4. **Description in coordinates.** Choose a local coordinate t on C and an adapted system of local coordinates $(t, z_1, z_2, \cdots, z_n)$ on $\mathcal{X}$ at a regular point of the map π such that the map π is given by $(t, z_1, z_2, \cdots, z_n) \mapsto t$. The set of vectors $\frac{\partial}{\partial t}, \frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}, \cdots, \frac{\partial}{\partial z_n}$ provides us with a local frame for $T_{\mathcal{X}}$. This defines relative homogeneous coordinates $[T_1 : A_1 : A_2 : \cdots : A_n]$ on $\mathcal{X}_1$.

A section s of π locally written as $t \mapsto (t, z_1(t), z_2(t), \cdots, z_n(t))$ is differentiated in

$$ds \ : \ \frac{\partial}{\partial t} \mapsto \frac{\partial}{\partial t} + z_1'(t)\frac{\partial}{\partial z_1} + z_2'(t)\frac{\partial}{\partial z_2} + \cdots + z_n'(t)\frac{\partial}{\partial z_n}.$$

The first order jet of the curve s is therefore locally written as $s_1 \ : \ C \to \mathcal{X}_1 = P(T_{\mathcal{X}})$,

$$s_1 \ : \ t \mapsto (t, z_1(t), z_2(t), \cdots, z_n(t), [1 : z_1'(t) : z_2'(t) : \cdots : z_n'(t)]).$$

It does not meet the divisor $\mathcal{D}_1 := P(T_{\mathcal{X}/C})$ locally given by $T_1 = 0$.

Outside this divisor, we get relative affine coordinates $a_1 := A_1/T_1, a_2 := A_2/T_1, \cdots, a_n := A_n/T_1$. Note that for the section s_1 we infer that $a_j(t) = z_j'(t)$. The set of vectors

$$\frac{\partial}{\partial t}, \frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}, \cdots, \frac{\partial}{\partial z_n}, \frac{\partial}{\partial a_1}, \frac{\partial}{\partial a_2}, \cdots, \frac{\partial}{\partial a_n}$$

provides us with a local frame for $T_{\mathcal{X}_1}$. The bundle $\mathcal{F}_1^\star$ is defined to be

$$\mathcal{F}_1^\star := \{(t, z, [A], v) \in T_{\mathcal{X}_1} \,/\, d\pi_{0,1}(v) \in [A] \subset T_{\mathcal{X}}\}.$$

It has a local frame built with $\frac{\partial}{\partial t} + a_1\frac{\partial}{\partial z_1} + a_2\frac{\partial}{\partial z_2} + \cdots + a_n\frac{\partial}{\partial z_n} \in \mathcal{O}_{\mathcal{X}_1}(-1)$ and $\frac{\partial}{\partial a_i} \in T_{\mathcal{X}_1/\mathcal{X}},\ 1 \leq i \leq n$. This defines relative homogeneous coordinates $[T_2 : B_1 : B_2 : \cdots : B_n]$ on $\mathcal{X}_2$.

The section $s_1 : C \to \mathcal{X}_1 - \mathcal{D}_1$ of π_1 is differentiated in

$$\begin{aligned} ds_1 \,:\, \frac{\partial}{\partial t} \mapsto & \\ & \tfrac{\partial}{\partial t} + z_1'(t)\tfrac{\partial}{\partial z_1} + z_2'(t)\tfrac{\partial}{\partial z_2} + \cdots + z_n'(t)\tfrac{\partial}{\partial z_n} + z_1''(t)\tfrac{\partial}{\partial a_1} + z_2''(t)\tfrac{\partial}{\partial a_2} + \cdots + z_n''(t)\tfrac{\partial}{\partial a_n} \\ = & \left(\tfrac{\partial}{\partial t} + a_1(t)\tfrac{\partial}{\partial z_1} + a_2(t)\tfrac{\partial}{\partial z_2} + \cdots + a_n(t)\tfrac{\partial}{\partial z_n}\right) + z_1''(t)\tfrac{\partial}{\partial a_1} + z_2''(t)\tfrac{\partial}{\partial a_2} + \cdots + z_n''(t)\tfrac{\partial}{\partial a_n}. \end{aligned}$$

The second order jet $s_2 : C \to \mathcal{X}_2$ is locally written as

$$s_2 : t \mapsto (t, z_1(t), z_2(t), \cdots, z_n(t), [1 : z_1'(t) : z_2'(t) : \cdots : z_n'(t)], [1 : z_1''(t) : z_2''(t) : \cdots : z_n''(t)]).$$

It does not meet the divisor $\mathcal{D}_2 := P(T_{\mathcal{X}_2/\mathcal{X}_1})$ locally given by $T_2 = 0$.

Coordinates in higher order jet spaces are defined similarly, over the regular points of the family π.

3. Vanishing criterion and algebraic Morse inequalities

This section is devoted to describe the tools needed to prove theorem 1 on the existence of differential equations fulfilled by the sections of the given family $\mathcal{X} \to C$ or its pull-back $\rho^\star\mathcal{X} \to C_\rho$ by a finite morphism $\rho : C_\rho \to C$.

3.1. **Vanishing criterion.** Consider the line bundles on the k-th order jet space $\mathcal{X}_k$ defined by

$$\mathcal{O}_{\mathcal{X}_k}(\underline{m}) := \pi_{1,k}^\star\mathcal{O}_{\mathcal{X}_1}(m_1) \otimes \pi_{2,k}^\star\mathcal{O}_{\mathcal{X}_2}(m_2) \otimes \cdots \otimes \mathcal{O}_{\mathcal{X}_k}(m_k)$$

and

$$\mathcal{O}_{\mathcal{X}_k}(\underline{M\mathcal{D}}) := \pi_{1,k}^\star\mathcal{O}_{\mathcal{X}_1}(M_1\mathcal{D}_1) \otimes \pi_{2,k}^\star\mathcal{O}_{\mathcal{X}_2}(M_2\mathcal{D}_2) \otimes \cdots \otimes \mathcal{O}_{\mathcal{X}_k}(M_k\mathcal{D}_k).$$

Define

$$\begin{aligned} \chi_\rho &:= \int_{C_\rho} \mathrm{seg}_1(\Omega_{C_\rho}) = -\int_{C_\rho} c_1(T_{C_\rho}) = -2\int_{C_\rho} Todd(T_{C_\rho}) = -2\chi(C_\rho) \\ &= 2g(C_\rho) - 2 \geq (\deg\rho)(2g(C) - 2) \geq 0. \end{aligned}$$

Take a line bundle μ on the base curve C. Consider a section σ of the line bundle $\mathcal{O}_{\mathcal{X}_k}(\underline{m}) \otimes \mathcal{O}_{\mathcal{X}_k}(\underline{M\mathcal{D}}) \otimes \pi_k^\star\mu^{-1}$. Pull it back to $\mathcal{X}_{\rho,k}$ first and then to C_ρ via the k-th order map $s_k : C_\rho \to \mathcal{X}_{\rho,k}$ of a section s of the pulled-back family $\rho^\star\mathcal{X} \to C_\rho$. It gives a section $s_k^\star\sigma$ of the line bundle $\Omega_{C_\rho}^{\otimes|\underline{m}|} \otimes \rho^\star\mu^{-1}$. If the latter bundle has an ample dual bundle (i.e. if $\deg\rho\deg\mu >| \underline{m} \mid \chi_\rho$), then the section $s_k^\star\sigma$ has to vanish. This gives

Lemma 3.1 (The vanishing criterion)**.** *If a line bundle μ on the base curve C has degree*

$$\deg\mu >| \underline{m} \mid \frac{\chi_\rho}{\deg\rho}$$

then for every section σ of (a multiple of) the line bundle $\mathcal{O}_{\mathcal{X}_k}(\underline{m})\otimes\mathcal{O}_{\mathcal{X}_k}(\underline{M\mathcal{D}})\otimes\pi_k^\star\mu^{-1}$ on $\mathcal{X}_k$ and every section s of the pulled-back family $\rho^\star\mathcal{X}\to C_\rho$, the k^{th}-order jet s_k of s lies in the zero locus of $\rho^\star\sigma$

$$s_k(C_\rho)\subset Zero(\rho^\star\sigma)\subset\mathcal{X}_{\rho,k}.$$

Note that we have considered only those bundles having zero components along the Picard group of $\mathcal{X}/C$. For example, in the case of a family of hypersurfaces of $\mathbb{P}^{n+1}$, bounding the intersection number $s(C)\cdot\mathcal{O}_{\mathbb{P}^{n+1}}(1)$, called the height of the section s, is a main step in proving Lang's conjecture. We could alternatively allow a negative part along the Picard group of $\mathcal{X}/C$. This will give the height estimates in section 5.

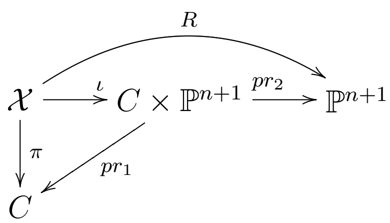
3.2. **Algebraic Morse inequalities.** We therefore have to try and produce sections of bundles with negative components along the Picard group of $\mathcal{X}/C$. We will use the algebraic form of holomorphic Morse inequalities [Demailly-85] [Trapani-95] to achieve this.

Proposition 3.2 (Algebraic Morse inequalities)**.** *Take a line bundle L on a projective manifold of dimension D that can be written as the difference of two nef line bundles $L=A-B$ where furthermore the intersection number $A^D-DA^{D-1}\cdot B$ is positive. Then, L is big.*

There are three elements to settle to get the proof of theorem 1, the construction of nef line bundles A and B on a jet spaces $\mathcal{X}_k$, $L=A-B$ having negative component on $Pic(\mathcal{X}_k/C)$ and being of weight $\underline{m}$, secondly the inequality $\deg\mu>\chi_\rho\mid\underline{m}\mid/\deg\rho$ for the negative part μ^{-1} of L coming from the base curve C needed to apply the vanishing criterion and finally the positivity of the intersection number $A^{\dim\mathcal{X}_k}-\dim\mathcal{X}_k A^{\dim\mathcal{X}_k-1}\cdot B$.

4. THE NEF CONES

We will from now on restrict to the situation of a family of hypersurfaces in $\mathbb{P}^{n+1}$ given by a section F_0 of an ample line bundle L_0 on $C\times\mathbb{P}^{n+1}$. We will assume that the genus of C and the relative dimension n are at least 2.



This gives the further sequence on $\mathcal{X}$

$$0 \to L_{0|\mathcal{X}}^{\star} \xrightarrow{{}^t dF_0} \Omega_C \boxplus \Omega_{\mathbb{P}^{n+1}|\mathcal{X}} \xrightarrow{{}^t d\iota} \Omega_{\mathcal{X}} = \mathcal{F}_0 \to 0. \tag{4.1}$$

From Leray-Hirsch theorem, we know that $Pic(C \times \mathbb{P}^{n+1}) = pr_1^\star\, Pic\, C \oplus pr_2^\star\, Pic\, \mathbb{P}^{n+1}$. In particular, we will write L_0 as $\lambda_0 \boxtimes \mathcal{O}_{\mathbb{P}^{n+1}}(d_0) = pr_1^\star \lambda_0 \otimes pr_2^\star \mathcal{O}_{\mathbb{P}^{n+1}}(d_0)$. Note that $\mathcal{O}_{\mathbb{P}^{n+1}}(d_0) = (L_0)_{|pr_1^{-1}b}$ is ample ($d_0 > 0$) and $(pr_1)_\star L_0 = \lambda_0 \otimes S^{d_0}\mathbb{C}^{n+2}$ is effective ($\deg \lambda_0 \geq 0$).

4.1. **The nef cone of $\mathcal{X}$.** For the line bundle L_0 is assumed to be ample and $\mathcal{X}$ is of dimension at least 3, Lefschetz hyperplane theorem reads

$$Pic\, \mathcal{X} = \iota^\star\, Pic(C \times \mathbb{P}^{n+1}) = \pi^\star\, Pic\, C \oplus R^\star\, Pic\, \mathbb{P}^{n+1}.$$

For a line bundle λ on C and an integer d, we will denote by $\mathcal{O}_{\mathcal{X}}(\lambda, d) = \pi^\star \lambda \otimes R^\star \mathcal{O}_{\mathbb{P}^{n+1}}(d)$ the restriction to $\mathcal{X}$ of the line bundle $\lambda \boxtimes \mathcal{O}_{\mathbb{P}^{n+1}}(d)$.

The line bundle $\pi^\star \mathcal{O}_B(1)$, nef but not ample, has its Chern class lying on a vertex of the nef cone of $\mathcal{X}$. If the morphism $R : \mathcal{X} \to \mathbb{P}^{n+1}$ is not finite (e.g. the section defining $\mathcal{X}$ does not involve all the homogeneous coordinates on $\mathbb{P}^{n+1}$) then the line bundle $R^\star \mathcal{O}_{\mathbb{P}^{n+1}}(1)$ gives the second vertex. This is not the general case.

The top intersection number of the first Chern class $c_1(\mathcal{O}_{\mathcal{X}}(\lambda, d)) \in NS(\mathcal{X})$ is given by

$$\begin{aligned} & c_1(\mathcal{O}_{\mathcal{X}}(\lambda, d))^{n+1} \\ &= \iota^\star \left[pr_1^\star c_1(\lambda) + pr_2^\star c_1(\mathcal{O}_{\mathbb{P}^{n+1}}(d))\right]^{n+1} \\ &= \left[c_1(\lambda_0) + c_1(\mathcal{O}_{\mathbb{P}^{n+1}}(d_0))\right].\left[c_1(\mathcal{O}_{\mathbb{P}^{n+1}}(d))^{n+1} + (n+1)c_1(\lambda) pr_2^\star c_1(\mathcal{O}_{\mathbb{P}^{n+1}}(d))^n\right] \\ &= d^n \left[d \deg(\lambda_0) + (n+1) d_0 \deg(\lambda)\right]. \end{aligned}$$

It has to be non-negative on the nef cone. We hence get in $NS_{\mathbb{R}}(\mathcal{X}) \equiv \mathbb{R}^2$

$$\begin{aligned} \left\{(l,d)/d \geq 0, \quad l \geq 0\right\} &= \iota^\star Nef(B \times \mathbb{P}^{n+1}) \\ &\subset Nef(\mathcal{X}) \subset \left\{(l,d)/d \geq 0, \quad l \geq -\left(\frac{\deg \lambda_0}{n+1}\right)\frac{d}{d_0}\right\}. \end{aligned}$$

4.2. **The pseudo-effective cone of $\mathcal{X}$.** We now compute the pseudo-effective cone, $Psef(\mathcal{X}) \supset Nef(\mathcal{X})$. Take $\deg \lambda < 0$ and $d > 0$. The push-forward by pr_1 of the sequence defining the structure sheaf of $\mathcal{X}$ tensorised by $\lambda \boxtimes \mathcal{O}_{\mathbb{P}^{n+1}}(d)$, reads

$$0 \to (pr_1)_\star \left(\lambda \otimes \lambda_0^\star \boxtimes \mathcal{O}_{\mathbb{P}^{n+1}}(d - d_0)\right) \to (pr_1)_\star(\lambda \boxtimes \mathcal{O}_{\mathbb{P}^{n+1}}(d)) \to \pi_\star \mathcal{O}_{\mathcal{X}}(\lambda, d) \to 0$$

that is

$$0 \to \lambda \otimes \lambda_0^\star \otimes S^{d-d_0}\mathbb{C}^{n+2} \to \lambda \otimes S^d \mathbb{C}^{n+2} \to \pi_\star \mathcal{O}_{\mathcal{X}}(\lambda, d) \to 0.$$

For $\deg \lambda < 0$, the associated long exact sequence gives

$$\begin{aligned} & 0 \to H^0(\mathcal{X}, \mathcal{O}_{\mathcal{X}}(\lambda, d)) \to \\ H^1(C, \lambda \otimes \lambda_0^\star) \otimes S^{d-d_0}\mathbb{C}^{n+2} \to H^1(C, \lambda) \otimes S^d \mathbb{C}^{n+2} & \to H^1(C, \pi_\star \mathcal{O}_{\mathcal{X}}(\lambda, d)) \to 0. \end{aligned}$$

Note that if ℓ is large, $\mathcal{R}^1\pi_\star\mathcal{O}_{\mathcal{X}}(\lambda^{\otimes\ell},\ell d)$ vanishes, so that $H^1(C,\pi_\star\mathcal{O}_{\mathcal{X}}(\lambda^{\otimes\ell},\ell d))$ and $H^1(\mathcal{X},\mathcal{O}_{\mathcal{X}}(\lambda^{\otimes\ell},\ell d))$ become isomorphic. We infer

$$\begin{aligned} h^0(\mathcal{X},\mathcal{O}_{\mathcal{X}}(\lambda,d)) \geq & h^1(C,\lambda\otimes\lambda_0^\star)\otimes S^{d-d_0}\mathbb{C}^{n+2} - h^1(C,\lambda)\otimes S^d\mathbb{C}^{n+2} \\ \geq & -\chi(C,\lambda\otimes\lambda_0^\star)\otimes S^{d-d_0}\mathbb{C}^{n+2} + \chi(C,\lambda)\otimes S^d\mathbb{C}^{n+2} \\ \geq & [\deg\lambda + 1 - g(C)]\binom{d+n+1}{n+1} \\ & - [\deg\lambda - \deg\lambda_0 + 1 - g(C)]\binom{d-d_0+n+1}{n+1}. \end{aligned}$$

We find that if $\deg\lambda > -\left(\frac{\deg\lambda_0}{n+1}\right)\frac{d}{d_0}$, for large ℓ, $H^0(\mathcal{X},\mathcal{O}_{\mathcal{X}}(\lambda^{\otimes\ell},\ell d)) \neq 0$. Hence

$$\Big\{d\geq 0, \quad l\geq 0\Big\} \subset Nef(\mathcal{X}) \subset \Big\{d\geq 0, \quad l \geq -\left(\frac{\deg\lambda_0}{n+1}\right)\frac{d}{d_0}\Big\} \subset Psef(\mathcal{X}).$$

4.3. **The cones in the very general case.** The ideas described here are due to Claire Voisin. The key result is the following

Lemma 4.1. *Let $\mathcal{Y}\subset T\times P\to T$ be a family of complex algebraic ample hypersurfaces of dimension at least 3 of a projective manifold P. Assume that the fibre Y_0 over 0 in T is irreducible and that the nef cone and the pseudo-effective cone of its normalisation coincide. Then, the nef cone and the pseudo-effective cone of a very general fibre of $\mathcal{Y}\to T$ do also coincide.*

"Very general" refers to the family being chosen outside a countable union of proper algebraic subsets of the parameter space.

Proof. The Picard group of any general member Y_t is induced by that of P by Lefschetz theorem. Take a numerical class c in $NS(P)$ and a line bundle $\mathcal{L}$ on P in the class c. Using semi-continuity theorems for the universal bundle over $Pic^0(\mathcal{Y}/T)$ twisted by $\mathcal{L}$ and the properness and flatness of the relative Picard scheme $Pic^0(\mathcal{Y}/T)\to T$ over the smooth locus of $\mathcal{Y}\to T$, we see that the locus Z_c of T where the line bundle $\mathcal{L}_{|Y_t}$ is algebraically equivalent to an effective line bundle is Zariski closed. Define Z to be the non-smooth locus of $\mathcal{Y}\to T$ together with the countable union of all those Z_c that are strict in T. Removing the countable union Z' of images in T of components of the Hilbert scheme of vertical curves in $\mathcal{Y}$ that do not dominate T, we can ensure that every curve C in Y_t for $t\in T-Z'$ deforms locally around t, and by properness of components of the Hilbert schemes, specialises to a curve C_0 at 0.

Take a $\tau\in T-Z-Z'$. Take a line bundle $\mathcal{L}\in Pic(P)$ whose restriction to Y_τ is effective and a curve C in Y_τ. We have to check that the degree $\deg\mathcal{L}_{|C}$ is non-negative. The line bundle $\mathcal{L}_{|Y_t}$ is algebraically equivalent to an effective line bundle on the whole of $T-Z-Z'$ and therefore $\mathcal{L}_{|Y_0}$ pulls back to a nef line bundle on the normalisation of Y_0, by hypothesis. Here we use the irreducibility of Y_0 to make sure that the gotten section do not identically vanish on some irreducible component of Y_0. For $\deg \nu^\star \mathcal{L}_{|\nu^{-1}C_0} \geq 0$, we infer using intersection theory for line bundles on the singular fibre Y_0 and especially the projection formula, that the integer $\deg \mathcal{L}_{|C}$ is non-negative. □

In our setting, making an assumption on the shape of the defining line bundle L_0, this leads to the

Proposition 4.2. *Take a line bundle $L_0 = \lambda_0^{n+1} \boxtimes \mathcal{O}_{\mathbb{P}^{n+1}}(d_0)$ on $C \times \mathbb{P}^{n+1}$, where λ_0 is a line bundle on C having a pencil of sections and d_0 any positive integer. If $\mathcal{X}$ is general in the linear system $|L_0|$, then*

$$Nef(\mathcal{X}) = Psef(\mathcal{X}) = \Big\{(l,d)/ \quad d \geq 0, \quad l \geq -\deg \lambda_0 \frac{d}{d_0}\Big\}.$$

Proof. Take a rational function $f\ : C \to \mathbb{P}^1$ given by a pencil of sections of λ_0 and a generic hypersurface X of $\mathbb{P}^{n+1}$ defined by a polynomial F of degree d_0. Construct the finite map gotten from Segre embedding and a generic projection

$$\phi\ :\ C \times X \to \mathbb{P}^1 \times \mathbb{P}^{n+1} \to \mathbb{P}^{2n+1} \to \mathbb{P}^{n+1}.$$

and consider the map $\Phi = (Id_C, \phi)\ :\ C \times X \to C \times \mathbb{P}^{n+1}$. Denote its image by $\mathcal{X}_0$. The map $\mathbb{P}^1 \times \mathbb{P}^{n+1} \to \mathbb{P}^{2n+1} \to \mathbb{P}^{n+2}$ is explicitly given in terms of coordinates by

$$\begin{aligned}([X_0 : X_1], [Y_0 : Y_1 : \cdots : Y_{n+1}]) \mapsto [2X_1Y_0 : X_0Y_0 - X_1Y_1 : X_0Y_1 - X_1Y_2 : \cdots \\ \cdots : X_0Y_n - X_1Y_{n+1} : 2X_0Y_{n+1}].\end{aligned}$$

If $F(1,0,0,0) \neq 0$ and F is generic, we can project to get a finite map

$$\begin{aligned}\mathbb{P}^1 \times X \to & \mathbb{P}^{n+1} \\ ([X_0 : X_1], [Y_0 : Y_1 : \cdots : Y_{n+1}]) \mapsto & [X_0Y_0 - X_1Y_1 : X_0Y_1 - X_1Y_2 : \cdots \\ & \cdots : X_0Y_n - X_1Y_{n+1} : 2X_0Y_{n+1}].\end{aligned}$$

The equation of the image $\mathcal{X}_0$ of Φ

$$F(X_0^{n+1}U_0 + X_0^nX_1U_1 + X_0^{n-1}X_1^2U_2 + \cdots + X_1^{n+1}\frac{U_{n+1}}{2} : \cdots$$
$$\cdots : X_0^{n+1}U_{n-1} + X_0^nX_1U_n + X_0^{n-1}X_1^2\frac{U_{n+1}}{2} : X_0^{n+1}U_n + X_0^nX_1\frac{U_{n+1}}{2} : X_0^{n+1}\frac{U_{n+1}}{2}) = 0$$

is in the linear system $|\ \lambda_0^{(n+1)d_0} \boxtimes \mathcal{O}_{\mathbb{P}^{n+1}}(d_0)\ |$. Note that $n+1$ is the degree of the image of $\mathbb{P}^1 \times \mathbb{P}^n$, considered as a divisor in $\mathbb{P}^1 \times \mathbb{P}^{n+1}$, by the Segre map to $\mathbb{P}^{2n+1}$.

The nef cone of $C \times X$, the normalisation of its image $\mathcal{X}_0$, is equal to its effective cone. By lemma 4.1, we infer that the same holds true for very general deformations of the image $\mathcal{X}_0$.

We can now apply this to get more linear systems than just those of type $|\ \lambda_0^{(n+1)d_0} \boxtimes \mathcal{O}_{\mathbb{P}^{n+1}}(d_0)\ |$. Take $\mathcal{X}$ to be a general hypersurface in the linear system $|\ \lambda_0^{(n+1)} \boxtimes \mathcal{O}_{\mathbb{P}^{n+1}}(1)\ |$ whose nef cone and the pseudo-effective cone coincide. Consider the Frobenius like finite morphism $\psi\ :\ \mathbb{P}^{n+1} \to \mathbb{P}^{n+1}$ gotten by raising homogeneous coordinates to the power δ_0, and the hypersurface $\mathcal{X}' := (Id_{|B}, \psi)^{-1}(\mathcal{X})$. It is a smooth ample hypersurface of $C \times \mathbb{P}^{n+1}$ in the linear system $\mid \lambda_0^{(n+1)} \boxtimes \mathcal{O}_{\mathbb{P}^{n+1}}(\delta_0) \mid$. By Lefschetz theorem, its $\mathbb{Q}$-Neron Severi group coincide with that of $C \times \mathbb{P}^{n+1}$. Take a curve C' and an effective divisor D' in $\mathcal{X}'$. Its multiple $\delta_0 D'$ pulls back from an effective divisor D in $\mathcal{X}$, which is nef by hypothesis.

$$\delta_0 D' \cdot C' = \psi^{-1}(D) \cdot C' = \delta_0^{n+1} D.\psi(C') \geq 0.$$

The hypersurface $\mathcal{X}'$ may be not general, but applying the lemma again, we infer that the nef cone and the pseudo-effective cone of a very general hypersurface in the linear system $\mid \lambda_0^{(n+1)} \boxtimes \mathcal{O}_{\mathbb{P}^{n+1}}(\delta_0) \mid$ coincide. □

4.4. **A nef line bundle on $\mathcal{X}_1$.** By Leray-Hirsch theorem, the Picard group of $\mathcal{X}_1 = \mathbb{P}(\mathcal{F}_0)$ is the group

$$Pic\,\mathcal{X}_1 = Pic\,\mathcal{X} \oplus \mathbb{Z}\mathcal{O}_{\mathcal{X}_1}(1) = Pic\,C \oplus Pic\,\mathbb{P}^{n+1} \oplus \mathbb{Z}\mathcal{O}_{\mathcal{X}_1}(1).$$

Accordingly, we will use the notation $\mathcal{O}_{\mathcal{X}_1}(\lambda, d; m_1)$. The bundle $\Omega_{\mathbb{P}^{n+1}} = \Lambda^n T_{\mathbb{P}^{n+1}} \otimes K_{\mathbb{P}^{n+1}}$ is a quotient of $(\Lambda^n \mathcal{O}_{\mathbb{P}^{n+1}}(1)^{\oplus n+1}) \otimes K_{\mathbb{P}^{n+1}} = \left(\Lambda^n \mathcal{O}_{\mathbb{P}^{n+1}}^{\oplus n+1}\right) \otimes \mathcal{O}_{\mathbb{P}^{n+1}}(-2)$. Hence, for Ω_C is globally generated, the quotient (see 4.1) $\mathcal{F}_0 \otimes \mathcal{O}_{\mathbb{P}^{n+1}}(2)$ and therefore the bundle $\mathcal{L}_1 := \mathcal{O}_{\mathcal{X}_1}(0, 2; 1)$ also are.

4.5. **A nef line bundle on $\mathcal{X}_{k+1}$.** Generally, the bundle $\Omega_{\mathcal{X}_k/\mathcal{X}_{k-1}} = \Lambda^{n-1} T_{\mathcal{X}_k/\mathcal{X}_{k-1}} \otimes K_{\mathcal{X}_k/\mathcal{X}_{k-1}}$ is a quotient of

$$\Lambda^{n-1}\left(\pi_{k-1,k}^{\star}\mathcal{F}_{k-1}^{\star} \otimes \mathcal{O}_{\mathcal{X}_k}(1)\right) \otimes \mathcal{O}_{\mathcal{X}_k}(-n-1) \otimes \pi_{k-1,k}^{\star} \det \mathcal{F}_{k-1} = \pi_{k-1,k}^{\star}\mathcal{F}_{k-1} \otimes \mathcal{O}_{\mathcal{X}_k}(-2).$$

Assuming that $\mathcal{O}_{\mathcal{X}_{k-1}}(\underline{m_{k-1}})$ and $\mathcal{O}_{\mathcal{X}_k}(\underline{m_{k-1}}, 1)$ are nef, we infer from the defining sequence of $\mathcal{F}_k$

$$\begin{aligned} 0 \to \mathcal{O}_{\mathcal{X}_k}(3\underline{m_{k-1}}, 3) \;\; \to \;\; & \mathcal{F}_k \otimes \mathcal{O}_{\mathcal{X}_k}(2) \otimes \pi_{k-1,k}^{\star}\mathcal{O}_{\mathcal{X}_{k-1}}(3\underline{m_{k-1}}) \\ & \to \Omega_{\mathcal{X}_k/\mathcal{X}_{k-1}} \otimes \mathcal{O}_{\mathcal{X}_k}(2) \otimes \pi_{k-1,k}^{\star}\mathcal{O}_{\mathcal{X}_{k-1}}(3\underline{m_{k-1}}) \to 0 \end{aligned}$$

setting $\underline{m_k} := (3\underline{m_{k-1}}, 2) = 2(\underline{m_{k-1}}, 1) + (\underline{m_{k-1}}, 0)$ that $\mathcal{O}_{\mathcal{X}_k}(\underline{m_k})$ and $\mathcal{O}_{\mathcal{X}_{k+1}}(\underline{m_k}, 1)$ are nef. We find that the line bundle

$$\mathcal{L}_k := \mathcal{O}_{\mathcal{X}_k}(0, 2 \cdot 3^{k-1}; 2 \cdot 3^{k-2}, \cdots, 2 \cdot 3^2, 2 \cdot 3, 2, 1)$$

is nef of total degree 3^k.

5. Construction of differential equations

Recall the setting of a family of hypersurfaces of $\mathbb{P}^{n+1}$ cut out in $C \times \mathbb{P}^{n+1}$ by a section of the line bundle L.

$$\begin{array}{ccccc} & & \overset{R}{\frown} & & \\ \mathcal{X} & \xrightarrow{\iota} & C \times \mathbb{P}^{n+1} & \xrightarrow{pr_2} & \mathbb{P}^{n+1} \\ \downarrow{\scriptstyle \pi} & \swarrow{\scriptstyle pr_1} & & & \\ C & & & & \end{array}$$

5.1. **Definitions of Segre classes.** Recall that the total Segre class $\operatorname{seg}(E)$ of a complex vector bundle $E \to X$ of rank e is defined in the following way : its component $\operatorname{seg}_i(E)$ of degree $2i$ is computed as $p_\star c_1(\mathcal{O}_E(1))^{e-1+i}$, where $p : \mathbb{P}(E) \to X$ is the variety of rank one quotients of E. From this construction, one deduces that for a line bundle $L \to X$,

$$\operatorname{seg}_i(E \otimes L) = \sum_{j=0}^{i} \binom{e-1+i}{i-j} \operatorname{seg}_j(E) c_1(L)^{i-j}.$$

From Grothendieck defining relation for Chern classes

$$c_e(p^\star E^\star \otimes \mathcal{O}_E(1)) = \sum_{i=0}^{e} p^\star c_i(E^\star) c_1(\mathcal{O}_E(1))^{e-i} = 0$$

one infers that the total Segre class $\operatorname{seg}(E)$ is the formal inverse $c(E^\star)^{-1}$ of the total Chern class of the dual bundle $E^\star$. It is therefore multiplicative in short exact sequences.

5.2. **Computations on $\mathcal{X}$.** Write on $C \times \mathbb{P}^{n+1}$,

$$c_1(L) = d\, pr_2^\star c_1(\mathcal{O}_{\mathbb{P}^{n+1}}(1)) + r\, pr_1^\star c_1(\mathcal{O}_C(1)).$$

We have the relations $c_1(\mathcal{O}_{\mathbb{P}^{n+1}}(1))^{n+2} = 0$, $c_1(\mathcal{O}_C(1))^2 = 0$, $c_1(\mathcal{O}_{\mathbb{P}^{n+1}}(1))^{n+1} \cdot c_1(\mathcal{O}_C(1)) = 1$. Set on $\mathcal{X}$,

$$\alpha := R^\star c_1(\mathcal{O}_{\mathbb{P}^{n+1}}(1)) \text{ and } \beta := \pi^\star c_1(\mathcal{O}_B(1)).$$

Using $\alpha = \iota^\star pr_2^\star c_1(\mathcal{O}_{\mathbb{P}^{n+1}}(1))$ and $\beta = \iota^\star pr_1^\star c_1(\mathcal{O}_C(1))$, we find the relations

$$\begin{aligned} \alpha^{n+1} &= c_1(L) \cdot pr_2^\star c_1(\mathcal{O}_{\mathbb{P}^{n+1}}(1))^{n+1} = r, \\ \alpha^n \beta &= c_1(L) \cdot pr_2^\star c_1(\mathcal{O}_{\mathbb{P}^{n+1}}(1))^n \cdot pr_1^\star c_1(\mathcal{O}_C(1)) = d. \end{aligned} \tag{5.1}$$

Hence, r is the degree of the map $R := pr_2 \circ \iota$, and d is the degree of the generic member of the family $X_c \subset \mathbb{P}^{n+1}$. From the relation (4.1) and the Euler sequence on $\mathbb{P}^{n+1}$, we infer that the total Segre class of $\mathcal{F}_0 = \Omega_{\mathcal{X}}$ is

$$\begin{aligned} \operatorname{seg}(\mathcal{F}_0) &= \pi^\star \operatorname{seg}(\Omega_B) R^\star \operatorname{seg}(\Omega_{\mathbb{P}^3}) \iota^\star \operatorname{seg}(L_0^\star)^{-1} \\ &= \pi^\star \operatorname{seg}(\Omega_B) R^\star \operatorname{seg}(\mathbb{C}^{n+2} \otimes \mathcal{O}_{\mathbb{P}^{n+1}}(-1)) \iota^\star c(L_0) \\ &= \pi^\star \operatorname{seg}(\Omega_B) R^\star c(\mathcal{O}_{\mathbb{P}^{n+1}}(1))^{-(n+2)} \iota^\star c(L_0) \\ &= (1 + \chi\beta)(1+\alpha)^{-(n+2)}(1 + d\alpha + r\beta). \end{aligned}$$

We find that the Segre classes of $\mathcal{F}_0$ are polynomials in (α, β) with coefficients that are linear in (r, d). In particular,

$$\operatorname{seg}_1(\mathcal{F}_0) = (d - n - 2)\alpha + (r + \chi)\beta.$$

5.3. **A recursion formula.** Recall the defining relation for the bundles $\mathcal{F}_k$ on $\mathcal{X}_k$

$$0 \to \mathcal{O}_{\mathcal{X}_k}(1) \to \mathcal{F}_k \to \Omega_{\mathcal{X}_k/\mathcal{X}_{k-1}} \to 0 \tag{5.2}$$

still valid for $k = 0$, if we set $\mathcal{X}_{-1} = C$, $\mathcal{X}_0 = \mathcal{X}$, $\mathcal{F}_0 = \Omega_{\mathcal{X}}$, $\mathcal{O}_{\mathcal{X}}(1) = \pi^\star\Omega_C$, that is

$$0 \to \pi^\star\Omega_C \to \Omega_{\mathcal{X}} \to \Omega_{\mathcal{X}/C} \to 0.$$

We will also need the relative Euler sequence on $\mathcal{X}_k$

$$0 \to \Omega_{\mathcal{X}_k/\mathcal{X}_{k-1}} \to \pi^\star_{k-1,k}\mathcal{F}_{k-1} \otimes \mathcal{O}_{\mathcal{X}_k}(-1) \to \mathcal{O}_{\mathcal{X}_k} \to 0. \tag{5.3}$$

From the two previous sequences, we can compute the total Segre class of $\mathcal{F}_k$ in terms of that of $\mathcal{F}_{k-1}$ ($k \geq 1$). Set

$$\alpha_k := c_1(\mathcal{O}_{\mathcal{F}_{k-1}}(1)) = c_1(\mathcal{O}_{\mathcal{X}_k}(1)).$$

Remark to begin with, that the first Segre classes are easy to compute. We find

$$\mathrm{seg}_1(\mathcal{F}_k) = \pi^\star_{0,k}\,\mathrm{seg}_1(\mathcal{F}_0) - n\left(\alpha_k + \pi^\star_{k-1,k}\alpha_{k-1} + \cdots + \pi^\star_{1,k}\alpha_1\right). \tag{5.4}$$

In general,

$$\begin{aligned}
\mathrm{seg}(\mathcal{F}_k) &= \mathrm{seg}(\mathcal{O}_{\mathcal{X}_k}(1))\,\mathrm{seg}(\Omega_{\mathcal{X}_k/\mathcal{X}_{k-1}}) = \mathrm{seg}(\mathcal{O}_{\mathcal{X}_k}(1))\,\mathrm{seg}(\pi^\star_{k-1,k}\mathcal{F}_{k-1} \otimes \mathcal{O}_{\mathcal{X}_k}(-1)) \\
&= \sum_{\ell=0}^{(k+1)n+1}\sum_{i=0}^{\ell} \mathrm{seg}_{\ell-i}(\mathcal{O}_{\mathcal{X}_k}(1))\,\mathrm{seg}_i(\Omega_{\mathcal{X}_k/\mathcal{X}_{k-1}}) \\
&= \sum_{\ell=0}^{(k+1)n+1}\sum_{i=0}^{\ell} \mathrm{seg}_{\ell-i}(\mathcal{O}_{\mathcal{X}_k}(1))\,\mathrm{seg}_i(\pi^\star_{k-1,k}\mathcal{F}_{k-1} \otimes \mathcal{O}_{\mathcal{X}_k}(-1)) \\
&= \sum_{\ell=0}^{(k+1)n+1}\sum_{i=0}^{\ell} \alpha_k^{\ell-i}\sum_{j=0}^{i}\binom{n+i}{i-j}\pi^\star_{k-1,k}\,\mathrm{seg}_j(\mathcal{F}_{k-1})(-\alpha_k)^{i-j} \\
&= \sum_{\ell=0}^{(k+1)n+1}\sum_{j=0}^{\ell} \pi^\star_{k-1,k}\,\mathrm{seg}_j(\mathcal{F}_{k-1})\alpha_k^{\ell-j}\sum_{i=j}^{\ell}(-1)^{i-j}\binom{n+i}{i-j} \\
&= \sum_{\ell=0}^{(k+1)n+1}\sum_{j=0}^{\ell}\left[\sum_{i=0}^{\ell-j}(-1)^i\binom{n+j+i}{i}\right]\pi^\star_{k-1,k}\,\mathrm{seg}_j(\mathcal{F}_{k-1})\alpha_k^{\ell-j}.
\end{aligned}$$

Defining the numbers $\mathbb{L}_e^{f+e} := \sum_{i=0}^{f}(-1)^i\binom{e+i}{e}$, we get

$$\mathrm{seg}_\ell(\mathcal{F}_k) = \sum_{a+b=\ell}\mathbb{L}^{n+\ell}_{n+a}\pi^\star_{k-1,k}\,\mathrm{seg}_a(\mathcal{F}_{k-1})\alpha_k^b.$$

5.4. **Estimates for intersection numbers.** The idea comes from the reading of [Diverio-09]. Recall that the line bundle $\mathcal{L}_k := \mathcal{O}_{\mathcal{X}_k}(0, 2\cdot 3^{k-1}; 2\cdot 3^{k-2}, \cdots, 2\cdot 3^2, 2\cdot 3, 2, 1)$ is nef on $\mathcal{X}_k$. Note that its entries do depend neither on the degree d nor on the variation r. Its first Chern class is

$$l_k := \alpha_k + 2\pi^\star_{k-1,k}\alpha_{k-1} + 6\pi^\star_{k-2,k}\alpha_{k-2} + \cdots + 2\cdot 3^{j-1}\pi^\star_{k-j,k}\alpha_{k-j} + \cdots + 2\cdot 3^{k-2}\pi^\star_{1,k}\alpha_1 + 2\cdot 3^{k-1}\pi^\star_{0,k}\alpha.$$

We are in position to prove the

Lemma 5.1. *For $r, d \gg 1$,*

$$\begin{aligned} (\pi_{k-1,k})_\star l_k^{n+1} &\geq \pi^\star_{0,k-1}\, seg_1(\mathcal{F}_0). \\ seg_1(\mathcal{F}_0)^{n+1} &\sim (n+2)rd^{n+1}. \\ l_1^{m_1} l_2^{m_2} \cdots l_s^{m_s} \cdot \alpha &\leq A_{n+1}(r,d) \\ l_1^{m_1} l_2^{m_2} \cdots l_s^{m_s} \cdot \beta &\leq B_{n+1}(r,d). \end{aligned}$$

where A_{n+1} and B_{n+1} are polynomials in (r,d) of degree less or equal to $n+1$.

The output is that the leading numerical term comes from the relative canonical degree.

Proof. (1) Recall from the relation (5.4) that

$$\begin{aligned} (\pi_{k-1,k})_\star l_k^{n+1} &= \mathrm{seg}_1(\mathcal{F}_{k-1}) \\ &\quad + (n+1)(2\alpha_{k-1} + 6\alpha_{k-2} + \cdots + 2\cdot 3^{j-1}\alpha_{k-j} + \cdots + 2\cdot 3^{k-2}\alpha_1 + 2\cdot 3^{k-1}\alpha) \\ &= \pi^\star_{0,k-1}\,\mathrm{seg}_1(\mathcal{F}_0) - n(\alpha_{k-1} + \alpha_{k-2} + \cdots + \alpha_1) \\ &\quad + (n+1)(2\alpha_{k-1} + 6\alpha_{k-2} + \cdots + 2\cdot 3^{j-1}\alpha_{k-j} + \cdots + 2\cdot 3^{k-2}\alpha_1 + 2\cdot 3^{k-1}\alpha) \\ &= \pi^\star_{0,k-1}\,\mathrm{seg}_1(\mathcal{F}_0) + (n+2)\alpha_{k-1} + (5n+6)\alpha_{k-2} + \cdots \\ &\quad + \left((n+1)2\cdot 3^{j-1} - n\right)\alpha_{k-j} + \left((n+1)2\cdot 3^{j} - n\right)\alpha_{k-j-1} + \cdots \\ &\quad + \left((n+1)2\cdot 3^{k-2} - n\right)\alpha_1 + 2\cdot 3^{k-1}\alpha. \end{aligned}$$

The claim follows from the inequalities $(n+1)2\cdot 3^j - n \geq 3\left[(n+1)2\cdot 3^{j-1} - n\right]$ that ensure the nefness of $(\pi_{k-1,k})_\star l_k^{n+1} - \pi^\star_{0,k-1}\,\mathrm{seg}_1(\mathcal{F}_0)$.

(2) Just compute

$$\begin{aligned} \mathrm{seg}_1(\mathcal{F}_0)^{n+1} &= ((d-n-2)\alpha + (r+\chi)\beta)^{n+1} \\ &\sim d^{n+1}\alpha^{n+1} + (n+1)rd^n\alpha^n\beta = (n+2)rd^{n+1}. \end{aligned}$$

(3) It follows from the recursion formula that, computed in $\mathcal{X}$ of dimension $n+1$,

$$l_1^{m_1} l_2^{m_2} \cdots l_s^{m_s} \cdot \alpha = \sum_{k\leq n} C_I s_{i_1}(\mathcal{F}_0) s_{i_2}(\mathcal{F}_0) \cdots s_{i_k}(\mathcal{F}_0) \cdot \alpha.$$

Recall that the Segre classes of $\mathcal{F}_0$ are polynomials in (α, β) whose coefficients are linear in (r,d) and use the relations (5.1) to get

$$l_1^{m_1} l_2^{m_2} \cdots l_s^{m_s} \cdot \alpha = P(r,d)\alpha^{n+1} + Q(r,d)\alpha^n\beta = P(r,d)r + Q(r,d)d$$

where P and Q are polynomials in (r,d) of degree less or equal to n.

(4) In the previous computations, the class α can be substituted by the class β to get

$$l_1^{m_1} l_2^{m_2} \cdots l_s^{m_s} \cdot \beta = R(r,d)\alpha^n \beta = R(r,d)d$$

where R is a polynomial in (r,d) of degree less or equal to n.

□

5.5. **Choice of line bundles.** Fix $\rho : C_\rho \to C$. We choose to work on the jet space of order $\kappa = n+1$. We define on $\mathcal{X}_{n+1}$ the line bundle, tensor product of the nef line bundles constructed in section 4

$$A = \mathcal{L}_{n+1} \otimes \mathcal{L}_n \otimes \cdots \otimes \mathcal{L}_j \otimes \cdots \otimes \mathcal{L}_1$$

and we choose B so that $L := A - B$ has negative component along $Pic(\mathcal{X}/C)$ and fulfils the vanishing criterion that is, for some fixed positive rational number x, and some positive rational number y with $\frac{\chi_\rho}{\deg \rho} \sum_{j=1}^{n+1} (3^j - 2 \cdot 3^{j-1}) = \frac{\chi_\rho}{\deg \rho} \frac{3^{n+1}-1}{2} < y$.

$$\begin{aligned} B &= \mathcal{O}_{\mathbb{P}^{n+1}}\left(2 \cdot 3^{n+1-1} + \cdots + 2 \cdot 3^{j-1} + \cdots + 2 + x\right) \otimes \mathcal{O}_C(y) \\ &= \mathcal{O}_{\mathbb{P}^{n+1}}(3^{n+1} - 1 + x) \otimes \mathcal{O}_C(y) \end{aligned}$$

where $\mathcal{O}_C(y)$ is any line bundle on the curve C of degree y. The precise knowledge of the nef cone of $\mathcal{X}$ as in proposition 4.2 could improve, from effective point of view, the choice of A and B.

With $\kappa = n+1$, we have $\dim \mathcal{X}_\kappa = \kappa(n+1)$. Hence, for we only omit intersections of nef classes,

$$\begin{aligned} A^{\dim \mathcal{X}_\kappa} &= (l_\kappa + l_{\kappa-1} + \cdots + l_1)^{\dim \mathcal{X}_\kappa} \\ &\geq l_\kappa^{(n+1)} l_{\kappa-1}^{(n+1)} \cdots l_1^{(n+1)} \\ &\geq \pi_{0,\kappa-1}^\star \operatorname{seg}_1(\mathcal{F}_0) \cdot l_{\kappa-1}^{(n+1)} \cdots l_1^{(n+1)} \\ &\geq \pi_{0,\kappa-2}^\star \operatorname{seg}_1(\mathcal{F}_0)^2 \cdot l_{\kappa-2}^{(n+1)} \cdots l_1^{(n+1)} \\ &\vdots \\ &\geq \operatorname{seg}_1(\mathcal{F}_0)^\kappa \sim (n+2)rd^{n+1} \end{aligned}$$

thanks to lemma 5.1. On the other hand, thanks to the same lemma, $A^{\dim \mathcal{X}_k - 1} \cdot B$ is bounded above by a polynomial in (r,d) of degree less or equal to $n+1$. If r and d are chosen large enough, we find that $A^{\dim \mathcal{X}_\kappa} - \dim \mathcal{X}_k A^{\dim \mathcal{X}_k - 1} \cdot B$ is positive. Hence, the line bundle $A - B$ is big and the sections of its powers provide non zero equations for the jets of sections of the family $\mathcal{X}_\rho \to C_\rho$.

We have proved a precise version of theorem 1.

Theorem 2. *Fix $\rho : C_\rho \to C$. Fix a positive rational x and a large enough positive rational y. For $r, d \gg 1$, for a general family π of hypersurfaces of $\mathbb{P}^{n+1}$ of degree d and variation r, there is a non-trivial differential equation of order $n+1$ and total weight $\underline{m}$, given by a section of the line bundle*

$$\mathcal{O}_{\mathcal{X}_{n+1}}(\underline{m}) \otimes \mathcal{O}_{\mathbb{P}^{n+1}}(- \mid \underline{m} \mid x) \otimes \mathcal{O}_C(- \mid \underline{m} \mid y),$$

fulfilled by all sections of the pull-back family π_ρ.

5.6. **Height inequalities.** We now look for a statement that incorporates the dependence in the ramified cover $C_\rho \to C$. We work on $\mathcal{X}_{n+1}$ with $A = \mathcal{L}_{n+1} \otimes \mathcal{L}_n \otimes \cdots \otimes \mathcal{L}_j \otimes \cdots \otimes \mathcal{L}_1$ and we choose B so that $L := A - B$ has negative component on $Pic(\mathcal{X}/B)$, that is $B = \mathcal{O}_{\mathbb{P}^{n+1}}(3^{n+1} - 1 + x)$. The previous computations show that $A - B$ is big for large enough r and d. As a result, we obtain

Corollary 5.2. *Fix a positive integer x. For $r, d \gg 1$, for general family $\mathcal{X}$ of hypersurfaces in $\mathbb{P}^{n+1}$ of degree d and variation r, there exists a proper algebraic set $\mathcal{Y} \subset \mathcal{X}_{n+1}$ such that for every finite ramified cover $\rho : C_\rho \to C$ and every section s of $\rho^\star\mathcal{X} \to C_\rho$ whose $(n+1)$-th order jet does not lie in $\rho^\star\mathcal{Y}$, the following height inequality holds*

$$h(s(C)) := \frac{s(C_\rho) \cdot \mathcal{O}_{\mathbb{P}^{n+1}}(1)}{\deg \rho} \leq \frac{3^{n+1} - 1}{2x} \frac{\chi_\rho}{\deg \rho}.$$

This is an analog of the first part of Vojta's work [Vojta-78]. The deepest part, dealing with sections having $(n+1)$-jet inside $\mathcal{Y}$, would require an analog of Jouanoulou's result on foliations [Jouanolou-78], that seems out of reach now.

6. NON-ZARISKI DENSITY

We follow the ideas of Siu [Siu-04], described in details in [D-M-R-08]. Let d be a positive integer, C be a connected compact complex smooth curve, λ be a holomorphic line bundle on C of degree r. Consider the linear system $\mathbb{P}^N :=\mid \lambda \boxtimes \mathcal{O}_{\mathbb{P}^{n+1}}(d) \mid$ on $C \times \mathbb{P}^{n+1}$, the element F of which represents a family $\pi^F : \mathcal{X}^F \to C$ of degree d hypersurfaces in $\mathbb{P}^{n+1}$ parametrised by C with variation r. Consider the associated universal family

$$\begin{array}{ccc} \mathfrak{X} & \hookrightarrow & \mathbb{P}^N \times C \times \mathbb{P}^{n+1} \\ \Pi \downarrow & \swarrow & \\ \mathbb{P}^N & & \end{array}$$

The variable t will denote a coordinate on the curve C. Coordinates on $\mathbb{P}^{n+1}$ will be denoted with z's, while relative coordinates on $\mathcal{X}^F \to C$ will be denoted with x's. Constant sections are those whose first order jet lies inside $\{z'_1 = z'_2 = \cdots = z'_n = z'_{n+1} = 0\}$. We will denote by $\mathfrak{X}_\kappa$ the Π-relative κ-jets space of sections of the families $(\mathcal{X}^F \to C)_{F \in \mathbb{P}^N}$.

6.1. **Proof using vector fields on universal families.** Choose an integer $\kappa \geq n+1$. Fix positive rational numbers $x > 3^{n+1} - 1 + \kappa^2 + 2\kappa$ and $y > \frac{\chi_\rho}{\deg \rho} \frac{3^{n+1}-1}{2}$. Consider a family π^F. Define $\mu := \mathcal{O}_C(y)$ to be any line bundle of degree y on C. Select the line bundle on $\mathcal{X}^F_\kappa$

$$\mathcal{L}^F_{-\mu,-x,\underline{m}} := \mathcal{O}_{\mathcal{X}^F_\kappa}(\underline{m}) \otimes \mathcal{O}_{\mathbb{P}^{n+1}}(- \mid \underline{m} \mid x) \otimes \mu^{-|\underline{m}|}.$$

We assume $r, d \gg 1$, and $\underline{m}$ be a weight provided by theorem 2 so that $\mathcal{L}^F_{-\mu,-x,\underline{m}}$ is big and effective and fulfils the vanishing criterion. The push-forward $(\pi^F_{\kappa,0})_\star\sigma$ of a non-zero section σ of $\mathcal{L}^F_{-\mu,-x,\underline{m}}$ is a non-zero section of the vector bundle $(\pi^F_{\kappa,0})_\star\mathcal{L}^F_{-\mu,-x,\underline{m}} \to \mathcal{X}^F$. We can now prove the precise version of the main theorem.

Theorem 3. *Consider the line bundle $\mathcal{L}^F_{-\mu,-x,\underline{m}}$ in the previous notations. If r, d are large enough, if the equation F is general in $\mathbb{P}^N$, if $\mid \underline{m} \mid$ is large enough, then there is a non-zero section σ of the line bundle $\mathcal{L}^F_{-\mu,-x,\underline{m}}$ such that every non constant section s of π^F fulfils*

$$s(C) \subset \mathit{Zero}\left((\pi^F_{\kappa,0})_\star\sigma\right) \subset \mathcal{X}^F.$$

The point is that the generality assumption on F enables to transfer the constraints on the jet s_k given by the vanishing criterion into constraints on the section s itself.

Proof of theorem 3. We only sketch the proof, the details being close to that given in [D-M-R-08]. We argue by contradiction and assume that there exists a c_0 in C where $(\pi^F_{\kappa,0})_\star\sigma(s(c_0)) \neq 0$. Take another view point on the section σ and view it as a meromorphic function

$$\begin{array}{ccc} \mathcal{X}^F_\kappa & \to & \mathbb{C} \\ \zeta_\kappa & \mapsto & \sum_{wl(I)=m} q_I(t,x)(x'(\zeta_\kappa))^{i_1}\cdots(x^{(\kappa)}(\zeta_\kappa))^{i_\kappa} \end{array}$$

where the $q_I(t,x)$ are meromorphic functions on $\mathcal{X}^F$, holomorphic when viewed as local sections of $\mathcal{O}_{\mathbb{P}^{n+1}}(- \mid \underline{m} \mid \delta) \otimes \mu^{-|\underline{m}|}$. The assumption $(\pi^F_{\kappa,0})_\star\sigma(s(c_0)) \neq 0$ translates into the existence of a multi-index I_0 of weighted length m such that $q_{I_0}(s_\kappa(c_0)) \neq 0$.

We now use the semi-continuity theorem : for each multi-index $\underline{M}$, the set of parameters G in $\mathbb{P}^N$ such that the line bundle $\mathcal{L}^G_{-\mu,-x,\underline{M}}$ has a section is an algebraic subset of $\mathbb{P}^N$. The countable union of those contains the complement of a proper algebraic subset (parametrising singular total spaces $\mathcal{X}^F$). Hence there is a multi-index $\underline{m}$ valid for general hypersurfaces. The semi-continuity theorem again ensures that the rank of the corresponding direct image is positive. By the generality assumption on F, we may extend the section $(\pi^F_{\kappa,0})_\star\sigma$ to a section $(\Pi_{\kappa,0})_\star\widetilde{\sigma}$ of $(\Pi_{\kappa,0})_\star\mathfrak{L}_{-\mu,-x,\underline{m}} \to \mathfrak{X}$ on a neighbourhood U^F of $\mathcal{X}^F$ in $\mathfrak{X}$.

We now need vector fields to separate the coefficient q_{I_0} by repeated derivations.

Proposition 6.1. *Every vector of*

$$T\left(\mathcal{X}^F_\kappa/\mathcal{X}^F\right)_{(s_\kappa(c_0))} \subset \left(T\mathcal{X}^F_\kappa\right)_{(s_\kappa(c_0))} = T\left(\mathfrak{X}_\kappa/\mathbb{P}^N\right)_{(F,s_\kappa(c_0))} \subset (T\mathfrak{X}_\kappa)_{(F,s_\kappa(c_0))}$$

outside the set $\Pi^{-1}_{\kappa,1}\{z'_1 = z'_2 = \cdots = z'_n = z'_{n+1} = 0\}$ is the value of a meromorphic vector field on $\Pi^{-1}_{\kappa,0}(U^F) \subset \mathfrak{X}_\kappa$ holomorphic when viewed with values in $\Pi^\star_{\kappa,0}\mathcal{O}_{\mathbb{P}^{n+1}}(\kappa^2 + 2\kappa)$.

Take it for granted until the next subsection. When we differentiate the extended meromorphic function $\widetilde{\sigma}$ with the gotten meromorphic vector fields at most $\mid \underline{m} \mid$-times and restrict to the fibre over F, we get meromorphic functions on $\mathcal{X}^F$ that in turn can be viewed as a section of $\mathcal{L}^F_{-\mu,-x+(\kappa^2+2\kappa),\underline{m}}$. Recall that $-x + 3^{n+1} - 1 + (\kappa^2 + 2\kappa)$ is still negative, so that s_κ still has to fulfil this new equation. Having chosen the vector fields in a suitable way, thanks to the proposition 6.1, this contradicts $q_{I_0}(s_\kappa(c_0)) \neq 0$. □

The proof of the main theorem is now ended by the following. If images of constant sections would dominate the total space $\mathcal{X}^F$, because they have bounded height (they are constant in the product $C \times \mathbb{P}^{n+1}$), there would exist an algebraic set S, an Hilbert scheme component of the space of sections, and a dominant map $C \times S \to \mathcal{X}^F$ over C. The criterion of birational splitting of Maehara and Moriwaki [Moriwaki-94], that follows from positivity of direct images of pluricanonical line bundles, would show that the family has to be birationally trivial.

6.2. **Constructing vector fields on universal families.** In homogeneous coordinates, having chosen a basis for $\mathbb{C}^{n+2}$, the corresponding basis $(Z^\alpha)_\alpha$ of monomials for $\mid \mathcal{O}_{\mathbb{P}^{n+1}}(d) \mid$, and a basis $(\Phi_\beta)_\beta$ for $\mid \lambda \mid$, the hypersurface $\mathfrak{X}$ of $\mathbb{P}^N \times C \times \mathbb{P}^{n+1}$ is defined by the equation

$$\sum_{\alpha,\beta} \mathfrak{A}^\beta_\alpha \Phi_\beta Z^\alpha = 0.$$

On the open set $\{\mathfrak{A}^0_{0,d,0,0,\cdots,0} \neq 0\} \times \{\Phi_0(b) \neq 0\} \times \{Z^0 \neq 0\}$ the equation rewrites in inhomogeneous coordinates

$$\mathfrak{F} = z_1^d + \sum_{\substack{\alpha \in \mathbb{N}^{n+1}, |\alpha| \leq d \\ \alpha \neq (d,0,0,\cdots,0) \\ \beta \geq 1}} a^\beta_\alpha \varphi_\beta(b) z^\alpha = 0.$$

Over this open set, the natural open set of the Π-relative κ-jets space $\mathfrak{X}_\kappa$ is given inside $\mathbb{C}^N \times U \times \mathbb{C}^{n+1} \times \underbrace{\mathbb{C}^{n+1} \times \cdots \times \mathbb{C}^{n+1}}_{\kappa \text{ times}}$ in terms of the operator

$$\mathfrak{D} := \frac{\partial}{\partial t} + \sum_{\nu=0}^{\kappa} \sum_{j=1}^{n+1} z_j^{(\nu+1)} \frac{\partial}{\partial z_j^{(\nu)}}$$

by the following set of equations

$$\sum_{\substack{\alpha \in \mathbb{N}^{n+1} \\ |\alpha| \leq d \\ \beta \geq 1}} a^\beta_\alpha \varphi_\beta(t) z^\alpha = \mathfrak{D}\left(\sum a^\beta_\alpha \varphi_\beta(t) z^\alpha\right) = \mathfrak{D}^2\left(\sum a^\beta_\alpha \varphi_\beta(t) z^\alpha\right) = \quad \cdots$$

$$\cdots = \mathfrak{D}^\kappa\left(\sum a^\beta_\alpha \varphi_\beta(t) z^\alpha\right) \quad = 0.$$

Those are the equations one infers from the derivatives of the relation $\sum a^\beta_\alpha \varphi_\beta(t) z^\alpha(t) = 0$ fulfilled by sections $t \mapsto (t, z_1(t), \cdots, z_{n+1}(t))$ of a family $\Pi^{-1}(F)$, after substituting $z_j^{(\nu)} := \frac{\partial^\nu z_j(t)}{\partial t^\nu}$.

Denote the partial sum $\sum_{\alpha \in \mathbb{N}^{n+1}, |\alpha| \leq d} a^\beta_\alpha z^\alpha$ by $\mathfrak{F}_\beta$. The equations for a vector field T of the special shape $T := \left(\sum_\beta T_\beta\right) + T_z$, where $T_\beta := \sum_\alpha A^\beta_\alpha \frac{\partial}{\partial a^\beta_\alpha}$ and $T_z := \sum_{\nu=0}^{\kappa} \sum_{j=1}^{n+1} P^\nu_j \frac{\partial}{\partial z_j^{(\nu)}}$, to be tangent to $\mathfrak{X}_\kappa$ rewrite, thanks to Leibniz formula and the fact that when $\beta \neq \gamma$, $T_\beta \cdot \mathfrak{D}^a \mathfrak{F}_\gamma = 0$, in terms of the operator $D := \sum_{\nu=0}^{\kappa} \sum_{j=1}^{n+1} z_j^{(\nu+1)} \frac{\partial}{\partial z_j^{(\nu)}}$

as

$$
\begin{aligned}
\sum_{\beta\geq 1}\varphi_\beta(t)(T_\beta+T_z)\cdot\mathfrak{F}_\beta &= 0\\
\sum_{\beta\geq 1}\varphi_\beta(t)(T_\beta+T_z)\cdot D\left(\mathcal{F}_\beta\right)+\varphi'_\beta(t)(T_\beta+T_z)\cdot\mathfrak{F}_\beta &= 0\\
\sum_{\beta\geq 1}\varphi_\beta(t)(T_\beta+T_z)\cdot D^2\left(\mathcal{F}_\beta\right)+2\varphi'_\beta(t)(T_\beta+T_z)\cdot D\left(\mathcal{F}_\beta\right)+\varphi''_\beta(t)(T_\beta+T_z)\cdot\mathfrak{F}_\beta &= 0\\
&\vdots\\
\sum_{\beta\geq 1}\sum_{a=0}^{\kappa}\binom{\kappa}{a}\varphi_\beta^{(\kappa-a)}(T_\beta+T_z)\cdot D^a\left(\mathfrak{F}_\beta\right) &= 0.
\end{aligned}
$$

A set of sufficient conditions is therefore

$$\forall\beta,\ \ (T_\beta+T_z)\cdot\mathfrak{F}_\beta=(T_\beta+T_z)\cdot D\left(\mathfrak{F}_\beta\right)=\cdots=(T_\beta+T_z)\cdot D^\kappa\left(\mathfrak{F}_\beta\right)=0$$

reducing to the absolute case. Note however that, for theorem 2 provides us with a differential equation of order $n+1$, we need to consider $(n+1)$-th order jets of hypersurfaces in $\mathbb{P}^{n+1}$, whereas the by now well settled results are for n-th order jets.

6.3. **Constructing vector fields in the absolute case.** We follow the ideas of Siu [Siu-04], Păun [Păun-08], Rousseau [Rousseau-07] and Merker [Merker-09]. For notational simplicity, we will replace β by a dot in the following. The exponents in brackets will be relative to the absolute operator D.

Write $(T_\cdot+T_z)\cdot D^{l+1}(\mathfrak{F}_\cdot)=[T_\cdot+T_z,D]D^l(\mathfrak{F}_\cdot)+D\left((T_\cdot+T_z)\cdot D^l(\mathfrak{F}_\cdot)\right)$ to infer that a set of sufficient conditions for the special vector field $T_\cdot+T_z$ to contribute to a tangent to $\mathfrak{X}_\kappa$ is

$$
\begin{aligned}
(T_\cdot+T_z)\cdot\mathfrak{F}_\cdot=[T_\cdot+T_z,D]\cdot\mathfrak{F}_\cdot=[T_\cdot+T_z,D]\cdot D(\mathfrak{F}_\cdot)=\cdots\\
\cdots=[T_\cdot+T_z,D]\cdot D^{\kappa-1}(\mathfrak{F}_\cdot)=0.
\end{aligned}
$$

We will now further restrict the shape of the chosen vector field to simplify its commutator with D.

Lemma 6.2. *Let $A^\cdot_\alpha$ and P be functions in the $(z_i^{(\nu)})$ variables. The commutator of the very special vector field $T_\cdot+T_z=\sum_\alpha A^\cdot_\alpha\frac{\partial}{\partial a^\cdot_\alpha}+\sum_{\nu=0}^{\kappa}P^{(\nu)}\frac{\partial}{\partial z_j^{(\nu)}}$ with D is*

$$[T_\cdot+T_z,D]=-\sum_\alpha(A^\cdot_\alpha)'\frac{\partial}{\partial a^\cdot_\alpha}-P^{(\kappa+1)}\frac{\partial}{\partial z_j^{(\kappa)}}.$$

Proof. Simply check that

$$T_\cdot(D)=0\qquad D(T_\cdot)=\sum_\alpha(A^\cdot_\alpha)'\frac{\partial}{\partial a^\cdot_\alpha}$$

$$T_z(D)=\sum_{\nu=1}^{\kappa}P^{(\nu)}\frac{\partial}{\partial z_j^{(\nu-1)}}=\sum_{\nu=0}^{\kappa-1}P^{(\nu+1)}\frac{\partial}{\partial z_j^{(\nu)}}\qquad D(T_z)=\sum_{\nu=0}^{\kappa}P^{(\nu+1)}\frac{\partial}{\partial z_j^{(\nu)}}.$$

□

We infer that a set of sufficient conditions for the very special vector field $T_{\cdot} + T_z$ to contribute to a tangent vector field to $\mathfrak{X}_\kappa$ is

$$\begin{aligned} \sum_\alpha A_\alpha^{\cdot} z^\alpha + P \sum_\alpha a_\alpha^{\cdot} \frac{\partial z^\alpha}{\partial z_j} = 0 \\ -\sum_\alpha (A_\alpha^{\cdot})' z^\alpha = -\sum_\alpha (A_\alpha^{\cdot})'(z^\alpha)' = -\sum_\alpha (A_\alpha^{\cdot})'(z^\alpha)^{(2)} = \cdots \\ \cdots = -\sum_\alpha (A_\alpha^{\cdot})'(z^\alpha)^{(\kappa-1)} = 0 \end{aligned}$$

or equivalently, using the formula $D^{l+1}\left(\sum_\alpha A_\alpha z^\alpha\right) = \sum_\alpha A_\alpha (z^\alpha)^{(l+1)} + \sum_\alpha A'_\alpha (z^\alpha)^{(l)} + \sum_{k=0}^{l-1} D^{l-k}\left(\sum_\alpha A'_\alpha (z^\alpha)^{(k)}\right)$,

$$\begin{aligned} \sum_\alpha A_\alpha^{\cdot} z^\alpha &= -P \sum_\alpha a_\alpha^{\cdot} \frac{\partial z^\alpha}{\partial z_j} \\ \sum_\alpha A_\alpha^{\cdot} (z^\alpha)' &= \left(-P \sum_\alpha a_\alpha^{\cdot} \frac{\partial z^\alpha}{\partial z_j}\right)' \\ \sum_\alpha A_\alpha^{\cdot} (z^\alpha)^{(2)} &= \left(-P \sum_\alpha a_\alpha^{\cdot} \frac{\partial z^\alpha}{\partial z_j}\right)^{(2)} \\ \sum_\alpha A_\alpha^{\cdot} (z^\alpha)^{(3)} &= \left(-P \sum_\alpha a_\alpha^{\cdot} \frac{\partial z^\alpha}{\partial z_j}\right)^{(3)} \\ &\vdots \\ \sum_\alpha A_\alpha^{\cdot} (z^\alpha)^{(\kappa)} &= \left(-P \sum_\alpha a_\alpha^{\cdot} \frac{\partial z^\alpha}{\partial z_j}\right)^{(\kappa)} \end{aligned} \tag{6.1}$$

or also

$$\begin{aligned} \sum_\alpha A_\alpha^{\cdot} z^\alpha = -P \sum_\alpha a_\alpha^{\cdot} \frac{\partial z^\alpha}{\partial z_j} \\ \sum_\alpha (A_\alpha^{\cdot})' z^\alpha = \sum_\alpha (A_\alpha^{\cdot})'' z^\alpha = \sum_\alpha (A_\alpha^{\cdot})^{(3)} z^\alpha = \cdots \\ \cdots = \sum_\alpha (A_\alpha^{\cdot})^{(\kappa)} z^\alpha = 0. \end{aligned} \tag{6.2}$$

When P is of degree less than 2, the first equation in the set (6.2) can be fulfilled with constant $A_\alpha^{\cdot}$, making the other equations tautological.

When P is of the form $P = z_i^k$, because the only non-zero term in the right hand side of (6.2) can be written as

$$\sum_{|\beta|\le d} b^{\cdot}_{\beta} z^{\beta} + \sum_{\ell=1}^{k-1} \sum_{|\beta|=d} b^{\cdot\ell}_{\beta} z^{\beta+\ell\epsilon_i},$$

we look for $A^{\cdot}_{\alpha}$ in the form

$$A^{\cdot}_{\alpha} := \sum_{\substack{\gamma,|\gamma|\le\kappa \\ |\alpha+\gamma|\le d}} A^{\cdot\gamma}_{\alpha} z^{\gamma} + \sum_{\ell=1}^{\min(\alpha_i,k-1)} \sum_{\substack{\gamma,|\gamma|\le\kappa \\ |\alpha+\gamma-\ell\epsilon_i|=d}} A^{\cdot\ell,\gamma}_{\alpha} z^{\gamma}.$$

Note that for $\alpha_i \ge \ell$, the multiindex $\alpha+\gamma-\ell\epsilon_i$ is non-negative. Then the set (6.2) rewrites, after recursive simplifications of all terms involving a $z_l^{(k)}$-variable with $k>1$, as a set of systems, one for each multiindex $\mu+\ell\epsilon_i$ where μ is a multiindex of length $\mid \mu \mid\le d$ when $\ell=0$, or $\mid \mu \mid= d$ when $1\le\ell\le k-1$. They have disjoint sets of indeterminates $(A^{\cdot\ell,\gamma}_{\alpha})_{\substack{\alpha+\gamma=\mu+\ell\epsilon_i \\ |\alpha|\le d, |\gamma|\le\kappa}}$. Note that for $\alpha_i\ge\ell$, the equality $\alpha+\gamma=\mu+\ell\epsilon_i$ implies $\gamma\le\mu$. The coefficient on the row indexed by the multiindex $\delta\le\mu$ of length $\mid\delta\mid\le\kappa$ and the column indexed by $\gamma\le\mu$ of length $\mid\gamma\mid\le\kappa$ is $z^{\mu+\ell\epsilon_i-\gamma}\frac{\partial^{|\delta|}z^{\gamma}}{(\partial z)^{\delta}}$. Its determinant is checked, as in [Păun-08], to be non-zero, for otherwise there would exist a non-zero polynomial of multidegree less or equal to μ and total degree less or equal to κ with all derivatives of order less or equal to μ and total order less or equal to κ vanishing. Let the polynomial P run over the set of polynomials in z_i of degree less or equal to κ. Over the set $\{z_i'\neq 0\}$, the determinant, computed by induction using $(z_i^j)^{(l)} = (jz_i^{j-1}z_i')^{(l-1)} = j\sum_{a=0}^{l-1}\binom{l-1}{a}(z_i^{j-1})^{(a)}z_i^{(l-a)}$ and combinaison of rows,

$$\det\begin{pmatrix} 1 & z_i & z_i^2 & \cdots & z_i^{\kappa} \\ 1' & (z_i)' & (z_i^2)' & \cdots & (z_i^{\kappa})' \\ & & \vdots & & \\ & & \vdots & & \\ (1)^{(\kappa)} & (z_i)^{(\kappa)} & (z_i^2)^{(\kappa)} & \cdots & (z_i^{\kappa})^{(\kappa)} \end{pmatrix} = 1!2!\cdots\kappa!(z_i')^{\frac{\kappa(\kappa+1)}{2}}$$

does not vanish. This shows that every vector of

$$T\left(\mathcal{X}^F_{\kappa}/\mathcal{X}^F\right)_{(s_\kappa(b_0))} \subset \left(T\mathcal{X}^F_{\kappa}\right)_{(s_\kappa(b_0))} = T\left(\mathfrak{X}_{\kappa}/\mathbb{P}^N\right)_{(A,s_\kappa(b_0))} \subset \left(T\mathfrak{X}_{\kappa}\right)_{(A,s_\kappa(b_0))}$$

is, up to "horizontal vectors", the value of a meromorphic vector field on $\Pi^{-1}_{\kappa,0}(U^F)\subset\mathfrak{X}_{\kappa}$ holomorphic when viewed with values in $\Pi^{\star}_{\kappa,0}\mathcal{O}_{\mathbb{P}^{n+1}}(\kappa)$.

For "horizontal vectors" (i.e. when $P=0$), we use the set (6.1). By Cramer formulae, over the set $\{z_i'\neq 0\}$, for any given set of $(A^{\cdot}_{\alpha})_{\substack{|\alpha|\le\kappa \\ \alpha\neq l\epsilon_i}}$ there exists $(A^{\cdot}_{l\epsilon_i})_l$ that fulfil the previous equations. Their pole order is less or equal to $\kappa^2+2\kappa$. The missing directions $(A^{\cdot}_{\alpha})_{|\alpha|>\kappa}$ are obtained with even smaller pole order, by considering some universal relations in the differential algebra of polynomials. Details for this last paragraph can be red in [Merker-09].

7. Appendix : Using Morse inequalities for families of surfaces

We check that in the case of surfaces, the bound $\kappa = n+1$ is optimal to find differential equations using holomorphic Morse inequalities.

Remark first that the numbers $\mathbb{L}$, that appeared in the recursion formula for the Segre classes of the bundles $\mathcal{F}_k$, can easily be computed writing Pascal triangle.

$\mathbb{L}_e^f$	$e=0$	$e=1$	$e=2$	$e=3$	$e=4$	$e=5$	$e=6$	$e=7$	$e=8$	$e=9$
$f=0$	1									
$f=1$	0	1								
$f=2$	1	-1	1							
$f=3$	0	2	-2	1						
$f=4$	1	-2	4	-3	1					
$f=5$	0	3	-6	7	-4	1				
$f=6$	1	-3	9	-13	11	-5	1			
$f=7$	0	4	-12	22	-24	16	-6	1		
$f=8$	1	-4	16	-34	46	-40	22	-7	1	
$f=9$	0	5	-20	50	-80	86	-62	29	-8	1

They also fulfil the relations

$$\mathbb{L}_e^f - \mathbb{L}_{e+1}^f = \mathbb{L}_{e+1}^{f+1}.$$

7.1. **On $\mathcal{X}_1$.** We choose ε to be equal to the bound we found when computing the generic nef cone of $\mathcal{X}$, that is $\varepsilon := \frac{r}{3d}$. Then, we take $A = \mathcal{O}_{\mathcal{X}_1}(0,2;1) \otimes \mathcal{O}_{\mathcal{X}}(-\varepsilon x, x)$ and $B = \mathcal{O}_{\mathcal{X}}(0, 2+x)$ with first Chern class $a = \alpha_1 + 2\alpha + x(\alpha - \varepsilon\beta) = \alpha_1 + (2+x)\alpha - x\varepsilon\beta$ and $b = (2+x)\alpha$. We find

$$\begin{aligned} A^5 - 5A^4B &= (\alpha_1 - \varepsilon x\beta)^5 - 10(\alpha_1 - \varepsilon x\beta)^3(2+x)^2\alpha^2 - 20(\alpha_1 - \varepsilon x\beta)^2(2+x)^3\alpha^3 \\ &= s_3 - 5\varepsilon x s_2\beta - 10(2+x)^2 s_1\alpha^2 - 20(2+x)^3\alpha^3 + 30\varepsilon(2+x)^2 x\alpha^2\beta \end{aligned}$$

whose dominant term

$$[-4\chi + 20\varepsilon x]\, d^2 + 20\left[1 - (2+x)^2\right] rd \;=\; -4\chi d^2 - 20\left[3 + 11/3x + x^2\right] rd$$

is negative.

7.2. **On $\mathcal{X}_2$.** Here we take $A = \mathcal{O}_{\mathcal{X}_2}(0,6;2,1) \otimes \mathcal{O}_{\mathcal{X}_1}(0,2y;y) \otimes \mathcal{O}_{\mathcal{X}}(-\varepsilon x, x)$ and $B = \mathcal{O}_{\mathcal{X}}(0, 6+2y+x)$ with first Chern class

$$\begin{aligned} a &= (\alpha_2 + 2\alpha_1 + 6\alpha) + y(\alpha_1 + 2\alpha) + x(\alpha - \varepsilon\beta) \\ &= \alpha_2 + (2+y)\alpha_1 + (6+2y+x)\alpha - \varepsilon x\beta \end{aligned}$$

and $b = (6+2y+x)\alpha$. The bundle $A - B$ is $\mathcal{O}_{\mathcal{X}_2}(-\varepsilon x, 0; 2+y, 1)$.

We compute only the term $(A^7 - 7A^6B)_{dom}$ in $A^7 - 7A^6B$ of degree 3 in (r,d). From the computation of the direct images on $\mathcal{X}$ of the Segre classes of $\mathcal{F}_1$, and from the Segre numbers of $\mathcal{F}_0$ on $\mathcal{X}$ we infer that the contributions have to contain a part in

$\mathrm{seg}_1\,\mathrm{seg}_2$ or seg_1^2 and should therefore contain only one power of α or β. We find that the dominant term is, viewed in $\mathcal{X}_2$

$$\begin{aligned}(A^7-7A^6B)_{dom} &= [\alpha_2+(2+y)\alpha_1]^7+7[\alpha_2+(2+y)\alpha_1]^6[(6+2y+x)\alpha-\varepsilon x\beta]\\ &\quad -7[\alpha_2+(2+y)\alpha_1]^6(6+2y+x)\alpha\\ &= [\alpha_2+(2+y)\alpha_1]^7-7\varepsilon x[\alpha_2+(2+y)\alpha_1]^6\beta\end{aligned}$$

viewed in $\mathcal{X}_1$

$$\begin{aligned}(A^7-7A^6B)_{dom} = \ & \mathrm{seg}_5(\mathcal{F}_1) + 7(2+y)\,\alpha_1\ \mathrm{seg}_4(\mathcal{F}_1) - 7\varepsilon x\ \mathrm{seg}_4(\mathcal{F}_1)\beta\\ &+ 21(2+y)^2\alpha_1^2\ \mathrm{seg}_3(\mathcal{F}_1) - 7\times 6\varepsilon x(2+y)\ \mathrm{seg}_3(\mathcal{F}_1)\alpha_1\beta\\ &+ 35(2+y)^3\alpha_1^3\ \mathrm{seg}_2(\mathcal{F}_1) - 7\times 15\varepsilon x(2+y)^2\ \mathrm{seg}_2(\mathcal{F}_1)\alpha_1^2\beta\\ &+ 35(2+y)^4\alpha_1^4\ \mathrm{seg}_1(\mathcal{F}_1) - 7\times 20\varepsilon x(2+y)^3\ \mathrm{seg}_1(\mathcal{F}_1)\alpha_1^3\beta\\ &+ 21(2+y)^5\alpha_1^5 - 7\times 15\varepsilon x(2+y)^4\ \ \alpha_1^4\beta\end{aligned}$$

This leads to the following expression for the dominant term, viewed in $\mathcal{X}$

$$\begin{aligned}(A^7-7A^6B)_{dom} &= [-2-14(2+y)+63(2+y)^2-70(2+y)^3+35(2+y)^4]s_1s_2\\ &\quad -7\varepsilon x[-13+42(2+y)-45(2+y)^2+20(2+y)^3]s_1^2\beta\\ &= [-2-14(2+y)+63(2+y)^2-70(2+y)^3+35(2+y)^4](\chi d^3-12rd^2)\\ &\quad -7\varepsilon x[-13+42(2+y)-45(2+y)^2+20(2+y)^3]d^3\\ &= \left(222+518\,y+483\,y^2+210\,y^3+35\,y^4\right)(\chi d^3-12rd^2)\\ &\quad -7\varepsilon x\left(51+102\,y+75\,y^2+20\,y^3\right)d^3\end{aligned}$$

We can apply the vanishing criterion provided $\varepsilon x>\chi(3+2y)$. This would lead to

$$\begin{aligned}(A^7-7A^6B)_{dom} &\le \left(222+518\,y+483\,y^2+210\,y^3+35\,y^4\right)(\chi d^3-12rd^2)\\ &\quad -7\chi(3+2y)\left(51+102\,y+75\,y^2+20\,y^3\right)d^3\\ &\le -\left(849+2338\,y+2520\,y^2+1260\,y^3+245\,y^4\right)\chi d^3\\ &\quad -\left(2664+6216\,y+5796\,y^2+2520\,y^3+420\,y^4\right)rd^2\end{aligned}$$

7.3. **On $\mathcal{X}_3$.** Here we take A and B with first Chern class

$$\begin{aligned}a &= (\alpha_3+2\alpha_2+6\alpha_1+18\alpha)+z(\alpha_2+2\alpha_1+6\alpha)\\ &\quad +y(\alpha_1+2\alpha)+x(\alpha-\varepsilon\beta)\\ &= \alpha_3+(2+z)\alpha_2+(6+2z+y)\alpha_1+(18+6z+2y+x)\alpha-\varepsilon x\beta\end{aligned}$$

and

$$b=(18+6z+2y+x)\alpha.$$

The bundle $A-B$ is $\mathcal{O}_{\mathcal{X}_2}(-\varepsilon x,0;6+2z+y,2+z,1)$. In order to apply the vanishing criterion, we choose $\varepsilon x=9+3z+y$. The dominant term of A^9-9A^8B is (computed with Maple)

$(34272\,y^3z+3304896\,z^3+17136\,z^6+25200\,y^2z^4+1332648+906336\,y+3997944\,z+495936\,y^2z+34272\,yz^5+181440\,y^2z^3+222768\,z^5+212544\,y^2+2416896\,yz+$

$1391040\,yz^3 + 1189440\,z^4 + 5016096\,z^2 + 352800\,yz^4 + 17136\,y^3 + 25200\,y^3z^2 + 6720\,y^3z^3 + 2613744\,yz^2 + 450576\,y^2z^2)rd^3$

$-(869904\,y^3z + 44108988\,z^3 + 559608\,z^6 + 32130\,y^4z + 772380\,y^2z^4 + 16542612 + 12428586\,y + 49627836\,z + 8196300\,y^2z + 18900\,y^4z^2 + 1085616\,yz^5 + 3507840\,y^2z^3 + 3780\,y^4z^3 + 4306554\,z^5 + 3512700\,y^2 + 30564\,z^7 + 33142896\,yz + 21170016\,yz^3 + 19278\,y^4 + 18008802\,z^4 + 63329508\,z^2 + 6674220\,yz^4 + 434952\,y^3 + 664020\,y^3z^2 + 221760\,y^3z^3 + 26460\,y^3z^4 + 36642312\,yz^2 + 65016\,y^2z^5 + 7663572\,y^2z^2 + 71316\,z^6y)\chi d^3.$

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