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Géométrie des variétés de Fano : sous-faisceaux du fibré tangent et diviseur fondamental

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Résumé

Cette thèse est consacrée à l'étude de la géométrie des variétés de Fano complexes en utilisant les propriétés des sous-faisceaux du fibré tangent et la géométrie du diviseur fondamental. Les résultats principaux compris dans ce texte sont :

(i) Une généralisation de la conjecture de Hartshorne : une variété lisse projective est isomorphe à un espace projectif si et seulement si son fibré tangent contient un sous-faisceau ample.

(ii) Stabilité du fibré tangent des variétés de Fano lisses de nombre de Picard un : à l'aide de théorèmes d'annulation sur les espaces hermitiens symétriques irréductibles de type compact M, nous montrons que pour presque toute intersection complète générale dans M, le fibré tangent est stable. La même méthode nous permet de donner une réponse sur la stabilité de la restriction du fibré tangent de l'intersection complète à une hypersurface générale.

(iii) Non-annulation effective pour des variétés de Fano et ses applications : nous étudions la positivité de la seconde classe de Chern des variétés de Fano lisses de nombre de Picard un. Ceci nous permet de montrer un théorème de non-annulation pour les variétés de Fano lisses de dimension n et d'indice $n-3$. Comme application, nous étudions la géométrie anticanonique des variétés de Fano et nous calculons les constantes de Seshadri des diviseurs anticanoniques des variétés de Fano d'indice grand.

(iv) Diviseurs fondamentaux des variétés de Moishezon lisses de dimension trois et de nombre de Picard un : nous montrons l'existence d'un diviseur lisse dans le système fondamental dans certain cas particulier.

Mots clés : Variétés de Fano, espaces projectifs, faisceaux amples, feuilletages, stabilité, espaces hermitiens symétriques, théorèmes d'annulation, intersections complètes, propriétés de Lefschetz, non-annulation, seconde classe de Chern, birationalité, diviseurs fondamentaux, constante de Seshadri, variétés de Moishezon, singularités, courbes rationnelles, théorie de Mori

Abstract

This thesis is devoted to the study of complex Fano varieties via the properties of subsheaves of the tangent bundle and the geometry of the fundamental divisor. The main results contained in this text are :

(i) A generalization of Hartshorne's conjecture : a projective manifold is isomorphic to a projective space if and only if its tangent bundle contains an ample subsheaf.

(ii) Stability of tangent bundles of Fano manifolds with Picard number one : by proving vanishing theorems on irreducible Hermitian symmetric spaces of compact type M, we prove that the tangent bundles of almost all general complete intersections in M are stable. Moreover, the same method also gives an answer to the problem of stability of the restriction of the tangent bundle of a complete intersection on a general hypersurface.

(iii) Effective non-vanishing for Fano varieties and its applications : we study the positivity of the second Chern class of Fano manifolds with Picard number one, this permits us to prove a non-vanishing result for n-dimensional Fano manifolds with index $n-3$. As an application, we study the anticanonical geometry of Fano varieties and calculate the Seshadri constants of the anticanonical divisors of Fano manifolds with large index.

(iv) Fundamental divisors of smooth Moishezon threefolds with Picard number one : we prove the existence of a smooth divisor in the fundamental linear system in some special cases.

Key words : Fano varieties, projective spaces, ample sheaves, foliations, stability, Hermitian symmetric spaces, vanishing theorems, complete intersects, Lefschetz properties, non-vanishing, second Chern class, birationality, fundamental divisors, Seshadri constants, Moishezon manifolds, singularities, rational curves, Mori theory

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Résumé en français

Dans cette thèse, nous étudions la géométrie des variétés de Fano complexes. Elles consistent une partie fondamentale de la classification des variétés projectives. D'après les progrès de Birkar-Cascini-Hacon-M$^{\mathrm{c}}$Kernan sur le programme des modèles minimaux, chaque variété uniréglée X est birationnellement équivalente à une variété X' avec une fibration $X' \to Y$ dont la fibre générale est une variété de Fano (à singularités terminales). Contrairement aux variétés de type general, il y a « très peu » de variétés de Fano. Étant donné la dimension, en utilisant la géométrie de courbes rationnelles, Kollár-Miyaoka-Mori ont montrés dans [KMM92] que les variétés de Fano lisses forment une famille bornée. Récemment, Birkar a confirmé la conjecture de Borisov-Alexeev-Borisov dans [Bir16b] : étant donné la dimension, les variétés de Fano à singularités ε-lc forment une famille bornée pour ε fixé. Il y deux approches différentes pour comprendre mieux la géométrie des variétés de Fano. L'une est d'introduire des notions de posivité algébrique sur le fibré tangent et ses sous-faisceaux et l'autre est consiste à étudier le système pluri-anticanonique. La difficulté de la seconde approche est que les éléments généraux dans le système pluri-anticanonique sont peut-être très singuliers.

Partie I : Sous-faisceaux du fibré tangent

Une stratégie standard en géométrie algébrique est d'obtenir des informations sur la structure d'une variété projective à partir des informations sur son fibré tangent. Le plus célèbre résultat dans cette direction est la conjecture de Hartshorne qui a été montrée par Mori : une variété projective lisse est isomorphe à un espace projectif si et seulement si son fibré tangent est ample. Il y a un nombre de généralisations de ce résultat en considérant les sous-faisceaux du fibré tangent et ses puissances extérieures (voir [Wah83, CP98, AW01, ADK08, AKP08] etc.). On rappelle le notion d'amplitude pour un faisceau cohérent sur une variété projective.

Définition. *Soient X une variété projective normale et $\mathcal{E}$ un faisceau cohérent sans torsion de rang positif sur X. Notons par $\mathbb{P}(\mathcal{E})$ le fibré projectif associé* $\mathrm{Proj}(\oplus_{m\geq 0} Sym^m(\mathcal{E}))$ *au sens de Grothendick. Alors $\mathcal{E}$ est appelé ample si $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ est ample.*

D'après un résultat de Miyaoka [Miy87a], une variété projective lisse dont le fibré tangent contient un sous-faisceau ample est uniréglée. En particulier, elle admet une famille dominante minimale de courbes rationelles. Notre premier résultat principal est inspiré de travaux de Araujo-Druel sur les feuilletages de Fano (cf. [AD13])

Théorème (= Theorem 2.3.9). *Soit X une variété projective lisse de dimension n telle que son fibré tangent contient un sous-faisceau ample $\mathcal{E}$. Alors $X \cong \mathbb{P}^n$ et $\mathcal{E}$ est isomorphe à $T_{\mathbb{P}^n}$ ou $\mathcal{O}_{\mathbb{P}^n}(1)^{\oplus r}$.*

Ce résultat a été montré avant avec des hypothèses supplémentaires : $\mathcal{E}$ est un fibré en droites [Wah83], $\mathcal{E}$ est localement libre et son rang est grand [CP98], $\mathcal{E}$ est localement libre [AW01], le nombre de Picard $\rho(X) = 1$ [AKP08]. Comme conséquence, on donnera une réponse positive d'une conjecture de Beltrametti-Sommese d'après les travaux de Litt [Lit17].

Théorème (= Theorem 2.4.2). *Soit X une variété projective lisse de dimension $n \geq 3$ et soit A un diviseur ample sur X. Supposons que A soit un fibré projectif, $p\colon A \to B$, au-dessus d'une variété projective lisse B de dimension $b > 0$. Alors (X, A) est isomorphe à une des paires suivantes.*

(1) $(\mathbb{P}(E), H)$ *pour un fibré vectoriel ample* E *au-dessus de* B *tel que* $H \in |\mathcal{O}_{\mathbb{P}(E)}(1)|$*, et* p *est la restriction à* A *de la projection induite* $\mathbb{P}(E) \to B$.
(2) $(\mathbb{P}(E), H)$ *pour un fibré vectoriel ample* E *au-dessus de* $\mathbb{P}^1$ *tel que* $H \in |\mathcal{O}_{\mathbb{P}(E)}(1)|$, $H = \mathbb{P}^1 \times \mathbb{P}^{n-2}$ *et* p *est la deuxième projection.*
(3) (Q^3, H)*, où* Q^3 *est une hypersurface quadrique de dimension* 3 *et* H *est une surface quadrique lisse avec* $H \in |\mathcal{O}_{Q^3}(1)|$*, et* p *est une des projections* $H \cong \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$.
(4) $(\mathbb{P}^3, H)$*, où* H *est une surface quadrique lisse et* $H \in |\mathcal{O}_{\mathbb{P}^3}(2)|$*, et* p *est une des projections* $H \cong \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$.

Nous étudions ensuite le problème de la stabilité du fibré tangent des variétés de Fano lisses de nombre de Picard un (voir Définition 0.2.7). Plus précisément, nous travaillons sur la conjecture suivante.

Conjecture ([Pet01, §3]). *Soit* X *une variété de Fano lisse de nombre de Picard un. Alors* T_X *est stable.*

Cette conjecture a été montré dans un certain nombre de cas particuliers, mais elle est encore ouverte dans le cas général (voir [Ram66, Rei77, PW95, Hwa98, Hwa01] etc.). Dans le résultat suivant, on élargit la liste (voir §3.1.1 pour la définition).

Théorème (= Theorem 3.3.5). *Soit* M *un espace hermitien symétrique irréductible de type compact et de dimension* n*. Notons par* $\mathcal{O}_M(1)$ *le générateur ample de* $\mathrm{Pic}(M)$*. Soit* Y *une sous-variété lisse de* M *telle que l'application* $\mathrm{Pic}(M) \to \mathrm{Pic}(Y)$ *soit surjective. Posons* $\mathcal{O}_Y(1) = \mathcal{O}_M(1)|_Y$*. Alors le fibré tangent* T_Y *est stable si* Y *est isomorphe à l'une des variétés suivantes.*

(1) *Il existe des hypersurfaces* $H_i \in |\mathcal{O}_M(d_i)|$ *pour* $1 \leq i \leq r \leq n-1$ *telles que* $d_i \geq 2$, $Y = H_1 \cap \cdots \cap H_r$ *et les intersections complètes* $H_1 \cap \cdots \cap H_j$ *soient lisses pour tout* $1 \leq j \leq r$.
(2) Y *est une hypersurface lisse.*

De plus, il est bien connu que si X est une hypersurface de degré d dans Y avec T_Y stable, alors la restriction $T_Y|_X$ est aussi stable si $d \gg 1$. Néanmoins, si d est petit, ceci n'est pas vrai en général. Par exemple, si Y est une hypersurface quadrique et X est une section linéaire, alors la restriction $T_Y|_X$ est semi-stable, mais elle n'est pas stable. Dans le théorème suivant, nous montrons un résultat d'effectivité liée aux stabilités de restrictions de fibrés tangents.

Théorème (= Theorem 3.3.8). *Soit* M *un espace hermitien symétrique irréductible de type compact et de dimension* $n + r$*. Supposons que* $n \geq 3$ *and* $r \geq 1$*. Soit* $\mathcal{O}_M(1)$ *le générateur ample de* $\mathrm{Pic}(M)$*. Soient* $H_i \in |\mathcal{O}_M(d_i)|$ $(1 \leq i \leq r)$ *des hypersurfaces telles que* $2 \leq d_1 \leq \cdots \leq d_r$ *et les intersections complètes* $H_1 \cap \cdots \cap H_j$ *soient lisses pour tout* $1 \leq j \leq r$*. Notons* $H_1 \cap \cdots \cap H_r$ *par* Y*. Soit* $X \in |\mathcal{O}_Y(d)|$ *une hypersurface générale. Supposons de plus que la composition d'applications*

$$\mathrm{Pic}(M) \to \mathrm{Pic}(Y) \to \mathrm{Pic}(X)$$

soit surjective. Alors la restriction $T_Y|_X$ *est stable si elle vérifie l'une des conditions suviantes.*

(1) Y *est une variété de Fano et* M *n'est ni l'espace projectif* $\mathbb{P}^{n+r}$ *ni une hypersurface quadrique lisse* $Q^{n+r} \subset \mathbb{P}^{n+r+1}$.
(2) Y *est une variété de Fano,* M *est l'espace projectif* $\mathbb{P}^{n+r}$ *avec* $n + r \geq 5$ *et* $d \geq d_1$.
(3) Y *est une variété de Fano,* M *est une hypersurface quadrique lisse et* $d \geq 2$.
(4) $d > d_r - r_Y/n$*, où* r_Y *est l'entier tel que* $\omega_Y \cong \mathcal{O}_Y(-r_Y)$.

Si Y est une hypersurface générale d'un espace projectif, on peut donner une réponse complète pour la question d'effectivité en utilisant la propriété de Lefschetz de l'algèbre de Milnor (voir §3.2.2 pour les détails).

Théorème (= Theorem 3.3.9). *Soit* Y *une hypersurface générale dans l'espace projectif* $\mathbb{P}^{n+1}$ *avec* $n \geq 3$*. Soit* $X \in |\mathcal{O}_Y(d)|$ *une hypersurface générale de degré* d *dans* Y *telle que l'application* $\mathrm{Pic}(Y) \to \mathrm{Pic}(X)$ *soit surjective. Alors* $T_Y|_X$ *est stable sauf si* $d = 1$ *et* Y *est isomorphe à* $\mathbb{P}^n$ *ou* Q^n.

Partie II : Géométrie de diviseurs fondamentaux

Soit X une variété de Fano à singularités log terminales. Alors le groupe de Picard $\mathrm{Pic}(X)$ est sans torsion et il existe un diviseur unique de Cartier ample H tel que $-K_X \sim r_X H$, où r_X est l'indice de X. Le diviseur H est appelé le *diviseur fondamental* de X. Comme $-K_X$ est ample, on peut utiliser certaines estimations effectives liées à la conjecture de Fujita pour mesurer la posivité globale de $-K_X$. Plus précisément, nous étudions les deux questions naturelles suivantes.

Question. *Soit X une variété Gorenstein à singularités canoniques de dimension n telle que $-K_X$ soit nef et gros.*

(1) Trouver la constante optimale $f(n)$ telle que le système linéaire complet $|-mK_X|$ soit sans point base pour tout entier $m \geq f(n)$.
(2) Trouver la constante optimale $b(n)$ telle que l'application rationnelle $\Phi_{|-mK_X|}$ soit birationnelle pour tout entier $m \geq b(n)$.

D'après les travaux de Reider et Fukuda [Rei88, Fuk91], on sait que $f(2) = 2$ et $b(2) = 3$. En dimension supérieure, il y a un nombre de travaux sur les variants de cette question ([And85, Fuk91, Che11, CJ16] etc.). Une approche naturelle de cette question est de trouver un élément dans $|H|$ qui n'a pas de point « très singulier », après nous répétons le processus en construisant une suite décroissante de sous-variétés de X. Ainsi nous pouvons réduire le problème à des variétés de Calabi-Yau de dimension petite. L'existence d'une telle suite de sous-variétés des variétés de Fano faibles de dimension n et de l'indice $r_X \geq n-2$ a été montrée dans [Amb99] et pour les variétés de Fano faibles Gorenstein de dimension quatre à singulariétés canoniques par Kawamata dans [Kaw00] (voir aussi [Flo13]). En particulier, d'après les travaux de Fukuda, Reider, Oguiso-Peternell et Jiang [Rei88, Fuk91, OP95, Jia16], on peut obtenir : $f(3) = 2$, $b(3) = 3$, $f(4) \leq 7$ et $b(4) = 5$ (voir Theorem 5.3.2 et Theorem 5.3.4). Dans le cas général, le premier pas de l'approche est la non-annulation effective de H.

Théorème (= Theorem 5.3.6). *Soit X une variété lisse de Fano de dimension $n \geq 4$ et d'indice $n-3$. Soit H le diviseur fondamental. Alors $h^0(X, H) \geq n-2$.*

Ce théorème est une conséquence d'une inégalité de type Bogomolov pour les variétés de Fano lisses X avec $\rho(X) = 1$.

Théorème (= Theorem 5.2.2). *Soit X une variété de Fano lisse de dimension $n \geq 7$ avec $\rho(X) = 1$. Soient H le diviseur fondamental de X et r_X l'indice de X.*

(1) Si $r_X = 2$, alors
$$c_2(X) \cdot H^{n-2} \geq \frac{11n-16}{6n-6} H^n.$$
(2) Si $3 \leq r_X \leq n$, alors
$$c_2(X) \cdot H^{n-2} \geq \frac{r_X(r_X-1)}{2} H^n.$$

Pour une variété de Fano X à singularités canoniques de dimension n avec $-K_X \sim (n-3)H$ pour certain diviseur de Cartier ample H, l'existence d'éléments à singularités canoniques dans $|H|$ a été montrée par Floris dans [Flo13]. De plus, en utilisant les travaux de Oguiso-Peternell et Jiang [OP95, Jia16] sur les variétés de Calabi-Yau de dimension trois, nous déduirons le théorème suivant.

Théorème (= Theorem 5.3.8). *Soient X une variété de Fano lisse de dimension n et d'indice $n-3$, et H le diviseur fondamental. Alors*

(1) le système linéaire complet $|mH|$ est sans point base pour tout entier $m \geq 7$;
(2) le système linéaire complet $|mH|$ définit une application birationnelle pour tout entier $m \geq 5$.

Nous étudions ensuite la posivité locale du diviseur fondamental des variétés de Fano lisses. La positivité locale d'un diviseur ample est mesuré par la *constante de Seshadri* introduit par Demailly dans [Dem92].

Définition. *Soient X une variété projective lisse et L un fibré en droites nef au-dessus de X. Pour chaque point $x \in X$, on peut définir le nombre suivant*

$$\varepsilon(X, L; x) := \inf_{x \in C} \frac{L \cdot C}{\nu(C, x)},$$

qui est appelé la constante de Seshadri de L en x. Ici la borne inférieure porte sur les courbes passant par le point x et $\nu(C, x)$ est la multiplicité de C en x.

Ein, Lazarsfeld et Küchle ont montré que les constantes de Seshadri jouissaient d'une surprenante propriété de minoration universelle si l'on se restreint à des points en position dite « très générale », c'est-à-dire des points en dehors d'une union dénombrable de sous-variétés strictes. Nous la noterons par $\varepsilon(X, L; 1)$. Si X est une variété de Fano de dimension n, on sait que $\varepsilon(X, -K_X; 1) \leq n+1$ avec l'égalité si et seulement si $X \cong \mathbb{P}^n$ (cf. [BS09]). En dimension deux, Ein et Lazarsfeld avaient précédemment montré dans [EL93] que la constante de Seshadri d'un diviseur ample A sur une surface S lisse vérifiaient $\varepsilon(S, A; 1) \geq 1$. La conjecture suivante est donc naturelle.

Conjecture ([Laz04, Conjecture 5.2.4]). *Soit X une variété projective lisse, L un diviseur ample sur X. Alors $\varepsilon(X, L; 1) \geq 1$.*

En particulier, cette conjecture prédit que $\varepsilon(X, -K_X; 1) \geq r_X$ pour une variété de Fano lisse X. Cet énoncé a été montrée par Broustet si $r_X \geq n-2$ (cf. [Bro09]). Nous généralisons ce résultat au cas $r_X = n-3$ (voir Theorem 6.3.1). Une autre question naturelle est de demander quand l'égalité $\varepsilon(X, -K_X; 1) = 1$ est vrai. En dimension deux, le résultat dans [Bro06] donne la réponse suivante.

Théorème ([Bro06, Théorème 1.3]). *Soit S une surface de del Pezzo lisse. Alors $\varepsilon(S, -K_S; 1) = 1$ si et seulement si S est une surface de del Pezzo de degré un, ou de façon équivalente, $r_S = 1$ et $\mathrm{Bs}\,|-K_X|$ n'est pas vide.*

En dimension trois, si X est une variété de Fano lisse avec $\rho(X) = 1$ telle que X est très générale dans sa famille de déformation, la constante de Seshadri $\varepsilon(X, -K_X; 1)$ est calculée par Ito dans [Ito14]. En utilisant l'existence de droites et l'existence de scindages libres de diviseurs anticanoniques, nous pouvons calculer les constantes de Seshadri des diviseurs anticanoniques des variétés de Fano de dimension trois et de nombre de Picard au moins deux. Dans le théorème suivant, nous suivons les numérotations dans [MM81] et [MM03] (voir aussi Appendice B).

Théorème (= Theorem 6.5.16). *Soit X une variété de Fano lisse de dimension 3 avec $\rho(X) \geq 2$.*

(1) $\varepsilon(X, -K_X; 1) = 1$ si et seulement si X admet une fibration en surfaces de del Pezzo de degré 1 (nº 1 dans Tableau 2 et nº 8 dans Tableau 5).
(2) $\varepsilon(X, -K_X; 1) = 4/3$ si et seulement si X admet une fibration en surfaces de del Pezzo de degré 2 (nº 2, 3 dans Tableau 2, et nº 7 dans Tableau 5).
(3) $\varepsilon(X, -K_X; 1) = 3/2$ si et seulement si X admet une fibration en surfaces de del Pezzo de degré 3 (nº 4, 5 dans Tableau 2, nº 2 dans Tableau 3 et nº 6 dans Tableau 5).
(4) $\varepsilon(X, -K_X; 1) = 3$ si et seulement si X est isomorphe à l'éclatement de $\mathbb{P}^3$ le long d'une courbe plane C de degré au plus 3 (nº 28, 30, 33 dans Tableau 2).
(5) $\varepsilon(X, -K_X; 1) = 2$ sinon.

Une conséquence du théorème ci-dessus est un caractérisation des variétés de Fano X lisse de dimension trois avec $\varepsilon(X, -K_X; 1) = 1$.

Théorème (= Corollary 6.4.14). *Soit X une variété de Fano lisse de dimension trois qui est très générale dans sa famille de déformation. Alors $\varepsilon(X, -K_X; 1) = 1$ si et seulement si $r_X = 1$ et $\mathrm{Bs}\,|-K_X|$ n'est pas vide.*

Le dernier chapitre est consacré à l'étude la géométrie anticanonique des variétés de Moishezon lisses. Plus précisément, nous étudions la question suivante.

Question. *Soit X une variété de Moishezon lisse de dimension n et de nombre de Picard un. Soit L le générateur gros de* $\mathrm{Pic}(X)$. *Suppons que $-K_X \sim (n-1)L$. Est-ce qu'il existe un élément lisse $D \in |L|$?*

Cette question est inspirée par la rigidité des variétés de Fano lisses de nombre de Picard un et l'existence de diviseur lisse dans le système fondamental des variétés lisses de del Pezzo (voir §7.1 et [Fuj77a]). En fait, les variétés de Moishezon lisses de dimension trois et de nombre de Picard un avec diviseur anticanonique gros sont étudiées dans un nombre de travaux (cf. [Pet85, Pet86b, Pet86a, Pet86c, Nak87, Nak88, Kol91b, Nak96] etc.). En particulier, si X est de dimension trois et de nombre de Picard un et $-K_X \sim r_X L$ pour un entier positif $r_X \geq 3$ et un fibré en droites gros L, Kollár a montré que X est projective. Dans le théorème suivant, nous considérons le cas $r_X = 2$.

Théorème (= **Theorem 7.3.11**). *Soit X une variété de Moishezon lisse de dimension trois telle que* $\mathrm{Pic}(X) = \mathbb{Z}L$ *pour un fibré en droites gros L et $-K_X \sim 2L$. Supposons que $h^0(X, L) \geq 3$. Soit D_1, D_2 deux éléments généraux dans $|H|$. Soit C l'intersection complète $D_1 \cap D_2$. Alors C contient au moins une composante irréductible mobile A. De plus, si A intersecte avec l'union des autres composantes de $D_1 \cap D_2$ en au moins deux points, alors un élément général D de $|L|$ est lisse.*

En particulier, en utilisant un résultat de Kollár (cf. Theorem 7.2.13), on obtient le résultat suivant.

Théorème (= **Corollary 7.3.13**). *Soit X une variété de Moishezon lisse de dimension trois telle que* $\mathrm{Pic}(X) = \mathbb{Z}L$ *pour un fibré en droites gros L et $-K_X \sim 2L$. Supposons que $h^0(X, L) \geq 5$. Alors il existe un élément lisse $D \in |L|$.*

Summary in English

The subject of this thesis is to study the geometry of complex Fano varieties. They constitute a fundamental part of the classification of algebraic varieties. By a straightfoward of Birkar-Cascini-Hacon-M^cKernan's works on minimal model program, every uniruled variety is birational to a fiberspace whose general fiber is a Fano variety (with terminal singularities). In contrast with varieties of general type, there are "very few" Fano varieties. Kollár-Miyaoka-Mori proved in [KMM92] that the smooth Fano varieties of fixed dimension form a bounded family by using the geometry of rational curves. Recently, Birkar solved the so-called Borisov-Alexeev-Borisov conjecture in [Bir16b] : for any given ε, Fano varieties with ε-lc singularities of fixed dimension form a bounded family. To better understand the geometry of Fano varieties, there are two different approaches. In the first approach we introduce appropriate algebraic notions of positivity of the tangent bundle and its subsheaves to obtain a refinement classification. In the second approach, one studies the pluri-anticanonical system to create a particular kind of subvarieties and then one can use induction by restricting to these subvarieties. A difficulty in the second approach is that a general member of the pluri-anticanonical system may have bad singularities.

Part I : Subsheaves of the tangent bundle

A basic strategy in algebraic geometry is to deduce properties of a projective manifold from the properties of its tangent bundle. The most famous result in this direction is Hartshorne's conjecture solved by Mori : a projective manifold is isomorphic to a projective space if and only if its tangent bundle is ample. There are many efforts to generalize this theorem by considering certain kinds of positive subsheaves of the tangent bundle and its exterior powers (see [Wah83, CP98, AW01, ADK08, AKP08] etc.). Before giving the precise statement, we introduce the notion of ampleness for coherent sheaves over projective varieties.

Definition. *Let X be a normal projective variety, and let $\mathcal{E}$ be a torsion free coherent sheaf of positive rank over X. Denote by $\mathbb{P}(\mathcal{E})$ the Grothendieck projectivization* $\mathrm{Proj}(\oplus_{m\geq 0} Sym^m(\mathcal{E}))$*. Then $\mathcal{E}$ is said to be ample if $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ is an ample invertible sheaf over $\mathbb{P}(\mathcal{E})$.*

Thanks to an important result of Miyaoka [Miy87a], a projective manifold whose tangent bundle contains an ample subsheaf is uniruled and it carries a minimal covering family of rational curves. Inspired by the work of Araujo-Druel on Fano foliations over projective manifolds [AD13], we obtain a generalization of Hartshorne's conjecture.

Theorem (= **Theorem 2.3.9**). *Let X be a n-dimensional projective manifold such that its tangent bundle T_X contains an ample subsheaf $\mathcal{E}$. Then $X \cong \mathbb{P}^n$ and $\mathcal{E}$ is isomorphic to $T_{\mathbb{P}^n}$ or $\mathcal{O}_{\mathbb{P}^n}(1)^{\oplus r}$.*

This result has been proved before under different additional assumptions : $\mathcal{E}$ is a line bundle [Wah83], $\mathcal{E}$ is a locally free subsheaf of large rank [CP98], $\mathcal{E}$ is locally free [AW01] and the Picard number $\rho(X) = 1$ [AKP08]. As an application, the works of Litt [Lit17] together with our result solve a conjecture of Beltrametti-Sommese.

Theorem (= **Theorem 2.4.2**). *Let X be a projective manifold of dimension $n \geq 3$, and let A be an ample divisor on X. Assume that A is a $\mathbb{P}^r$-bundle, $p\colon A \to B$, over a manifold B of dimension $b > 0$. Then one of the following holds.*

(1) $(X, A) = (\mathbb{P}(E), H)$ *for some ample vector bundle* E *over* B *such that* $H \in |\mathcal{O}_{\mathbb{P}(E)}(1)|$*, and* p *is equal to the restriction to* A *of the induced projection* $\mathbb{P}(E) \to B$*.*
(2) $(X, A) = (\mathbb{P}(E), H)$ *for some ample vector bundle* E *over* $\mathbb{P}^1$ *such that* $H \in |\mathcal{O}_{\mathbb{P}(E)}(1)|$*,* $H = \mathbb{P}^1 \times \mathbb{P}^{n-2}$ *and* p *is the projection to the second factor.*
(3) $(X, A) = (Q^3, H)$*, where* Q^3 *is a smooth quadric threefold and* H *is a smooth quadric surface and* p *is the projection to one of the factors* $H \cong \mathbb{P}^1 \times \mathbb{P}^1$*.*
(4) $(X, A) = (\mathbb{P}^3, H)$*, where* H *is a smooth quadric surface and* p *is again a projection to one of the factors of* $H \cong \mathbb{P}^1 \times \mathbb{P}^1$*.*

Next we study the problem of the stability of the tangent bundle of Fano manifolds with Picard number one (see Définition 0.2.7). More precisely, we focus on the following long-standing conjecture.

Conjecture ([Pet01, §3]). *Let* X *be a Fano manifold with Picard number one. Then* T_X *is stable.*

Although known to be valid in many cases (see [Ram66, Rei77, PW95, Hwa98, Hwa01] etc.), this conjecture is wide open in general. We enlarge the list, proving the following result. For the definition of Hermitian symmetric spaces, we refer to §3.1.1.

Theorem (= Theorem 3.3.5). *Let* M *be a* n*-dimensional irreducible Hermitian symmetric space of compact type, and denote by* $\mathcal{O}_M(1)$ *the ample generator of* $\mathrm{Pic}(M)$*. Let* Y *be a submanifold of* M *such that the restriction* $\mathrm{Pic}(M) \to \mathrm{Pic}(Y)$ *is surjective. Then the tangent bundle* T_Y *is stable if one of the following conditions holds.*

(1) There exists a collection of hypersurfaces $H_i \in |\mathcal{O}_M(d_i)|$ *with* $d_i \geq 2$ *and* $1 \leq i \leq r \leq n-1$ *such that the complete intersections* $H_1 \cap \cdots \cap H_j$ *are smooth for all* $1 \leq j \leq r$ *and* $Y = H_1 \cap \cdots \cap H_r$*.*
(2) Y *is a smooth hypersurface.*

Moreover, it is well-known that if X is a general hypersurface of degree d on Y with T_Y stable, then the restriction $T_Y|_X$ is stable if $d \gg 1$. However, if d is small, in general this not true. For example, if Y is a smooth quadric hypersurface and X is linear section, then $T_Y|_X$ is just semi-stable, but not stable. In the following theorem, we derive some effective results for the stability of restriction.

Theorem (= Theorem 3.3.8). *Let* M *be a* $(n+r)$*-dimensional irreducible Hermitian symmetric space of compact type such that* $n \geq 3$ *and* $r \geq 1$*. Let* $H_i \in |\mathcal{O}_M(d_i)|$ $(1 \leq i \leq r)$ *be a collection of hypersurfaces such that* $2 \leq d_1 \leq \cdots \leq d_r$ *and the complete intersections* $H_1 \cap \cdots \cap H_j$ *are smooth for all* $1 \leq j \leq r$*. Denote* $H_1 \cap \cdots \cap H_r$ *by* Y*. Let* $X \in |\mathcal{O}_Y(d)|$ *be a general smooth hypersurface. Assume moreover that the composite of restrictions*

$$\mathrm{Pic}(M) \to \mathrm{Pic}(Y) \to \mathrm{Pic}(X)$$

is surjective. Then the restriction $T_Y|_X$ *is stable if one of the following conditions holds.*

(1) Y *is a Fano manifold and* M *is isomorphic to neither the projective space* $\mathbb{P}^{n+r}$ *nor a smooth quadric hypersurface* Q^{n+r}*.*
(2) Y *is a Fano manifold,* M *is isomorphic to the projective space* $\mathbb{P}^{n+r}$ *with* $n+r \geq 5$ *and* $d \geq d_1$*.*
(3) Y *is a Fano manifold,* M *is isomorphic to a smooth quadric hypersurface* Q^{n+r} *and* $d \geq 2$*.*
(4) $d > d_r - r_Y/n$*, where* r_Y *is the unique integer such that* $\omega_Y \cong \mathcal{O}_Y(-r_Y)$*.*

If Y is a general hypersurface of a projective space, we can go further and obtain a complete answer to the effective restriction problem by using the Lefschetz properties of the Milnor algebras of the general hypersurfaces (see § 3.2.2 for the details).

Theorem (= Theorem 3.3.9). *Let* Y *be a general smooth hypersurface in the projective space* $\mathbb{P}^{n+1}$ *of dimension* $n \geq 3$*. Let* $X \in |\mathcal{O}_Y(d)|$ *be a general smooth hypersurface of degree* d *on* Y *such that the restriction* $\mathrm{Pic}(Y) \to \mathrm{Pic}(X)$ *is surjective. Then* $T_Y|_X$ *is* $\mathcal{O}_X(1)$*-stable unless* $d = 1$ *and* Y *is isomorphic to either* $\mathbb{P}^n$ *or* Q^n*.*

Part II : Geometry of fundamental divisors

Let X be a Fano variety with at worst log terminal singularities. Then the Picard group $\mathrm{Pic}(X)$ of X is torsion-free and there exists an ample unique Cartier divisor H such that $-K_X \sim r_X H$, where r_X is the index of X. We call H the *fundamental divisor* of X. Since $-K_X$ is ample, we can use effective birationality and effective basepoint freeness to measure the global positivity of $-K_X$. More precisely, we will study the following two natural questions.

Question. *Let X be a n-dimensional weak Fano variety with at most canonical Gorenstein singularities.*

(1) Find the optimal constant $f(n)$ depending only on n such that the linear system $|-mK_X|$ is basepoint free for all $m \geq f(n)$.

(2) Find the optimal constant $b(n)$ depending only on n such that the rational map Φ_{-m} corresponding to $|-mK_X|$ is a birational map for all $m \geq b(n)$.

By the works of Reider and Fukuda [Rei88, Fuk91], we have $f(2) = 2$ and $b(2) = 3$. In higher dimension, there are many works on the variation of this question ([And85, Fuk91, Che11, CJ16] etc.). One natural approach of this question is to find a member in $|-K_X|$ with mild singularities to reduce the problem to lower Calabi-Yau varieties. The existence of a good divisor in $|-K_X|$ for $n \leq 4$ was proved by Kawamata in [Kaw00] (see also [Flo13]). In particular, by the works of Fukuda, Reider, Oguiso-Peternell and Jiang [Rei88, Fuk91, OP95, Jia16], we can derive the following results : $f(3) = 2$, $b(3) = 3$, $f(4) \leq 7$ and $b(4) = 5$ (see Theorem 5.3.2 and Theorem 5.3.4). In general case, the first step towards the existence of good ladder is to prove the existence of global section of H.

Theorem (= **Theorem 5.3.6**). *Let X be a Fano manifold of dimension $n \geq 4$ and index $n-3$. Let H be the fundamental divisor. Then $h^0(X, H) \geq n-2$.*

This non-vanishing theorem is a consequence of an inequality of Bogomolov type for Fano manifolds with Picard number one.

Theorem (= **Theorem 5.2.2**). *Let X be a n-dimensional Fano manifolds with $\rho(X) = 1$ such that $n \geq 7$. Let H be the fundamental divisor of X and let r_X be the index of X.*

(1) If $r_X = 2$, then

$$c_2(X) \cdot H^{n-2} \geq \frac{11n-16}{6n-6} H^n.$$

(2) If $3 \leq r_X \leq n$, then

$$c_2(X) \cdot H^{n-2} \geq \frac{r_X(r_X-1)}{2} H^n.$$

The existence of a good divisor in $|H|$ was proved by Floris in [Flo13]. Thus we get the existence of a ladder for n-dimensional Fano manifolds with index $n-3$. From the results of Oguiso-Peternell and Jiang [OP95, Jia16] on Calabi-Yau threefolds, we derive the following theorem.

Theorem (= **Theorem 5.3.8**). *Let X be a n-dimensional Fano manifold with index $n-3$ and let H be the fundamental divisor. Then*

(1) the linear system $|mH|$ is basepoint free for $m \geq 7$;

(2) the linear system $|mH|$ gives a birational map for $m \geq 5$.

Next we study the local positivity of the fundamental divisor of Fano manifolds. The local positivity of an ample line bundle is measured by the so-called *Seshadri constant* introduced by Demailly in [Dem92].

Definition. *Let X be a projective manifold and let L be a nef line bundle on X. To every point $x \in X$, we attach the number*

$$\varepsilon(X, L; x) : = \inf_{x \in C} \frac{L \cdot C}{\nu(C, x)},$$

which is called the Seshadri constant of L at x. Here the infimum is taken over all irreducible curves C passing through x and $\nu(C, x)$ is the multiplicity of C at x.

The Seshadri constant is a lower-continuous function over X in the topology where the closed sets are countable unions of Zariski closed sets. Moreover, there is a number, which we denote by $\varepsilon(X, L; 1)$, such that it is the maximal value of Seshadri constant on X. This maximum is attained for a very general point $x \in X$. If X is a n-dimensional Fano manifold, it is known that $\varepsilon(X, -K_X; 1) \leq n+1$ with equality if and only if $X \cong \mathbb{P}^n$. In dimension 2, Ein and Lazarsfeld showed in [EL93] that the Seshadri constant of an ample divisor A on a smooth surface satisfies $\varepsilon(S, A; 1) \geq 1$. Thus, the following conjecture is natural.

Conjecture ([Laz04, Conjecture 5.2.4]). *Let X be a projective manifold, and let L be an ample divisor on X. Then $\varepsilon(X, L; 1) \geq 1$.*

In particular, this conjecture predicts that we have $\varepsilon(X, -K_X; 1) \geq r_X$ for a Fano manifold X. This statement was confirmed by Broustet in the case $r_X \geq n-2$ in [Bro09]. By the existence of ladders on n-dimensional Fano manifolds with index $n-3$, we generalize this to the case $r_X = n-3$ (cf. Theorem 6.3.1). Another natural question is to ask when the equality $\varepsilon(X, -K_X; 1) = 1$ holds. In dimension two, as a consequence the explicit calculation of $\varepsilon(X, -K_X; x)$ given in [Bro06], we have the following result.

Théorème ([Bro06, Théorème 1.3]). *Let S be a del Pezzo surface. Then $\varepsilon(S, -K_S; 1) = 1$ if and only if S is a del Pezzo surface of degree one, or equivalently $r_S = 1$ and $|-K_S|$ is not basepoint free.*

In dimension three, the Seshadri constant $\varepsilon(X, -K_X; 1)$ is calculated by Ito in [Ito14] via toric degeneration for a very general smooth Fano threefold with Picard number one. Using the existence of lines and the existence of free splittings of anticanonical divisors, we can deal with Fano threefolds with Picard number at least two. In the following theorem, we follow the numbering in [MM81] and [MM03] (see also Appendix B).

Theorem (= Theorem 6.5.16). *Let X be a smooth Fano threefold with $\rho(X) \geq 2$.*

(1) $\varepsilon(X, -K_X; 1) = 1$ if and only if X carries a del Pezzo fibration of degree 1 ($n^o\, 1$ in Table 2 and $n^o\, 8$ in Table 5).
(2) $\varepsilon(X, -K_X; 1) = 4/3$ if and only if X carries a del Pezzo fibration of degree 2 ($n^o\, 2, 3$ in Table 2, and $n^o\, 7$ in Table 5).
(3) $\varepsilon(X, -K_X; 1) = 3/2$ if and only if X carries a del Pezzo fibration of degree 3 ($n^o\, 4, 5$ in Table 2, $n^o\, 2$ in Table 3 and $n^o\, 6$ in Table 5).
(4) $\varepsilon(X, -K_X; 1) = 3$ if X is isomorphic to the blow-up of $\mathbb{P}^3$ along a smooth plane curve C of degree at most 3 ($n^o\, 28, 30, 33$ in Table 2).
(5) $\varepsilon(X, -K_X; 1) = 2$ otherwise.

As a consequence, combining with Ito's result [Ito14, Theorem 1.8], one can derive the following characterization of Fano threefolds X with $\varepsilon(X, -K_X; 1) = 1$.

Theorem (= Corollary 6.4.14). *Let X be a smooth Fano threefold very general in its deformation family. Then $\varepsilon(X, -K_X; 1) = 1$ if and only if $r_X = 1$ and $|-K_X|$ is not basepoint free.*

The last chapter of this thesis is devoted to study the anticanonical geometry of Moishezon manifolds. More precisely, we consider the following question.

Question. *Let X be a n-dimensional Moishezon manifold such that $\rho(X) = 1$. Denote by L the ample generator of X and suppose that $-K_X \sim (n-1)L$. Does there exist a smooth element in $|L|$?*

This questions is inspired by the rigidity problem of Fano manifolds with Picard number one and the existence of smooth elements in the fundamental system of del Pezzo manifolds (see §7.1 and [Fuj77a]). In fact, the smooth Moishezon threefolds with Picard number one and big anticanonical divisor are investigated by many authors (cf. [Pet85, Pet86b, Pet86a, Pet86c, Nak87, Nak88, Kol91b, Nak96] etc.). In particular, if X is of dimension three and of Picard number one such that $-K_X \sim r_X L$ for some integer $r_X \geq 3$ and some big line bundle L, then Kollár proved that X is actually projective. In the following theorem, we consider the case $r_X = 2$.

Theorem (= **Theorem 7.3.11**). *Let X be a smooth Moishezon threefold such that $\mathrm{Pic}(X) = \mathbb{Z}L$ for some big line bundle L and $-K_X \sim 2L$. Assume moreover that $h^0(X, L) \geq 3$. Let D_1, D_2 be two general members of $|H|$, and let C be the complete intersection $D_1 \cap D_2$. Then C contains at least one mobile irreducible component A. Moreover, if A intersects the union of other components of $D_1 \cap D_2$ in at least two points, then a general member D of $|L|$ is smooth.*

In particular, combining this theorem with a result due to Kollár (cf. Theorem 7.2.13), we obtain the following result.

Theorem (= **Corollary 7.3.13**). *Let X be a smooth Moishezon threefold such that $\mathrm{Pic}(X) = \mathbb{Z}L$ for some big line bundle L and $-K_X \sim 2L$. Assume moreover that $h^0(X, L) \geq 5$. Then there exists a smooth element D in $|L|$.*

Chapitre 0

Notations et préliminaires

Nous commençons par donner des notations et résultats utilisés dans tout ce texte. Tous ces résultats sont classiques. Toutes les variétés sont définies sur $\mathbb{C}$ sauf indication contraire.

0.1 Diviseurs et systèmes linéaires

La référence pour ce paragraphe est [Deb01, Chapter 1]. Soit X une variété normale. En particulier, X est lisse en codimension un. Un *diviseur premier* D sur X est une sous-variété réduite et irréductible de X de codimension 1. Un *diviseur de Weil* sur X est une combinaison linéaire formelle $D = \sum d_i D_i$, à coefficients entiers, de diviseurs premiers D_i. Le groupe des diviseurs de Weil sur X à coefficients dans $\mathbb{Z}$ (resp. $\mathbb{Q}$ et $\mathbb{R}$) est noté $Z^1(X)_{\mathbb{Z}}$ (resp. $Z^1(X)_{\mathbb{Q}}$ et $Z^1(X)_{\mathbb{R}}$). Un $\mathbb{R}$-diviseur de Weil D est dit *effectif* lorsque tous les coefficients sont positifs ; on écrit alors $D \geq 0$.

Toute fonction rationnelle non nulle $f \in K(X)$ sur X a un diviseur, celui de ses pôles et zéros, noté $\mathrm{div}(f)$. On désigne par K_X un *diviseur canonique* sur X, c'est-à-dire le diviseur d'une forme différentielle méromorphe de degré maximal ; si X est lisse, on a $\mathcal{O}_X(K_X) \cong \omega_X$.

Un *diviseur de Cartier* sur X est un diviseur de Weil qui peut être défini localement par une seule équation. Le sous-groupe de $Z^1(X)_{\mathbb{Z}}$ formé des diviseurs de Cartier sur X est noté $\mathrm{Div}(X)$. Un $\mathbb{Q}$-diviseur (resp. $\mathbb{R}$-diviseur) de Weil est dit $\mathbb{Q}$-Cartier (resp. $\mathbb{R}$-Cartier) s'il est dans le $\mathbb{Q}$-sous-espace (resp. $\mathbb{R}$-sous-espace) vectoriel de $Z^1(X)_{\mathbb{Q}}$ (resp. $Z^1(X)_{\mathbb{R}}$) engendré par $\mathrm{Div}(X)$. L'ensemble de $\mathbb{R}$-diviseur de Weil $\mathbb{R}$-Cartier est noté par $\mathrm{Div}(X)_{\mathbb{R}}$.

Les diviseurs (resp. $\mathbb{Q}$-diviseurs de Weil) D_1 et D_2 de $Z^1(X)_{\mathbb{Z}}$ (resp. $Z^1(X)_{\mathbb{Q}}$) sont dits *linéairement équivalents* (resp. *$\mathbb{Q}$-linéairement équivalents*) et on note $D_1 \sim D_2$ (resp. $D_1 \sim_{\mathbb{Q}} D_2$) s'il existe une fonction rationnelle f (resp. une fonction rationnelle f et un rationel $r \in \mathbb{Q}$) telle que $D_1 - D_2 = \mathrm{div}(f)$ (resp. telles que $D_1 - D_2 = r\mathrm{div}(f)$). Si D est un diviseur de Weil, on note $|D| = \{D' \geq 0 | D \sim D'\}$ le système linéaire associé au D, et le *lieu base* de $|D|$ est

$$\mathrm{Bs}\,|D| = \bigcap_{D' \in |D|} \mathrm{Supp}(D').$$

Si X est projective, $D \in \mathrm{Div}(X)$ et $C \subset X$ est une courbe réduite et irréductible, on peut définir le *nombre d'intersection* $D \cdot C = \deg(\mathcal{O}_X(D)|_C)$. On note $Z_1(X)_{\mathbb{R}}$ l'ensemble des 1-cycles à coefficients dans $\mathbb{R}$ sur X. Alors le nombre d'intersection peut être défini pour les diviseurs de Weil $\mathbb{R}$-Cartier et les 1-cycles à coefficients dans $\mathbb{R}$. Les deux $\mathbb{R}$-diviseur de Weil $\mathbb{R}$-Cartier D_1 et D_2 sont dits *numériquement équivalents* et on note $D_1 \equiv D_2$ si $D_1 \cdot C = D_2 \cdot C$ pour tout 1-cycle $C \in Z_1(X)_{\mathbb{R}}$. On note $N^1(X)$ (resp. $N_1(X)$) l'espace vectoriel réel $\mathrm{Div}(X)_{\mathbb{R}}$ (resp. $Z_1(X)_{\mathbb{R}}$) modulo la relation d'équivalence numérique définie ci-dessus. L'espace vectoriel $N^1(X)$ est de dimension finie ; sa dimension est appelée le nombre de Picard de X et notée $\rho(X)$. Le cône convexe fermé de $N_1(X)$ engendré par les classes des 1-cycles effectifs de $N_1(X)$ est noté $\overline{\mathrm{NE}}(X)$. Un $\mathbb{R}$-diviseur de Weil $\mathbb{R}$-Cartier $D \in \mathrm{Div}(X)_{\mathbb{R}}$ est dit *nef* si pour tout $C \in \overline{\mathrm{NE}}(X)$, on a $D \cdot C \geq 0$.

Soit D un diviseur de Cartier. Alors il existe une application rationnelle $\phi_{|D|} \colon X \dashrightarrow \mathbb{P}(H^0(X, \mathcal{O}_X(D)))$ qui est définie dehors du lieu base Bs $|D|$ de $|D|$. Un $\mathbb{Q}$-diviseur de Weil $\mathbb{Q}$-Cartier D est dit *ample* (resp. *gros* et *semi-ample*) s'il existe un entier positif m tel que mD est un diviseur de Cartier et l'application rationnelle $\Phi_{|mD|} \dashrightarrow \mathbb{P}(H^0(X, \mathcal{O}_X(mD)))$ est un plongement (resp. birationnelle et un morphisme).

On note Eff(X) le cône convexe de $N^1(X)$ engendré par les classes des $\mathbb{R}$-diviseurs de Weil $\mathbb{R}$-Cartier effectifs et Psf(X) son adhérence. Un $\mathbb{R}$-diviseur de Weil $\mathbb{R}$-Cartier $D \in \mathrm{Div}(X)_{\mathbb{R}}$ est dit *pseudoeffectif* si sa classe dans $N^1(X)$ est dans Psef(X).

Soit $f \colon X \to Z$ un morphisme projectif de variétés quasi-projectives normales. Un $\mathbb{Q}$-diviseur de Weil $\mathbb{Q}$-Cartier D est dit f*-nef* si $D \cdot C \geq 0$ pour toute courbe irréductible C avec $f(C)$ un point. On dit que D est f*-ample* s'il existe un entier positif m tel que mD soit Cartier et le morphisme canonique

$$\rho \colon f^* f_* \mathcal{O}_X(mD) \to \mathcal{O}_X(mD)$$

soit surjectif et il définit un plongement de schémas au-dessus Z.

$$\begin{tikzcd}
X \arrow[rr, hook, "j"] \arrow[dr, "f"'] & & \mathbb{P}(f_* \mathcal{O}_X(mD)) \arrow[dl] \\
& Z &
\end{tikzcd}$$

0.2 Pente des faisceaux cohérents

Nous allons introduire ici des notions utiles à l'étude des faisceaux cohérents. La référence est [OSS11, §II.1]. Soit $\mathcal{F}$ un faisceau cohérent sur une variété algébrique normale X. L'*ensemble singulier* de $\mathcal{F}$ est donné par

$$\mathrm{Sing}(\mathcal{F}) \colon = \{x \in X | \mathcal{F}_x \text{ n'est pas un module libre sur } \mathcal{O}_{X,x}\}.$$

0.2.1. Proposition [OSS11, Corollary, p.145]. *L'ensemble singulier* Sing$(\mathcal{F})$ *d'un faisceau cohérent* $\mathcal{F}$ *sur une variété algébrique* X *est une sous-variété de codimension au moins* 1.

Ainsi, sur $X \setminus \mathrm{Sing}(\mathcal{F})$, $\mathcal{F}$ est localement libre. Si X est connexe, on peut définir le rang du faisceau cohérent $\mathcal{F}$ par

$$\mathrm{rg}(\mathcal{F}) \colon = \mathrm{rg}(\mathcal{F}|_{X \setminus \mathrm{Sing}(\mathcal{F})}).$$

0.2.2. Définition. *Un faisceau cohérent* $\mathcal{F}$ *sur une variété* X *est dit sans torsion si tout germe* $\mathcal{F}_x$ *est un* $\mathcal{O}_{X,x}$*-module sans torsion, i.e., si* $f \in \mathcal{F}_x$ *et* $a \in \mathcal{O}_{X,x}$ *sont tels que* $af = 0$*, alors, ou* $f = 0$ *ou* $a = 0$*.*

Les faisceaux localement libres sont sans torsion, les sous-faisceaux de faisceaux sans torsion sont sans torsion.

0.2.3. Proposition [OSS11, Corollary, p.148]. *L'ensemble singulier d'un faisceau cohérent sans torsion est au moins de codimension* 2.

Le dual d'un faisceau cohérent $\mathcal{F}$ est le faisceau $\mathcal{F}^\vee \colon = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$. Il y a un morphisme naturel $\mu \colon \mathcal{F} \to \mathcal{F}^{\vee\vee}$. Le noyau de ce morphisme est le sous-faisceau torsion $T(\mathcal{F})$ de $\mathcal{F}$ ([GR84, p.69] pour une preuve dans le cadre analytique).

0.2.4. Définition. *Le faisceau cohérent* $\mathcal{F}$ *est dit réflexif si le morphisme naturel* μ *de* $\mathcal{F}$ *vers son bidual* $\mathcal{F} \to \mathcal{F}^{\vee\vee}$ *est un isomorphisme.*

Le faisceau $\mathcal{F}^{\vee\vee}$ est un objet universel au sens suivant : soit $\nu \colon \mathcal{F} \to \mathcal{G}$ un morphisme de faisceaux cohérents, où $\mathcal{G}$ est réflexif, alors ν se factorise de façon unique par $\mu \colon \mathcal{F} \to \mathcal{F}^{\vee\vee}$. Les faisceaux localement libres sont des faisceaux réflexifs et les faisceaux réflexifs sont sans torsion.

0.2.5. Proposition [OSS11, Lemma 1.1.10, p.149]. *Supposons que X est lisse. L'ensemble singulier* $\mathrm{Sing}(\mathcal{F})$ *d'un faisceau réflexif est de codimension au moins* 3.

Le critère suivant dû à Hartshorne est utile.

0.2.6. Proposition [Har80, Proposition 1.1]. *Un faisceau cohérent $\mathcal{F}$ sur un schéma intégral et séparé X est réflexif si et seulement s'il peut-être inclu localement dans une suite exacte*

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{E} \longrightarrow \mathcal{Q} \longrightarrow 0,$$

où $\mathcal{E}$ est localement libre et $\mathcal{Q}$ est sans torsion. En particuler, le dual de tout faisceau cohérent est réflexif.

A tout diviseur D de Weil sur une variété projective, le faisceau $\mathcal{O}_X(D)$ est un faisceau réflexif de rang un. Deux faisceaux réflexifs $\mathcal{F}_1$ et $\mathcal{F}_2$ sont isomorphes et on note $\mathcal{F}_1 \cong \mathcal{F}_2$ s'il existe un $\mathcal{O}_X$-isomorphisme $f\colon \mathcal{F}_1 \to \mathcal{F}_2$. Alors, si X est une variété projective normale, on a un isomorphisme de groupes

$$Z^1(X)_{\mathbb{Z}}/\sim \longrightarrow \{\text{faisceaux réflexifs de rang un}\}/\cong .$$

En particulier, si X est lisse (ou plus généralement factorielle), un faisceau réflexif de rang un est un fibré en droites. Soit $\mathcal{F}$ un faisceau cohérent sans torsion de rang r sur une variété projective lisse X. Le fibré en droites déterminant associé à $\mathcal{F}$ est défini par $\det(\mathcal{F}) = (\wedge^r \mathcal{F})^{\vee\vee}$ et la première classe de Chern de $\mathcal{F}$ est définie par $c_1(\mathcal{F}) = c_1(\det(\mathcal{F}))$. Soit H un diviseur ample sur X. La pente de $\mathcal{F}$ par rapport à H est donnée par

$$\mu_H(\mathcal{F}) \colon = \frac{c_1(\mathcal{F}) \cdot H^{\dim(X)-1}}{\mathrm{rg}(\mathcal{F})}.$$

0.2.7. Définition. *Un faisceau cohérent sans torsion $\mathcal{F}$ non nul sur X est dit H-stable (resp. H-semistable) si pour tout sous-faisceau cohérent $\mathcal{E} \subset \mathcal{F}$, $0 < \mathrm{rg}(\mathcal{E}) < \mathrm{rg}(\mathcal{F})$, on a*

$$\mu_H(\mathcal{E}) < \mu_H(\mathcal{F}) \qquad (resp. \quad \mu_H(\mathcal{E}) \leq \mu_H(\mathcal{F})).$$

Rappelons qu'un morphisme injectif de faisceaux cohérents sans torsion de même rang $\alpha\colon \mathcal{E} \to \mathcal{E}'$ induit un morphisme injectif des fibrés en droites déterminant

$$\det(\alpha)\colon \det(\mathcal{E}) \to \det(\mathcal{E}').$$

En particulier, on a $\mu_H(\mathcal{E}) \leq \mu_H(\mathcal{E}')$, donc il suffit de considérer les sous-faisceaux réflexifs dans Définition 0.2.7 si $\mathcal{F}$ est réflexif.

0.3 Les courbes rationnelles

On rappelle dans ce paragraphe des résultats sur les courbes rationnelles. Les références sont [Kol96] et [Deb01]. Soit X une variété projective lisse. Grâce au résultat fondamental de Boucksom-Demailly-Paŭn-Peternell [BDPP13], le fibré canonique K_X n'est pas pseudoeffectif si et seulement s'il existe une famille couvrante de courbes $(C_t)_{t\in T}$ de X telle que $-K_X \cdot C_t > 0$. D'après le lemme du cassage suivant (« bend and break ») de Mori, on sait que X est recouverte par des courbes rationnelles.

0.3.1. Théorème [Mor82]. *Soit X une variété projective lisse et soit $C \subset X$ une courbe irréductible. Si $K_X \cdot C < 0$, alors par tout point de C passe une courbe rationnelle.*

On pourrait aussi se demander si on peut caractériser les variétés uniréglées par la positivité du fibré tangent. Avant d'énoncer le théorème, on introduit des notations.

0.3.2. Définition. *Soit X une variété projective normale de dimension n.*

(1) *Une courbe C est dite MR-générale si elle est obtenue comme intersection complète*

$$C := D_1 \cap \cdots \cap D_{n-1}$$

avec $D_i \in |m_i H_i|$ général pour $H_1, \cdots, H_{n-1}$ une collection de diviseurs amples et $m_i \gg 0$.

(2) *Soit $\mathcal{E}$ un faisceau cohérent sans torsion. On dit que $\mathcal{E}$ est génériquement nef si la restriction $\mathcal{E}|_C$ est nef pour toute courbe MR-générale C.*

Le résultat suivant montré par Miyaoka est important.

0.3.3. Théorème [Miy87b, Theorem 8.5]. *Soit X une variété projective lisse et soit C une courbe MR-générale. Si X n'est pas uniréglée, alors le fibré vectoriel $\Omega^1_X|_C$ est génériquement nef.*

Maintenant on fixe une polarisation sur X. Pour deux entiers $k \geq 0$ et $d \geq 0$, on note par $\mathrm{Chow}_{k,d}(X)$ la variété projective parmétrant les cycles effectifs de dimension k et de degré d sur X. L'ensemble

$$\mathrm{Chow}(X) := \bigsqcup_{k \geq 0, d \geq 0} \mathrm{Chow}_{k,d}(X)$$

est appelé la variété de Chow de X. Le sous-ensemble $\mathrm{RatCurves}(X)$ est un ouvert de $\mathrm{Chow}(X)$ définie comme suivant

$$\mathrm{RatCurves}(X) := \{[C] \in \mathrm{Chow}(X) \mid C \text{ une courbe rationnelle irréductible}\}.$$

$\mathrm{RatCurves}^n(X)$ est la normalisation de $\mathrm{RatCurves}(X)$. Une famille de courbes rationnelles sur X est une composante irréductible $\mathcal{V}$ de $\mathrm{RatCurves}^n(X)$. Soient $\mathcal{V}$ une famille de courbes rationnelles sur X et $\mathcal{U}$ la normalisation de la famille universelle au-dessus de $\mathcal{V}$. On note π et e

$$\begin{array}{ccc} \mathcal{U} & \xrightarrow{e} & X \\ \downarrow{\scriptstyle \pi} & & \\ \mathcal{V} & & \end{array}$$

les restrictions à $\mathcal{U}$ des projections de $\mathcal{V} \times X$ sur $\mathcal{V}$ et X respectivement.

0.3.4. Définition. *On dit qu'une famille de courbes rationnelles $\mathcal{V}$ est dominante si l'application d'évaluation $e\colon \mathcal{U} \to X$ de la famille universelle $\mathcal{U} \to \mathcal{V}$ est dominante. On dit qu'une famille dominante de courbes rationnelles $\mathcal{V}$ est minimale si pour un point général $x \in X$ le sous-ensemble $\mathcal{V}_x \subset$ RatCurves(X, x) qui paramètre les courbes de la famille $\mathcal{U}$ contenant x est propre.*

En particuler, une famille dominante de courbes rationnelles de degré minimal par rapport à une polarisation donnée est une famille minimale ([Kol96, Theorem 2.4]). Les familles minimales de courbes rationnelles ont des propriétés intéressantes : pour un membre général $[\ell]$ de la famille de $\mathcal{V}$ avec la normalisation $f\colon \mathbb{P}^1 \to \ell \subset X$ on a

$$f^*T_X \cong \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus p} \oplus \mathcal{O}_{\mathbb{P}^1}^{\oplus q},$$

on dit que c'est une courbe rationnelle standard. Soit $\mathcal{V}$ une famille dominante dont un membre général $f\colon \mathbb{P}^1 \to X$ est libre (disons que f^*T_X est nef), alors $\mathcal{V}$ est minimale si et seulement si $f\colon \mathbb{P}^1 \to X$ est une courbe rationnelle standard ([Kol96, Corollary 2.9]). Fixons maintenant une famille minimale $\mathcal{V}$ de courbes rationnelles sur X et x un point général de X. Soient $\bar{\mathcal{V}}_x$ la normalisation de $\mathcal{V}_x$ et τ_x l'application rationnelle

$$\begin{aligned} \tau_x\colon \bar{\mathcal{V}}_x &\dashrightarrow \mathbb{P}(T^\vee_{X,x}) \\ [\ell] &\mapsto \mathbb{P}(T^\vee_{\ell,x}) \end{aligned}$$

et on note $\mathcal{C}_x$ l'adhérence de l'image de τ_x. Alors l'application $\tau_x\colon \bar{\mathcal{V}}_x \dashrightarrow \mathcal{C}_x$ est finie et birationnelle ([Keb02, HM04]). En particulier, elle coïncide avec la normalisation de $\mathcal{C}_x$.

0.3.5. **Définition**. *La variété $\mathcal{C}_x$ est appelée la variété des tangentes en x (≪ variety of minimal rational tangents ≫ en anglais).*

Les variétés des tangentes ont de nombreuses applications dans l'étude des variétés projectives lisses ([Kol96, HM98, Hwa01, Keb02, HM05, Hwa06, Hwa07, DH17] etc.). Pour chaque famille dominante de courbes rationnelles, on peut associer une application presque régulière.

0.3.6. **Définition**. *Soient X et T des variétés projectives et quasi-projectives respectivement et $q\colon X \dashrightarrow T$ une application rationnelle. On dit que q est presque régulière s'il existe des ouverts denses X_0 et T_0 de X et T respectivement tels que la restriction de q à X_0 induise un morphisme propre et surjectif $q_0\colon X_0 \to T_0$.*

Soit $\mathcal{V}$ une famille dominante de courbes rationnelles sur X avec $\bar{\mathcal{V}}$ son adhérence dans $\mathrm{Chow}(X)$.

0.3.7. **Définition**. *Les points x et x' de X sont dits $\bar{\mathcal{V}}$-équivalents s'il existe des courbes $[C_1], \cdots, [C_k] \in \bar{\mathcal{V}}$ tels que $Supp(C_1) \cup \cdots \cup Supp(C_k)$ soit connexe et contienne x et x', où $Supp(C_i)$ désigne le support du cycle C_i.*

0.3.8. **Théorème [Cam92, KMM92]**. *Soient X une variété projective lisse et $\mathcal{V}$ une famille dominante de courbes rationnelles sur X. Il existe une variété quasi-projective T et une application rationnelle presque régulière $q\colon X \dashrightarrow T$ dont les fibres générales sont les classes d'équivalence pour la relation de $\bar{\mathcal{V}}$-équivalence. L'application q est unique à équivalence birationnelle près et est appelé le quotient $\bar{\mathcal{V}}$-rc de X.*

0.4 Les singulariés des paires

On rappelle dans ce sous-paragraphe quelques notations et résultats concernant les singularités des paires. Les références sont [Kol97] et [Kol13]. Si $\Delta = \sum a_i\Delta_i$ est un $\mathbb{Q}$-diviseur avec Δ_i distincts, on notera $\lfloor\Delta\rfloor = \sum_i \lfloor a_i\rfloor\Delta_i$ la partie entière, où $\lfloor x\rfloor$ désigne la partie entière d'un réel x.

0.4.1. **Definition**. *Une résolution logarithmique (ou log-résolution) de (X, Δ) est un morphisme birationnel $\mu\colon Y \to X$ tel que Y soit lisse, le lieu exceptionnel $Ex(\mu)$ de μ soit un diviseur et $Ex(\mu) \cup \mathrm{Supp}(\mu_*^{-1}\Delta)$ soit un diviseur à croisements normaux simples.*

L'existence de log-résolutions est montrée par Hironaka et elle est généralisée par Szabó.

0.4.2. **Théorème [Kol13, Theorem 4.45]**. *Soient X une variété quasi-projective normale et Δ un diviseur effectif de Weil sur X. Alors (X, Δ) admet une log-résolution $\pi\colon \widetilde{X} \to X$ telle que π est un isomorphisme au-dessus du lieu snc de (X, Δ).*

0.4.3. **Définition**. *Une paire (X, Δ) est la donnée d'une variété quasi-projective X normale et d'un $\mathbb{Q}$-diviseur de Weil effectif Δ dont les coefficients sont tous compris entre 0 et 1 tels que $K_X + \Delta$ soit $\mathbb{Q}$-Cartier. On dit que le diviseur Δ est une frontière.*

Soient (X, Δ) une paire et $\mu\colon Y \to X$ une résolution des singularités de (X, Δ). Ecrivons

$$K_Y + \mu_*^{-1}\Delta = \mu^*(K_X + \Delta) + \sum a(E, X, \Delta)E$$

où la somme porte sur l'ensemble des diviseurs premiers μ-exceptionnels. Les coefficients $a(E, X, \Delta)$ ne dépendent pas du choix des diviseurs canoniques K_Y et K_X par le lemme de négativité.

0.4.4. **Définition**. *Le nombre rationnel $a(E, X, \Delta)$ est appelé la discrépance du diviseur E relativement à la paire (X, Δ).*

Un diviseur F est dit exceptionnel sur X s'il existe une variété projective normale Y et un morphisme birationnel $\pi\colon Y \to X$ tels que $F \subset Y$ et $\pi(F)$ soit de codimension au moins 2. On définit la discrépance de (X, Δ) par

$$discrep(X, \Delta) := \inf\{a(F, X, \Delta) \mid F \text{ est exceptionnelle sur } X\}.$$

Les singularités de la paire (X, Δ) peuvent être classées selon la discrépance de (X, Δ).

0.4.5. Définition. *Une paire (X, Δ) est dite*

$$\begin{cases} \textit{terminale} \\ \textit{canonique} \\ \textit{lt pour log terminale} \\ \textit{klt pour Kawamata log terminale} \\ \textit{plt pour purement log terminale} \\ \textit{lc pour log canonique} \end{cases} \quad \textit{si} \quad discrep(X, \Delta) \begin{cases} > 0 \\ \geq 0 \\ > -1 \textit{ et } \Delta = 0 \\ > -1 \textit{ et } \lfloor \Delta \rfloor = 0 \\ > -1 \\ \geq -1 \end{cases}$$

On peut montrer que si $discrep(X, \Delta) < -1$ alors on a en fait $discrep(X, \Delta) = -\infty$. Soit (X, Δ) une paire. On s'intéresse au lieu où la paire (X, Δ) n'est pas klt. On note

$$nklt(X, \Delta) = \{x | (X, \Delta) \text{ n'est pas klt au voisinage de } x\}.$$

0.4.6. Définition. *On appelle centre de singularités log canoniques l'image W dans X d'un diviseur irréductible de discrépance -1 sur un modèle birationnel de X, tel que la paire (X, Δ) soit log-canonique au point générique de W.*

Si (X, Δ) est log-canonique, $nklt(X, \Delta)$ est égal à l'union des centres de singularités log canoniques de X. Pour une paire log canonique, ces centres sont en nombre fini. En considérant une log-résolution $\mu\colon Y \to X$ de la paire (X, Δ), on les obtient comme l'image d'une intersection quelconque de diviseurs de discrépance -1.

0.4.7. Définition. *Un centre de singulartiés log canoniques maximal est un élément maximal pour l'inclusion. Un centre de singularités log canoniques minimal est un élément minimal pour l'inclusion.*

Parmi les centres minimaux, on peut considérer la classe *a priori* beaucoup plus restreinte des centres de singularités log canoniques exceptionnelles.

0.4.8. Définition. *Soit (X, Δ) une paire log-canonique, $\mu\colon Y \to X$ une log-résolution de la paire (X, Δ). Un centre de singularités log-canoniques W et dit exceptionnel si les deux propriétés suivantes sont vérifiées :*

(1) il existe un unique diviseur E_W de discrépance -1 sur Y dont l'image dans X est W,
(2) pour tout diviseur $E' \neq E_W$ sur Y de discrépance -1, $f(E) \cap W = \emptyset$.

On remarque qu'un centre de singularités log-canoniques exceptionnel est une composante connexe du lieu non-klt de la paire (X, Δ). Cette dernière propriété nous sera fort utile pour construire des sections non nulles de fibrés en droites sur le lieu non-klt de certaines paires (X, Δ).

0.4.9. Théorème (Tie-breaking, [Bro09, Théorème 3.7]). *Soit (X, Δ) une paire klt et D un $\mathbb{Q}$-diviseur $\mathbb{Q}$-Cartier effectif tel que $(X, \Delta + D)$ soit log canonique et non klt. On note W un centre de singularités log canoniques minimales pour la paire $(X, \Delta + D)$ et H un diviseur de Cartier ample sur X. Pour tout rationnel $0 < r \leq 1$, il existe des rationnels $0 \leq c_1 \leq r$ et $0 \leq c_2 \leq r$ et un $\mathbb{Q}$-diviseur effectif $A \sim_{\mathbb{Q}} c_1 H$ tels que la paire $(X, \Delta + (1 - c_2 D) + A)$ soit log canonique et W soit un centre de singularités log canoniques exceptionnel pour $(X, \Delta + (1 - c_2)D + A)$.*

La singularité log-canonique peut être assez compliquée du point de vue cohomologique, car elle n'est pas rationnelle. Pour éviter ce problème, on introduit la notation dlt.

0.4.10. Définition. *Une paire (X, Δ) est dite de singularités dlt pour divisoriellement log terminale si les coefficients de Δ sont inférieurs à 1 et s'il existe un ouvert $X_0 \subset X$ tel que X_0 soit lisse, $\Delta|_{X_0}$ un diviseur dont le support est à croisements normaux simples et $a(E, X, \Delta) > -1$ pour tout diviseur exceptionnel E sur X dont l'image dans X est contenu dans $X \setminus X_0$.*

Soit (X, Δ) une paire dlt. Alors X est à singularités rationnelles. En plus, une paire dlt (X, Δ) est klt si et seulement si $\lfloor \Delta \rfloor = 0$. De plus, une paire dlt est « limite » d'une suite de paires klt. De façon plus précise, si A est un diviseur ample sur X, il existe un réel $c > 0$ et un $\mathbb{Q}$-diviseur $\Delta_1 \sim_{\mathbb{Q}} \Delta + cA$ tel que la paire $(X, (1 - \epsilon)\Delta + \epsilon\Delta_1)$ soit klt pour tout $0 < \epsilon \ll 1$. De plus, les composantes irréductibles de $\lfloor \Delta \rfloor$ sont normales. La proposition suivante explique le lien entre les singularités de paires plt et dlt.

0.4.11. Proposition [KM98, Proposition 5.51]. *Soit (X, Δ) une paire dlt. Alors (X, Δ) est plt si et seulement si $\lfloor \Delta \rfloor$ est réunion disjointe de ses composantes irréductibles.*

L'introduction des paires est motivée par la théorie de l'adjonction. En particulier, on a la formule suivante.

0.4.12. Théorème [Kol13, Theorem 4.9]. *Soient (X, Δ) une paire lc et $S \subset \lfloor \Delta \rfloor$ une composante de $\lfloor \Delta \rfloor$ à coefficient 1. Alors il existe un diviseur Δ_S sur S tel que*

(1) $(K_X + \Delta)|_S = K_S + \Delta_S$;
(2) si (X, Δ) est plt, alors (S, Δ_S) est klt ;
(3) si Δ est $\mathbb{Q}$-Cartier et S est klt au voisinage de S, alors (S, Δ_S) est lc ;
(4) supposons de plus que S est Cartier en codimension 2. Si (X, Δ) est dlt, alors (S, Δ_S) est dlt.

Une vaste généralisation du théorème d'adjonction pour les diviseurs lisses, démontrée d'abord par Kawamata dans [Kaw98] et puis simplée par Fujino et Gongyo [FG12], permet de munir d'une structure de paire, dépendant du (log)-diviseur canonique de (X, Δ), les centres log-canoniques minimaux de la paire (X, Δ).

0.4.13. Théorème [FG12, Theorem 1.2]. *Soient (X, Δ) une paire lc et W un centre lc minimal par rapport de (X, Δ). Alors W est normale et il existe une diviseur effective Δ_W sur W tel que*

$$(K_X + \Delta)|_W \sim_{\mathbb{Q}} K_W + \Delta_W$$

et la paire (W, Δ_W) soit klt. En particulier, W est à singularités rationnelles.

0.4.14. Definition. *Soit X une variété projective normale telle que K_X soit $\mathbb{Q}$-Cartier. Soit $g : Y \to X$ un morphisme birationnel propre On dit que $g : Y \to X$ est une modification terminale si Y est à singularités terminales et K_Y est g-nef.*

L'existence de modification terminale est montré dans [BCHM10, Corollary 1.4.4], mais en général elle n'est pas unique.

0.5 Programme du modèle minimal

Dans ce sous-paragraphe on discutera du programme du modèle minimal. Les textes de référence pour ce paragraphe sont [Kol91b] et [KM98]. Une des questions centrales en géométrie algébrique est celle de la classification des variétés à équivalence birationnelle près. Il s'agit d'une relation qui identifie deux variétés qui sont isomorphes le long d'un ouvert de Zariski, ou, de manière équivalente, qui diffèrent par des lieux de dimension strictement plus petite. Le programme de Mori ou programme des modèles minimaux ou encore MMP (« minimal model program » en anglais) est de construire un algorithme explicite qui permette à partir d'une variété X d'obtenir un « bon » représentant birationnel.

0.5.1. Définition. *Soient (X, Δ) et (X', Δ') deux paires klt avec une application birationnelle $\varphi : X \dashrightarrow X'$ telles que*

(1) φ^{-1} *ne contracte pas de diviseur ;*
(2) $\Delta' = \varphi_*\Delta$ *;*
(3) $a(F, X, \Delta) < a(F, X', \Delta')$ *pour tout diviseur premier* φ*-exceptionnel* F*.*

Alors, on dit que (X', Δ') *est un modèle minimal de* (X, Δ) *si* $K_{X'} + \Delta'$ *est nef.*

0.5.2. Définition. *Une fibration de Mori est une paire klt* (X, Δ) *avec une fibration à fibres connexes* $f \colon X \to Y$ *avec* Y *normale telle que* $\dim(Y) < \dim(X)$*,* $-(K_X + \Delta)$ *soit* f*-ample et* $\rho(X/Y) = 1$*.*

Étant donné une variété projective X, on peut supposer que X est lisse à une résolution près. Le MMP prédit l'existence d'une variété projective X' birationnellement équivalente à X, peu singulière, telle que ou bien X' soit un modèle minimal de X, ou bien X' admet une fibration de Mori. Le premier pas du MMP est le résultat suivant dit ≪ théorème du cône ≫.

0.5.3. Théorème [KM98, Theorem 3.7]. *Soit* (X, Δ) *une paire dlt.*

(1) *Il existe une famille au plus dénombrable* $\{C_i\}_{i \in I}$ *de courbes rationnelles* $C_i \in Z_1(X)$*, telle que* $0 < -(K_X + \Delta) \cdot C_i \leq 2\dim(X)$*,* $R_i = \mathbb{R}_{\geq 0}[C_i]$ *une arête du cône* $\overline{NE}(X)$ *et*

$$\overline{NE}(X) = \overline{NE}(X)_{K_X + \Delta \geq 0} + \sum_{i \in I} R_i.$$

(2) *Pour tout diviseur ample* A*, il n'y a qu'un nombre fini d'arêtes du cône* $\overline{NE}(X)$ *contenues dans le demi-espace* $N_1(X)_{K_X + \Delta + A < 0}$*.*
(3) *Soit* $i \in I$*. Il existe un morphisme projectif à fibres connexes* $\varphi_i \colon X \to X_i$ *tel que, pour toute courbe complète* $C \subset X$*,* $\dim(\varphi_i(C)) = 0$ *si et seulement si* $[C] \in R_i$ *; le morphisme* φ_i *est appelé la contraction de* R_i*.*
(4) *Si* $\varphi_i \colon X \to X_i$ *est comme ci-dessus et si* L *est un fibré en droites sur* X *tel que* $L \cdot C = 0$ *pour toute* $[C] \in R_i$*, alors il existe un fibré en droites* L_i *sur* X_i *tel que* $L \cong \varphi_i^* L_i$*.*

Soit $\varphi_i \colon X \to X_i$ un morphisme comme ci-dessus. Il y a trois possibilités pour la variété X_i.

(1) $\dim(X_i) = \dim(X)$ et le lieu exceptionnel $Ex(\varphi_i)$ est de codimension 1. La contraction φ_i est dit divisorielle.
(2) $\dim(X_i) = \dim(X)$ et le lieu exceptionnel $Ex(\varphi_i)$ est de codimension au moins 2. La contraction φ_i est dit petite.
(3) $\dim(X_i) < \dim(X)$. Alors φ_i est une fibration de Mori dont la fibré générale est une variété de Fano.

Quand $\dim(X_i) = \dim(X)$, alors le morphisme φ_i est birationnel et on peut poser $\Delta_i = \varphi_{i*}\Delta$. Si φ_i est petit, la situation se complique singulièrement : le diviseur $K_{X_i} + \Delta_i$ n'est pas $\mathbb{Q}$-Cartier. Il faut introduire une nouvelle transformation birationnelle : le *flip* de φ_i.

0.5.4. Définition. *Le flip de* φ_i *est un morphisme birationnel projectif* $\varphi_i^+ \colon X_i^+ \to X_i$ *où* X_i^+ *est une variété normale, dont le lieu exceptionnel* $Ex(\varphi_i^+)$ *est de codimension au moins deux dans* X_i^+ *et tel que* $K_{X_i^+} + \Delta_i^+$ *soit* $\mathbb{Q}$*-Cartier et* φ_i^+*-ample où* Δ_i^+ *est le transformé strict de* Δ_i *dans* X_i^+*. De plus, la paire* (X_i^+, Δ_i^+) *est encore dlt.*

Si $f \colon X \dashrightarrow X'$ est une contraction divisorielle ou un flip associée à une contraction petite, alors X' est au plus aussi singulière que X. Plus précicément :

0.5.5. Proposition [KM98, Corollary 3.42]. *Soit* (X, Δ) *une paire klt (resp. canonique, terminale) telle que* X *soit* $\mathbb{Q}$*-factorielle. Soit* $f \colon X \to X'$ *une contraction petite. Soit* $X^+ \to X'$ *un* $(K_X + \Delta)$*-flip de* f*, alors la paire* (X^+, Δ^+) *est klt (resp. canonique, terminale), où* Δ^+ *est la tranformée stricte de* Δ *dans* X^+*.*

0.5.6. Proposition [KM98, Corollary 3.43]. *Soit* (X, Δ) *une paire klt telle que* X *soit* $\mathbb{Q}$*-factorielle. Soit* $f \colon X \to X'$ *une contraction divisorielle. Alors* X' *est aussi* $\mathbb{Q}$*-factorielle et* (X', Δ') *est aussi klt, où* Δ' *est la transformée stricte de* Δ *dans* X'*.*

Soit (X, Δ) une paire klt telle que X soit projective $\mathbb{Q}$-factorielle. Le MMP pour la paire (X, Δ) est une suite de transformations birationnelles

$$(X, \Delta) = (X_0, \Delta_0) \overset{\varphi_0}{\dashrightarrow} (X_1, \Delta_1) \overset{\varphi_1}{\dashrightarrow} (X_2, \Delta_2) \overset{\varphi_2}{\dashrightarrow} \cdots \overset{\varphi_{k-1}}{\dashrightarrow} (X_k, \Delta_k)$$

telle que φ_i soit une contraction divisorielle ou une contraction petite, Δ_{i+1} soit la transformée stricte de Δ_i et (X_k, Δ_k) soit un modèle minimal ou une fibration de Mori. Si $X_i \to X_{i+1}$ est une contraction divisorielle, alors on a $\rho(X_{i+1}) = \rho(X_i) - 1$. Comme conséquence, pour que le MMP soit complet, il reste à montrer l'existence des flips et l'aboutissement de flips. L'existence des flips en toutes dimensions est montrée dans [BCHM10] et la non-existence de suite infinie de flips est dû à Shokurov si $\dim(X) = 3$ et elle est une question encore ouverte en dimension ≥ 4. Par contre, on peut considérer le MMP dirigé qui est une version du MMP où les arêtes contractées ne sont pas choisies de façon arbitraire. En particulier, Birkar-Cascini-Hacon-M^{c}Kernan ont montré dans [BCHM10] la non-existence de suite infinie de flips dirigés lorsque Δ est gros et ils ont obtenus les résultats suivants.

0.5.7. Théorème [BCHM10, Theorem 1.1 et Corollary 1.3.3]. *Soit (X, Δ) une paire klt.*

(1) Si Δ est gros et $K_X + \Delta$ est pseudo-effectif, alors (X, Δ) a un modèle minimal.
(2) Si $K_X + \Delta$ n'est pas pseudo-effectif, un MMP dirigé pour (X, Δ) donne une paire klt (X', Δ') et une fibration de Mori $f\colon X' \to Y$.

Première partie

Subsheaves of the tangent bundle

Chapitre 1

Introduction to Part I

1.1 Main results

Special subsheaves $\mathcal{E}$ of the tangent bundle T_X of a projective manifold X often carry important geometric information on X. The most important special properties $\mathcal{E}$ can have are :
– maximality of slope with respect to some polarization
– integrability
– positivity
Maximality means that $\mathcal{E}$ is a destabilizing subsheaf with respect to a given polarization. This leads to the slope stability properties of tangent bundle. An integrable subsheaf defines a foliation on X, and then the structure of the leaves of the foliation will give information on the structure of X. Positivity means that $\mathcal{E}$ is a coherent sheaf such that the invertible sheaf $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ satisfies some positivity conditions in the sense of algebraic geometry (e.g. ampleness, semi-ampleness, pseudo-effectivity, nefness or bigness etc.). In the special case that $\mathcal{E} = T_X$ and $\mathcal{E}$ is ample, this is the famous Hartshorne's conjecture proved by Mori.

Theorem A [Mor79, Theorem 8]. Let X be a projective manifold defined over an algebraically closed field **k** of characteristic arbitrary. Then X is a projective space if and only if T_X is ample.

This result has been generalized, over the field of complex numbers, by several authors (see [Wah83, CP98, AW01]).

Theorem B [AW01, Theorem]. Let X be a complex projective manifold of dimension n. If T_X contains an ample locally free subsheaf $\mathcal{E}$ of rank r, then $X \cong \mathbb{P}^n$ and $\mathcal{E} \cong \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus r}$ or $\mathcal{E} \cong T_{\mathbb{P}^n}$.

This theorem was successively proved for $r = 1$ by Wahl [Wah83] and latter for $r \geq n-2$ by Campana and Peternell [CP98]. The proof was finally completed by Andreatta and Wiśniewski [AW01]. We generalize these results by dropping the locally freeness condition.

1.1.1. Theorem (= Theorem 2.3.9). *Let X be a complex projective manifold of dimension n. Suppose that T_X contains an ample subsheaf $\mathcal{F}$ of positive rank r, then $(X, \mathcal{F})$ is isomorphic to $(\mathbb{P}^n, T_{\mathbb{P}^n})$ or $(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus r})$.*

We refer to § 2.1.1 for the basic definition and properties of ample sheaves. Comparing with Theorem B, we do not require *a priori* the locally freeness of the subsheaf $\mathcal{F}$ in Theorem 1.1.1. In the case where the Picard number of X is one, Theorem 1.1.1 was proved in [AKP08]. In fact, in [AKP08], it was shown that the subsheaf $\mathcal{F}$ must be locally free under the additional assumption $\rho(X) = 1$, and then Theorem B immediately implies Theorem 1.1.1. In particular, to prove Theorem 1.1.1, it suffices to show that X is isomorphic to some projective space if its tangent bundle contains an ample subsheaf $\mathcal{F}$; then, the locally freeness of $\mathcal{F}$ follows from [AKP08]. An interesting and important special case of Theorem 1.1.1 is when $\mathcal{F}$ comes from the image of an ample vector bundle E over X. This confirms a conjecture of Litt [Lit17, Conjecture 2].

1.1.2. Corollary (= Corollary 2.3.10). *Let X be a projective manifold of dimension n, E an ample vector bundle on X. If there exists a non-zero map $E \to T_X$, then $X \cong \mathbb{P}^n$.*

As an application, we derive the classification of projective manifolds containing a $\mathbb{P}^r$-bundle as an ample divisor. This problem attracted a great deal of interest over the past few decades (see [Som76, Bǎd84, FSS87, BS95, BI09] etc.). Recently, in [Lit17, Corollary 7], Litt proved that it can be reduced to Corollary 1.1.2 . To be more precise, we have the following classification theorem.

1.1.3. Theorem (= Theorem 2.4.2). *Let X be a projective manifold of dimension $n \geq 3$, let A be an ample divisor on X. Assume that A is a $\mathbb{P}^r$-bundle, $p\colon A \to B$, over a manifold B of dimension $b > 0$. Then one of the following holds :*

(1) $(X, A) = (\mathbb{P}(E), H)$ for some ample vector bundle E over B such that $H \in |\mathcal{O}_{\mathbb{P}(E)}(1)|$, and p is equal to the restriction to A of the induced projection $\mathbb{P}(E) \to B$.
(2) $(X, A) = (\mathbb{P}(E), H)$ for some ample vector bundle E over $\mathbb{P}^1$ such that $H \in |\mathcal{O}_{\mathbb{P}(E)}(1)|$, $H = \mathbb{P}^1 \times \mathbb{P}^{n-2}$ and p is the projection to the second factor.
(3) $(X, A) = (Q^3, H)$, where Q^3 is a smooth quadric threefold, H is a smooth quadric surface and p is the projection to one of the factors $H \cong \mathbb{P}^1 \times \mathbb{P}^1$.
(4) $(X, A) = (\mathbb{P}^3, H)$. H is a smooth quadric surface and p is again a projection to one of the factors of $H \cong \mathbb{P}^1 \times \mathbb{P}^1$.

Next we turn our attention to the stability of tangent bundle. To study which Fano manifolds admit a Kähler-Einstein metric has been one of the main problems in Kähler geometry. The celebrated Yau-Tian-Donaldson conjecture asserts that a Fano manifold admits a Kähler-Einstein metric if and only if it is K-polystable. This conjecture has been solved (see [CDS14, CDS15, Tia15]). The slope stability of tangent bundle is a weaker algebraic notation related to the existence of Kähler-Einstein metric. In general, one can destroy the stability by blow-up, therefore it is natural to restrict to Fano manifolds with $b_2 = 1$ and we have the following conjecture.

1.1.4. Conjecture [Pet01, §3]. *Let X be a Fano manifold with Picard number one, then the tangent bundle T_X is stable.*

By the works of Ramanan-Umemura-Azad-Biswas, Reid, Peternell-Wiśniewski and Hwang, the stability of tangent bundle was confirmed for homogeneous spaces [Ram66, Ume78, AB10], Fano manifolds with index one [Rei77], Fano manifolds of dimension at most six [PW95, Hwa98], complete intersections in $\mathbb{P}^N$ [PW95] and Fano manifolds with large index [Hwa01]. The following theorem is a generalization of [PW95, Theorem 1.5] in the Hermitian symmetric spaces cases (see §3.1.1).

1.1.5. Theorem (= Theorem 3.3.5). *Let M be a n-dimensional irreducible Hermitian symmetric space of compact type, and denote by $\mathcal{O}_M(1)$ the ample generator of $\mathrm{Pic}(M)$. Let Y be a submanifold of M such that the restriction $\mathrm{Pic}(M) \to \mathrm{Pic}(Y)$ is surjective. Then the tangent bundle T_Y is stable if one of the following conditions holds.*

(1) There exists a collection of hypersurfaces $H_i \in |\mathcal{O}_M(d_i)|$ with $d_i \geq 2$ and $1 \leq i \leq r \leq n-2$ such that the complete intersections $H_1 \cap \cdots \cap H_j$ are smooth for all $1 \leq j \leq r$ and $Y = H_1 \cap \cdots \cap H_r$.
(2) Y is a smooth hypersurface.

The proof of this theorem is based on the study the vanishings of the cohomological groups of the form $H^q(Y, \Omega_Y^p(\ell))$. Let $M = G/P$ be a rational homogeneous space of Picard number one and let $\mathcal{O}_M(1)$ be the ample generator of $\mathrm{Pic}(M)$. We consider the bundle of twisted holomorphic p-forms $\Omega_M^p(\ell) = \Omega_M^p \otimes \mathcal{O}_M(\ell)$. Some vanishing theorems for the cohomology of $\Omega_M^p(\ell)$ were first obtained by Bott in [Bot57] (e.g. $X \cong \mathbb{P}^n$), and then this is extended to Grassmannians by Le Potier in [LP75] and to quadric hypersurfaces by Shiffman and Sommese in [SS85]. Making use of the work by Kostant in [Kos61], Snow developed an algorithm in [Sno86] and [Sno88] to determine whether a given cohomology group $H^q(M, \Omega_M^p(\ell))$ vanishes when M is an irreducible Hermitian symmetric space of compact type. Manivel and Snow extended these results to arbitrary homogeneous spaces in [MS96]. Based on the work of Snow, a vanishing theorem for irreducible Hermitian symmetric spaces of compact type was proved by Biswas-Chaput-Mourougane in [BCM18].

1.1.6. Theorem ([BCM18, Theorem D]). *Let M be an irreducible Hermitian symmetric space of compact type which is not isomorphic to a projective space. Let ℓ and p be two positive integers such that $H^q(M, \Omega_M^p(\ell)) \neq 0$ for some $q \geq 0$, then we have*

$$\ell + q \geq p \frac{r_M}{\dim(M)},$$

where r_M is the index of M, i.e., $\mathcal{O}_M(K_M) \cong \mathcal{O}_M(-r_M)$.

Biswas-Chaput-Mourougane's inequality is enough to derive the first part of Theorem 1.1.5. However, for the second part of Theorem 1.1.5, we need a slightly stronger version. In fact, if $q = 0$, using Snow's algorithm one can give an explicit upper bound for p in terms of ℓ, r_M and $\dim(M)$. In the case $q > 0$, we show the following improvement of Biswas-Chaput-Mourougane's inequality (see Theorem 3.1.5, 3.1.13, 3.1.19 and 3.1.24).

1.1.7. Theorem (= Theorem 3.1.1). *Let M be an irreducible Hermitian symmetric space of compact type. Let ℓ and p be two positive integers such that $H^q(M, \Omega_M^p(\ell)) \neq 0$ for some $q > 0$, then we have*

$$\ell + q - 2 \geq (p-2) \frac{r_M}{\dim(M)}. \tag{1.1}$$

Recall that we always have $r_M \leq \dim(M)$ if M is not isomorphic to a projective space, so our inequality (1.1) is indeed stronger than Biswas-Chaput-Mourougane's inequality. On the other hand, let (Z, H) be a polarized projective manifold, and let $Y \in |dH|$ be a general smooth hypersurface of degree d. Let $\mathcal{E}$ be a torsion-free coherent sheaf over Z. Then it is easy to see that, if $\mathcal{E}$ is an H-unstable sheaf, then $\mathcal{E}|_Y$ is $H|_Y$-unstable. Equivalently, $\mathcal{E}$ is H-semistable if $\mathcal{E}|_Y$ is $H|_Y$-semistable. Nevertheless, the converse is false in general.

1.1.8. Example. The tangent bundle $T_{\mathbb{P}^n}$ of $\mathbb{P}^n$ is $\mathcal{O}_{\mathbb{P}^n}(1)$-stable with $\mu(T_{\mathbb{P}^n}) = (n+1)/n$. However, if Y is a hyperplane, then the restriction $T_{\mathbb{P}^n}|_Y$ is unstable since T_Y is a subbundle of $T_{\mathbb{P}^n}|_Y$ with $\mu(T_Y) = n/(n-1)$.

However, by a result of Mehta-Ramanathan, if we choose d to be an integer large enough, then the restriction of a (semi-)stable sheaf is (semi-)stable. In general, we have the following important effective restriction theorem (cf. [Fle84, MR84, Lan04]).

1.1.9. Theorem [Lan04, Theorem 5.2 and Corollary 5.4]. *Let (Z, H) be a polarized projective manifold of dimension n. Let $\mathcal{E}$ be a torsion-free H-(semi-)stable sheaf of rank $p \geq 2$. Let $Y \in |dH|$ be a general smooth hypersurface. If*

$$d > \frac{p-1}{p} \Delta(\mathcal{E}) H^{n-2} + \frac{1}{p(p-1)H^n},$$

then $\mathcal{E}|_Y$ is $H|_Y$-(semi-)stable. Here $\Delta(\mathcal{E}) = 2pc_2(\mathcal{E}) - (p-1)c_1^2(\mathcal{E})$ is the discriminant of $\mathcal{E}$.

In [BCM18], a sharp effective restriction theorem of cotangent bundle was proved for irreducible Hermitian symmetric spaces of compact type. Recall that a vector bundle E over a polarized projective manifold (X, H) is H-stable (resp. H-semistable) if and only if its dual $E^\vee$ is H-stable (resp. H-semistable).

1.1.10. Theorem [BCM18, Theorem A]. *Let M be an irreducible n-dimensional Hermitian symmetric space of compact type such that $n \geq 3$, and let Y be a smooth hypersurface such that the restriction $\mathrm{Pic}(M) \to \mathrm{Pic}(Y)$ is an isomorphism. Then the restriction $T_M|_Y$ is stable unless Y is a linear section and M is isomorphic to either $\mathbb{P}^n$ or Q^n.*

Based on Biswas-Chaput-Mourougane's inequality (or the inequality (1.1)), we reduce the effective restriction problem of tangent bundles of general complete intersections to the existence of certain twisted vector fields. As a consequence, we can derive the following effective restriction theorem.

1.1.11. Theorem (= Theorem 3.3.8). *Let M be a $(n+r)$-dimensional irreducible Hermitian symmetric space of compact type such that $n \geq 3$ and $r \geq 1$. Let $H_i \in |\mathcal{O}_M(d_i)|$ $(1 \leq i \leq r)$ be a collection of hypersurfaces such that $2 \leq d_1 \leq \cdots \leq d_r$ and the complete intersections $H_1 \cap \cdots \cap H_j$ are smooth for all $1 \leq j \leq r$. Denote $H_1 \cap \cdots \cap H_r$ by Y. Let $X \in |\mathcal{O}_Y(d)|$ be a general smooth hypersurface. Assume moreover that the compoisite of restrictions*

$$\mathrm{Pic}(M) \to \mathrm{Pic}(Y) \to \mathrm{Pic}(X)$$

is surjective. Then the restriction $T_Y|_X$ is stable if one of the following conditions holds.

(1) Y is a Fano manifold and M is isomorphic to neither the projective space $\mathbb{P}^{n+r}$ nor a smooth quadric hypersurface Q^{n+r}.
(2) Y is a Fano manifold, $d \geq d_1$ and M is isomorphic to the projective space $\mathbb{P}^{n+r}$ with $n+r \geq 5$.
(3) Y is a Fano manifold, $d \geq 2$ and M is isomorphic to a smooth quadric hypersurface Q^{n+r}.
(4) $d > d_r - r_Y/n$, where r_Y is the unique integer such that $\omega_Y \cong \mathcal{O}_Y(-r_Y)$.

In the case where Y is a general smooth hypersurface in $\mathbb{P}^n$, using the strong Lefschetz property of the Milnor algebra of Y (see § 3.2.2 for details), we can prove an extension theorem for twisted vector fields on X (see Theorem 3.2.13), and a complete answer to the effective restriction problem is given in this setting.

1.1.12. Theorem (= Theorem 3.3.9). *Let Y be a general smooth hypersurface in the projective space $\mathbb{P}^{n+1}$ of dimension $n \geq 3$. Let $X \in |\mathcal{O}_Y(d)|$ be a general smooth hypersurface of degree d on Y. Assume furthermore that the restriction homomorphism $\mathrm{Pic}(Y) \to \mathrm{Pic}(X)$ is surjective, then $T_Y|_X$ is stable unless $d=1$ and Y is isomorphic to either $\mathbb{P}^n$ or Q^n.*

In each exceptional case, the tangent bundle of X will destabilizes $T_Y|_X$, so our result above is sharp. The stability of restriction of tangent bundle with an increase of Picard group was also considered in [BCM18]. According to Lefschetz's hyperplane theorem, the map $\mathrm{Pic}(Y) \to \mathrm{Pic}(X)$ is always surjective if $n \geq 4$. In fact, Lefschetz proved an even more general version, the so-called Noether-Lefschetz theorem, in [Lef21] : a very general complete intersection surface X in $\mathbb{P}^N$ contains only curves that are themselves complete intersections unless X is an intersection of two quadric threefolds in $\mathbb{P}^4$, or a quadric surface in $\mathbb{P}^3$, or a cubic surface in $\mathbb{P}^3$ (see also [Gre88, Kim91]). In these exceptional cases, the possibilities of the pair (Y, X) are as follows :

(1) Y is the projective space $\mathbb{P}^3$ and X is a quadric surface or a cubic surface.
(2) $Y \subset \mathbb{P}^4$ is a quadric threefold and X is a linear section or a quadric section of Y.
(3) $Y \subset \mathbb{P}^4$ is a cubic threefold and X is a linear section of Y.

When Y is a quadric threefold or a projective space, in view of [BCM18, Theorem B], the restriction $T_Y|_X$ is $\mathcal{O}_X(1)$-semistable unless Y and X are both projective spaces, and $T_Y|_X$ is $\mathcal{O}_X(1)$-stable unless X is a linear section. In the following result, we address the stability of the restriction $T_Y|_X$ in the case where Y is a cubic threefold and X is a linear section.

1.1.13. Theorem (= Theorem 3.4.5). *Let $Y \subset \mathbb{P}^4$ be a general cubic threefold and $X \in |\mathcal{O}_Y(1)|$ a general smooth linear section. Then the restriction $T_Y|_X$ is stable with respect to $\mathcal{O}_X(1)$.*

1.2 Organization

This part is organized as follows. Chapter 2 is devoted to study ample subsheaves of tangent bundle and Chapter 3 is devoted to study the stability of the tangent bundles of complete intersections in irreducible Hermitian symmetric spaces and their effective restrictions. More precisely, the organization of each chapter is as follows.

Chapter 2 : In Section 2.1, we collect some basic properties about ample sheaves and we explain how the existence of positive subsheaves gives a strong restriction on the global geometry using rational curves. In Section 2.2, we recall the basic definitions of foliations, and we prove that an ample subsheaf must define a foliation on the total space. In Section 2.3, we prove a characterization of projective spaces (cf. Theorem 1.1.1) which is the main theorem in this chapter. In Section 2.4, we give two applications of our main theorem, in particular, we explain Litt's argument to show how to derive Theorem 1.1.3 from Theorem 1.1.1.

Chapter 3 : In Section 3.1, we study vanishing theorems on Hermitian symmetric spaces, in particular, we will prove Theorem 1.1.7. In Section 3.2, we investigate the twisted vector fields over complete intersections of Hermitian symmetric spaces and we prove some extension results in different settings. In Section 3.3, we address the stability of the tangent bundles of complete intersections in Hermitian symmetric spaces and study the effective restriction problem. In particular, we prove Theorem 1.1.5, Theorem 1.1.11 and Theorem 1.1.12. In Section 3.4, we consider the case where the Picard number increases and we prove Theorem 1.1.13. In Section 3.5, we consider the surfaces in $\mathbb{P}^3$.

1.3 Convention and notations

If L is a line bundle over a projective variety X, we denote also by L^{-1} the dual bundle $L^\vee$ of L. By (semi-)stability of a vector bundle we mean *slope (semi-)stability* with respect to same fixed polarization.

If $\mathcal{F}$ is coherent sheaf over a projective variety X, we denote by $\mathcal{F}(x) = \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} k(x)$ the fiber of $\mathcal{F}$ at $x \in X$, where $k(x) = \mathcal{O}_{X,x}/\mathfrak{m}_{X,x}$. Moreover, we denote by $\mathbb{P}(\mathcal{F})$ the Grothendieck projectivization Proj $(\oplus_{m\geq 0}\mathrm{Sym}^m\mathcal{F})$ (see [Gro61, §4]). For an invertible sheaf $\mathcal{O}_X(1)$ and a coherent sheaf $\mathcal{F}$ on X by $\mathcal{F}(\ell)$ we will denote $\mathcal{F}\otimes\mathcal{O}_X(\ell)$, and the number $h^i(X,\mathcal{F})$ is the dimension of $H^i(X,\mathcal{F})$ over $\mathbb{C}$. For a subvariety Y of a polarized manifold $(X,\mathcal{O}_X(1))$, we denote by $\mathcal{O}_Y(1)$ the restriction $\mathcal{O}_X(1)|_Y$. Moreover, if $i\colon Y \hookrightarrow X$ is an immersion, then we denote by $\mathcal{F}|_Y$ the pull-back $i^*\mathcal{F}$. The canonical divisor $K_{\mathcal{F}}$ is a Weil divisor associated to $\det(\mathcal{F})^\vee$. If the polarization is clear from the context, we will abbreviate the slope $\mu_H(\mathcal{F})$ to $\mu(\mathcal{F})$.

If X is a n-dimensional projective normal variety, we denote by Ω_X^1 the sheaf of Kähler differentials and we denote by ω_X the canonical sheaf $(\wedge^n\Omega_X^1)^{\vee\vee}$. The canonical divisor, denoted by K_X, is a Weil divisor associated to ω_X. Let us denote the tangent bundle (sheaf) $(\Omega_X^1)^\vee$ of X by T_X. For a subvariety $Y\subset X$, the conormal sheaf I_Y/I_Y^2 of Y is denote by $N_{Y/X}^*$ and its normal sheaf $N_{Y/X}$ is defined to be $(N_{Y/X}^*)^\vee$. The Picard group of X is denoted by $\mathrm{Pic}(X)$.

If $f\colon X \to Y$ is a morphism between projective normal varieties, we denote by $\Omega_{X/Y}^1$ the relative Kähler differential sheaf. Moreover, if Y is smooth, we denote by $K_{X/Y}$ the relative canonical divisor $K_X - f^*K_Y$ and by $\omega_{X/Y}$ the reflexive sheaf $\omega_X\otimes f^*\omega_Y^\vee$.

Chapitre 2

Characterization of projective spaces and $\mathbb{P}^r$-bundles as ample divisors

The aim of this chapter is to study the projective manifolds whose tangent bundle contains an ample subsheaf, and it is based on the paper [Liu17a]. We have seen in [Ara06] that the existence of ample subsheaves implies that the ambient space can be realized as a projective bundle over an open subset. We will prove that an ample subsheaf actually defines an algebraically integrable foliation. The global strategy to prove Theorem 1.1.1 is to show that the general leaves shall pass through a common point. If the foliation is regular, we can conclude it by the positivity of relative anticanonical divisor. If the foliation is not regular, we will work over some open subset which is disjoint from the singular locus.

2.1 Ample sheaves and rational curves

Let X be a projective manifold. In this section, we gather some results about the behavior of an ample subsheaf $\mathcal{F} \subset T_X$ with respect to a covering family of minimal rational curves on X.

2.1.1 Ample sheaves

We recall the following useful criterion of freeness of coherent sheaves on a reduced connected noetherian scheme.

2.1.1. Proposition [Har77, II, Exercise 5.8]. *Let X be a reduced connected noetherian scheme and $\mathcal{F}$ a coherent sheaf on X. Then $\mathcal{F}$ is locally free if and only if the function $\varphi(x) = \dim_{k(x)} \mathcal{F}(x)$ is constant.*

Recall that an invertible sheaf $\mathcal{L}$ on a quasi-projective variety X is said to be *ample* if for every coherent sheaf $\mathcal{G}$ on X, there is an integer $n_0 > 0$ such that for every $n \geq n_0$, the sheaf $\mathcal{G} \otimes \mathcal{L}^n$ is generated by its global sections (see [Har77, Section II.7]). In general, a non-zero torsion free coherent sheaf $\mathcal{F}$ on a quasi-projective variety X is said to be *ample* if the invertible sheaf $\mathcal{O}_{\mathbb{P}(\mathcal{F})}(1)$ is ample on the Grothendieck projectivization. We list some well-known properties of ample sheaves in the following.

(1) A coherent sheaf $\mathcal{F}$ on a quasi-projective variety X is ample if and only if, for any coherent sheaf $\mathcal{G}$ on X, $\mathcal{G} \otimes \mathbf{Sym}^m \mathcal{F}$ is globally generated for $m \gg 1$ (see [Kub70, Theorem 1]).
(2) If $i \colon Y \to X$ is an immersion, and $\mathcal{F}$ is an ample sheaf on X, then $i^*\mathcal{F}$ is an ample sheaf on Y (see [Kub70, Proposition 6]).
(3) If $\pi \colon Y \to X$ is a finite morphism with X and Y quasi-projective varieties, and $\mathcal{F}$ is a coherent sheaf on X, then $\mathcal{F}$ is ample if and only if $\pi^*\mathcal{F}$ is ample. Note that $\mathbb{P}(\pi^*\mathcal{F}) = \mathbb{P}(\mathcal{F}) \times_X Y$ and $\mathcal{O}_{\mathbb{P}(\mathcal{F})}(1)$ pulls back, by a finite morphism, to $\mathcal{O}_{\mathbb{P}(\pi^*\mathcal{F})}(1)$.
(4) Any nonzero quotient of an ample sheaf is ample (see [Kub70, Proposition 1]). In particular, the image of an ample sheaf under a nonzero map is also ample.

(5) If $\mathcal{F}$ is a locally free ample sheaf of rank r, then the s^{th} exterior power $\wedge^s \mathcal{F}$ is ample for any $1 \leq s \leq r$ (see [Har66, Corollary 5.3]). In particular, $\det(\mathcal{F})$ is ample.
(6) If $\mathcal{L}$ is an ample invertible sheaf on a quasi-projective variety X, then $\mathcal{L}^m$ is very ample for some $m > 0$; that is, there is an immersion $i\colon X \to \mathbb{P}^n$ for some n such that $\mathcal{L}^m = i^*\mathcal{O}_{\mathbb{P}^n}(1)$ (see [Har77, II, Theorem 7.6]).
(7) If $\mathcal{L}$ is an ample invertible sheaf over a quasi-projective variety X and $\mathcal{A}$ is an invertible sheaf over X, then there exists an integer $m > 0$ such that $\mathcal{L}^{\otimes m} \otimes \mathcal{A}$ is very ample over X (see [Gro61, Corollaire 4.5.11]).

2.1.2 Minimal rational curves and VMRTs

Let X be a uniruled projective manifold. Then X carries a minimal covering family of rational curves. We fix such a family $\mathcal{V} \subset \mathrm{RatCurves}^n(X)$, and let $[\ell] \in \mathcal{V}$ be a general point. Then the tangent bundle T_X can be decomposed on the normalization of ℓ as

$$\mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus d} \oplus \mathcal{O}_{\mathbb{P}^1}^{\oplus(n-d-1)},$$

where $d + 2 = \det(T_X) \cdot \ell \geq 2$ is called the *anticanonical degree* of $\mathcal{V}$. We denote by $\bar{\mathcal{V}}$ the closure of $\mathcal{V}$ in $\mathrm{Chow}(X)$. The following proposition is very convenient when discussing the generic property of $\mathcal{V}$, and we will use it frequently in the latter.

2.1.2. Proposition [Kol96, II, Proposition 3.7]. *Let X be a projective uniruled manifold with a minimal covering family $\mathcal{V}$ of rational curves. Given any subset $Z \subset X$ of codimension at least two, the general member of $\mathcal{V}$ is disjoint from Z.*

Recall that the *variety of minimal rational tangents* (VMRTs) $\mathcal{C}_x$ at x associated to $\mathcal{V}$ is the closure of the image of the tangent map $\tau_x\colon \mathcal{V}_x \dashrightarrow \mathbb{P}(T^\vee_{X,x})$, and the map $\mathcal{V}_x \dashrightarrow \mathcal{C}_x$ is in fact the normalization morphism at a general point x [Keb02, HM04]. Let $[\ell] \in \mathcal{V}_x$ be a general member, and we denote its normalization by $f\colon (\mathbb{P}^1, o) \to (X, x)$. Let $f^*T_X^+$ be the subbundle of f^*T_X defined by

$$f^*T_X^+ = \mathrm{im}[H^0(\mathbb{P}^1, f^*T_X(-1)) \to f^*T_X(-1)] \otimes \mathcal{O}(1) \hookrightarrow f^*T_X.$$

As ℓ is smooth at x, the fiber $(f^*T_X)_o$ of f^*T_X at o is naturally isomorphic to $T_{X,x}$. As a consequence, the subbundle $f^*T_X^+$ yields a linear subspace of $T_{X,x}$ which will be denoted by $T_xX_f^+$.

2.1.3. Proposition [Hwa01, Proposition 2.3]. *Let $[\ell] \in \mathcal{V}_x$ be a general element. Then the projective tangent space of $\mathcal{C}_x$ at $\tau_x([\ell])$ is $\mathbb{P}((T_xX_f^+)^\vee) \subset \mathbb{P}(T^\vee_{X,x})$.*

2.1.3 Distribution defined by VMRTs

The first step towards Theorem 1.1.1 is the following result due to Araujo [Ara06] which gives a structure theorem for projective manifolds whose VMRTs is linear.

2.1.4. Theorem [Ara06, Theorem 1.1]. *Suppose that $\mathcal{C}_x$ is a d-dimensional linear subspace of $\mathbb{P}(T^\vee_{X,x})$ for a general point x. Then there is a dense open subset X_0 of X and a $\mathbb{P}^{d+1}$-bundle $\varphi_0\colon X_0 \to T_0$ such that any curve on X parametrized by $\mathcal{V}$ and meeting X_0 is a line on a fiber of φ_0. In particular, φ_0 is exactly the $\bar{\mathcal{V}}$-rc quotient of X.*

To apply this theorem in our situation, we need to show that the VMRTs is linear if T_X contains an ample subsheaf $\mathcal{F}$. In fact, we prove a stronger result, assuming the ampleness of $\mathcal{F}$ only on certain rational curves. This was essentially proved in [Ara06]. In [Ara06], $\mathcal{F}$ is assumed to be locally free, but the proof also works in our more general situation. Before giving the precise statement, we recall a result due to Araujo which characterizes the cones and linear subspaces in projective spaces needed in the proof.

2.1.5. Lemma [Ara06, Lemma 4.2 and 4.3]. *Let Z be an irreducible closed subvariety of $\mathbb{P}^m$.*

(1) Assume that there is a dense open subset U of the smooth locus of Z and a point $z_0 \in \mathbb{P}^m$ such that $z_0 \in \bigcap_{z\in U} T_{Z,z}$, where $T_{Z,z}$ is the projective tangent space of Z at x. Then Z is a cone in $\mathbb{P}^m$ and z_0 lies in the vertex of this cone.

(2) If Z is a cone in $\mathbb{P}^m$ and the normalization of Z is smooth, then Z is a linear subspace.

2.1.6. Proposition. *Let X be a projective uniruled manifold, $\mathcal{V}$ a minimal covering family of rational curves on X. Assume that the tangent bundle T_X contains a subsheaf $\mathcal{F}$ such that $\mathcal{F}|_\ell$ is an ample vector bundle for a general member $[\ell] \in \mathcal{V}$. Let $d+2$ be the anti-canonical degree of $\mathcal{V}$. Then $\mathcal{C}_x$ is a union of linear subspace of dimension d in $\mathbb{P}(T^\vee_{X,x})$ at a general point $x \in X$ such that its every component contains the image of $\mathcal{F}|_\ell$ in $\mathbb{P}(T_{X,x})$ under τ_x.*

Proof. Let $x \in X$ be a general point such that $\mathcal{F}$ is locally free around x and the fiber $\mathcal{F}(x)$ is a subspace of $T_{X,x}$. Let $\mathcal{V}^i_x$ be a component of $\mathcal{V}_x$, and $[\ell] \in \mathcal{V}^i_x$ a general member. By Proposition 2.1.2 and our assumption, the torsion free sheaf $\mathcal{F}$ is locally free and ample over ℓ and hence $\mathcal{F}|_\ell$ is a subsheaf of $f^*T^+_X$, where $f\colon \mathbb{P}^1 \to \ell$ is the normalization. It follows that $\mathcal{F}(x)$ is a subspace of $T_xX^+_f$. By Proposition 2.1.3, we conclude

$$\mathbb{P}(\mathcal{F}(x)^\vee) \subset \mathbb{P}((T_xX^+_f)^\vee) = \overline{T_{\tau_x([\ell])}\mathcal{C}^i_x} \subset \mathbb{P}(T^\vee_{X,x}).$$

This holds for a general element $[\ell] \in \mathcal{V}^i_x$, so it also holds for a general element in $\mathcal{C}^i_x$. By Lemma 2.1.5 (1), the variety $\mathcal{C}^i_x$ is a cone containing $\mathbb{P}(\mathcal{F}(x)^\vee)$. Note that the normalization $\mathcal{V}^i_x$ of $\mathcal{C}^i_x$ is smooth because of the generality of x, therefore Lemma 2.1.5 (2) implies that $\mathcal{C}^i_x$ is a linear subspace containing the projective space $\mathbb{P}(\mathcal{F}(x)^\vee)$. □

Recall that the *singular locus* $\mathrm{Sing}(\mathcal{S})$ of a coherent sheaf $\mathcal{S}$ over X is the set of all points of X where $\mathcal{S}$ is not locally free.

2.1.7. Remark. The hypothesis in Proposition 2.1.6 that $\mathcal{F}$ is locally free over a general member of $\mathcal{V}$ is automatically satisfied. In fact, since $\mathcal{F}$ is torsion free and X is smooth, $\mathcal{F}$ is locally free in codimension one. By Proposition 2.1.2, a general member of $\mathcal{V}$ is disjoint from $\mathrm{Sing}(\mathcal{F})$; hence, $\mathcal{F}$ is locally free along a general member of $\mathcal{V}$.

The irreducibility of $\mathcal{C}_x$ follows immediately from the following result due to Hwang. In [Hwa07], X is assumed to be of Picard number one, but this assumption is not used in the proof.

2.1.8. Proposition [Hwa07, Proposition 2.2]. *Let X be a projective uniruled manifold. $\mathcal{V}$ is a minimal covering family of rational curves on X. If $\mathcal{C}_x$ is a union of linear subspaces of $\mathbb{P}(T^\vee_{X,x})$ for a general point $x \in X$, then the intersection of any two irreducible components of $\mathcal{C}_x$ is empty.*

Summarizing, we have proved the following theorem.

2.1.9. Theorem. *Let X be a projective uniruled manifold, $\mathcal{V}$ a minimal covering family of rational curves on X. If T_X contains a subsheaf $\mathcal{F}$ of rank $r > 0$ such that $\mathcal{F}|_\ell$ is ample for a general member $[\ell] \in \mathcal{V}$, then there exists a dense open subset X_0 of X and a $\mathbb{P}^{d+1}$-bundle $\varphi_0\colon X_0 \to T_0$ such that any curve on X parametrized by $\mathcal{V}$ and meeting X_0 is a line on a fiber of φ_0. In particular, φ_0 is the $\bar{\mathcal{V}}$-rc quotient of X.*

As an immediate application of Theorem 2.1.9, we can derive a weak version of [AKP08, Theorem 4.2].

2.1.10. Corollary. *Let X be a projective uniruled manifold with $\rho(X) = 1$, and let $\mathcal{V}$ be a minimal covering family of rational curves on X. If T_X contains a subsheaf $\mathcal{F}$ of positive rank such that $\mathcal{F}|_\ell$ is ample for a general member $[\ell] \in \mathcal{V}$, then $X \cong \mathbb{P}^n$.*

2.1.11. Corollary [AKP08, Corollary 4.3]. *Let X be a n-dimensional projective manifold with $\rho(X) = 1$. Assume that T_X contains an ample subsheaf, then $X \cong \mathbb{P}^n$.*

Proof. Since the tangent bundle T_X contains an ample subsheaf $\mathcal{F}$, X is uniruled (see [Miy87a, Corollary 8.6]) and it carries a minimal covering family $\mathcal{V}$ of rational curves . Note that the restriction $\mathcal{F}|_C$ is ample for any curve $C \subset X$, thus we can derive the result from Corollary 2.1.10. □

2.1.12. Remark. Our approach above is quite different from that in [AKP08]. The proof in [AKP08] is based on a careful analysis of the singular locus of $\mathcal{F}$ and the locally freeness of $\mathcal{F}$ has been proved. Even though our argument does not tell anything about the singular locus of $\mathcal{F}$, it has the advantage to give a rough description of the geometric structure of projective manifolds whose tangent bundle contains a "positive" subsheaf.

2.2 Foliations and Pfaff fields

Let $\mathcal{S}$ be a subsheaf of T_X on a quasi-projective manifold X. We denote by $\mathcal{S}^{\text{reg}}$ the largest open subset of X such that $\mathcal{S}$ is a subbundle of T_X over $\mathcal{S}^{\text{reg}}$. Note that in general the singular locus $\text{Sing}(\mathcal{S})$ of $\mathcal{S}$ is a proper subset of $X \setminus \mathcal{S}^{\text{reg}}$.

2.2.1. Definition. *Let X be a quasi-projective manifold and let $\mathcal{S} \subsetneq T_X$ be a coherent subsheaf of positive rank. $\mathcal{S}$ is called a foliation if it satisfies the following conditions.*

(1) $\mathcal{S}$ is saturated in T_X ; that is, $T_X/\mathcal{S}$ is torsion free.
(2) The sheaf $\mathcal{S}$ is closed under the Lie bracket.

In addition, $\mathcal{S}$ is called an algebraically integrable foliation if the following holds.

(3) For a general point $x \in X$, there exists a projective subvariety F_x passing through x such that

$$\mathcal{S}|_{F_x \cap \mathcal{S}^{\text{reg}}} = T_{F_x}|_{F_x \cap \mathcal{S}^{\text{reg}}} \subset T_X|_{F_x \cap \mathcal{S}^{\text{reg}}}.$$

We call F_x the $\mathcal{S}$-leaf through x.

2.2.2. Remark. Let X be a projective manifold and $\mathcal{S}$ a saturated subsheaf of T_X. To show that $\mathcal{S}$ is an algebraically integrable foliation, it is sufficient to show that it is an algebraically integrable foliation over a Zariski open subset of X.

2.2.3. Example. Let $X \to Y$ be a fibration with X and Y projective manifolds. Then $T_{X/Y} \subset T_X$ defines an algebraically integrable foliation on X such that the general leaves are the fibers.

2.2.4. Example [AD13, 4.1]. Let $\mathcal{F}$ be a subsheaf $\mathcal{O}_{\mathbb{P}^n}(1)^{\oplus r}$ of $T_{\mathbb{P}^n}$ on $\mathbb{P}^n$. Then $\mathcal{F}$ is an algebraically integrable foliation and it is defined by a linear projection $\mathbb{P}^n \dashrightarrow \mathbb{P}^{n-r}$. The set of points of indeterminacy S of this rational map is a $(r-1)$-dimensional linear subspace. Let $x \notin S$ be a point. Then the leaf passing through x is the r-dimensional linear subspace L of $\mathbb{P}^n$ containing both x and S.

2.2.5. Definition. *Let X be a projective variety, and r a positive integer. A Pfaff field of rank r on X is a nonzero map $\partial\colon \Omega_X^r \to \mathcal{L}$, where $\mathcal{L}$ is an invertible sheaf on X.*

2.2.6. Lemma [ADK08, Proposition 4.5]. *Let X be a projective variety and let $n\colon \widetilde{X} \to X$ be its normalization. Let $\mathcal{L}$ be an invertible sheaf on X, let r be a positive integer, and let $\partial\colon \Omega_X^r \to \mathcal{L}$ be a Pfaff field. Then ∂ can be extended uniquely to a Pfaff field $\widetilde{\partial}\colon \Omega^r_{\widetilde{X}} \to n^*\mathcal{L}$.*

Given a foliation $\mathcal{S} \subset T_X$ of rank r over a normal projective variety X, recall that $K_{\mathcal{S}}$ is the canonical class of $\mathcal{S}$. Moreover, if X is smooth, then there is a natural associated Pfaff field of rank r :

$$\Omega_X^r = \wedge^r(\Omega_X^1) = \wedge^r(T_X^\vee) = (\wedge^r T_X)^\vee \to \mathcal{O}_X(K_{\mathcal{S}}).$$

2.2.7. Lemma [AD13, Lemma 3.2]. *Let X be a projective manifold, and $\mathcal{S}$ an algebraically integrable foliation on X. Then there is a unique irreducible projective subvariety W of* Chow(X) *whose general point parametrizes a general leaf of $\mathcal{S}$.*

2.2.8. Remark. Let X be a projective manifold, and let $\mathcal{S}$ be an algebraically integrable foliation of rank r on X. Let W be the subvariety of $\mathrm{Chow}(X)$ provided by Lemma 2.2.7. Let $Z \subset W$ be a general closed subvariety of W and let $U \subset Z \times X$ be the universal cycle over Z. Let $\widetilde{Z}$ and $\widetilde{U}$ be the normalizations of Z and U, respectively. We claim that the Pfaff field $\Omega_X^r \to \mathcal{O}_X(K_{\mathcal{S}})$ can be extended to a Pfaff field $\Omega_{\widetilde{U}}^r \to n^*p^*\mathcal{O}_X(K_{\mathcal{S}})$ which factors through $\Omega_{\widetilde{U}/\widetilde{Z}}^r \to n^*p^*\mathcal{O}_X(K_{\mathcal{S}})$.

$$\begin{tikzcd}
\widetilde{U} \arrow[r, "n"] \arrow[d, "\widetilde{q}"'] & U \arrow[d, "q"] & \subset & Z \times X \arrow[r, "p"] \arrow[d, "q"] & X \\
\widetilde{Z} \arrow[r] & Z \arrow[rr, "="] & & Z &
\end{tikzcd}$$

Let $\partial\colon \Omega_X^r \to \mathcal{O}_X(K_{\mathcal{S}})$ be the Pfaff field associated to $\mathcal{S}$. Then it induces a Pfaff field on $Z \times X$:

$$\partial_{Z\times X}\colon \Omega_{Z\times X}^r = \wedge^r(q^*\Omega_Z^1 \oplus p^*\Omega_X^1) \twoheadrightarrow \wedge^r(p^*\Omega_X^1) \simeq p^*\Omega_X^r \to p^*\mathcal{O}_X(K_{\mathcal{S}}).$$

Let $\mathcal{K}$ be the kernel of the natural morphism $\Omega_{Z\times X}^r|_U \twoheadrightarrow \Omega_U^r$. Let F be a general fiber of $q\colon U \to Z$ such that its image is a $\mathcal{S}$-leaf. Then the composite map $\Omega_{Z\times X}^r|_F \twoheadrightarrow \Omega_U^r|_F \twoheadrightarrow \Omega_F^r$ implies that the composite map

$$\mathcal{K} \to \Omega_{Z\times X}^r|_U \to p^*\mathcal{O}_X(K_{\mathcal{S}})|_U$$

vanishes over a general fiber F since $p\colon F \to p(F)$ is an isomorphism. Note that the sheaf $p^*\mathcal{O}_X(K_{\mathcal{S}})|_U$ is torsion-free, it vanishes actually identically, i.e. we obtain a Pfaff field $\Omega_U^r \to p^*\mathcal{O}_X(K_{\mathcal{S}})|_U$ induced by the associated Pfaff field $\Omega_X^r \to \mathcal{O}_X(K_{\mathcal{S}})$. By Lemma 2.2.6, it can be uniquely extended to a Pfaff field $\Omega_{\widetilde{U}}^r \to n^*p^*\mathcal{O}_X(K_{\mathcal{S}})$.

Let $\mathcal{G}$ be the kernel of the morphism $\Omega_{\widetilde{U}}^r \twoheadrightarrow \Omega_{\widetilde{U}/\widetilde{Z}}^r$. As before, let F be a general fiber of $\widetilde{q}$ such that its image under $p \circ n$ is a $\mathcal{S}$-leaf and the morphism $p \circ n$ restricted on F is finite and birational. Let $x \in F$ be a point such that F is smooth at x and $p \circ n$ is an isomorphism at a neighborhood of x. Then the composite map $\Omega_{\widetilde{U}}^r|_F \twoheadrightarrow \Omega_{\widetilde{U}/\widetilde{Z}}^r|_F \twoheadrightarrow \Omega_F^r$ implies that the composite map

$$\mathcal{G} \to \Omega_{\widetilde{U}}^r \to n^*p^*\mathcal{O}_X(K_{\mathcal{S}})$$

vanishes in a neighborhood of x, hence it vanishes generically over $\widetilde{U}$. Since the sheaf $n^*p^*\mathcal{O}_X(K_{\mathcal{S}})$ is torsion-free, it vanishes identically and finally yields a morphism $\Omega_{\widetilde{U}/\widetilde{Z}}^r \to n^*p^*\mathcal{O}_X(K_{\mathcal{S}})$.

Let X be a projective manifold, and let $\mathcal{S} \subset T_X$ be a subsheaf. We define its saturation $\overline{\mathcal{S}}$ as the kernel of the natural surjection $T_X \twoheadrightarrow (T_X/\mathcal{S})/\mathrm{torsion}$. Then $\overline{\mathcal{S}}$ is obviously saturated.

2.2.9. Theorem. *Let X be a projective manifold. Assume that T_X contains an ample subsheaf $\mathcal{F}$ of positive rank $r < \dim(X)$. Then its saturation $\overline{\mathcal{F}}$ defines an algebraically integrable foliation on X, and the closure of the $\overline{\mathcal{F}}$-leaf passing through a general point is isomorphic to $\mathbb{P}^r$.*

Proof. Let $\varphi_0\colon X_0 \to T_0$ be as the morphism provided by Theorem 2.1.9. Since $\mathcal{F}$ is locally free in codimension one, we may assume that no fiber of φ_0 is completely contained in $\mathrm{Sing}(\mathcal{F})$.

The first step is to show that $\mathcal{F}|_{X_0} \subset T_{X_0/T_0}$. Since $\varphi_0\colon X_0 \to T_0$ is smooth, we get a short exact sequence of vector bundles,

$$0 \to T_{X_0/T_0} \to T_X|_{X_0} \to \varphi_0^*T_{T_0} \to 0.$$

The composite map $\mathcal{F}|_{X_0} \to T_X|_{X_0} \to \varphi_0^*T_{T_0}$ vanishes on a Zariski open subset of every fiber since $\mathcal{F}|_\ell$ is contained in $T_F|\ell$ where F is a fiber of φ_0 and ℓ is a line contained in F. Since $\varphi_0^*T_{T_0}$ is torsion free, it vanishes identically, and it follows $\mathcal{F}|_{X_0} \subset T_{X_0/T_0}$.

Next, we show that, after shrinking X_0 and T_0 if necessary, $\mathcal{F}$ is actually locally free over X_0. By the generic flatness theorem [Gro65, Théorème 6.9.1], after shrinking T_0, we can suppose that $(T_X/\mathcal{F})|_{X_0}$

is flat over T_0. Let $F \cong \mathbb{P}^{d+1}$ be an arbitrary fiber of φ_0. The following short exact sequence of sheaves

$$0 \to \mathcal{F}|_{X_0} \to T_X|_{X_0} \to (T_X/\mathcal{F})\,|_{X_0} \to 0$$

induces an exact sequence of sheaves

$$\mathcal{T}or((T_X/\mathcal{F})\,|_{X_0}, \mathcal{O}_F) \to \mathcal{F}|_F \to T_X|_F \to (T_X/\mathcal{F})\,|_F \to 0.$$

Since $(T_X/\mathcal{F})\,|_{X_0}$ is flat over T_0, it follows that $\mathcal{F}|_F$ is a subsheaf of $T_X|_F$; in particular, $\mathcal{F}|_F$ is torsion free. Without loss of generality, we may assume that the restrictions of $\mathcal{F}$ on all fibers of φ_0 are torsion-free. By Remark 2.1.12, the restrictions of $\mathcal{F}$ on all fibers of φ_0 are locally free. Note that $\mathrm{sing}(\mathcal{F})$ is a close subset of X_0 of codimension at least two, it follows from Nakayama's lemma that $\mathcal{F}$ is locally free over X_0 after shrinking T_0 and X_0 if necessary (cf. Proposition 2.1.1).

Now, we claim that $\overline{\mathcal{F}}$ actually defines an algebraically integrable foliation on X_0. Let $F \cong \mathbb{P}^{d+1}$ be an arbitrary fiber of φ_0. We know that $(F, \mathcal{F}|_F)$ is isomorphic to $(\mathbb{P}^{d+1}, T_{\mathbb{P}^{d+1}})$ or $(\mathbb{P}^{d+1}, \mathcal{O}_{\mathbb{P}^{d+1}}(1)^{\oplus r})$ (cf. Theorem B) ; therefore, $\mathcal{F}$ defines an algebraically integrable foliation over X_0 (cf. Example 2.2.4). Note that we have $\mathcal{F}|_{X_0} = \overline{\mathcal{F}}|_{X_0}$, since $\mathcal{F}|_{X_0}$ is saturated in T_{X_0}. Hence, $\overline{\mathcal{F}}$ also defines an algebraically integrable foliation over X (cf. Remark 2.2.2). □

2.2.10. Remark. Since $\mathcal{F}$ is locally free on X_0, it follows that $\mathcal{O}_X(-K_{\mathcal{F}})|_{X_0}$ is isomorphic to $\wedge^r(\mathcal{F}|_{X_0})$ and the invertible sheaf $\mathcal{O}_X(-K_{\mathcal{F}})$ is ample over X_0. Moreover, as $\mathcal{F}$ is locally free in codimension one, there exists an open subset $X' \subset X$ containing X_0 such that $\mathrm{codim}(X \setminus X') \geq 2$ and $\mathcal{O}_X(-K_{\mathcal{F}})$ is ample on X'.

2.3 Proof of main theorem

The aim of this section is to prove Theorem 1.1.1. Let X be a normal quasi-projective variety, let D a be $\mathbb{Q}$-Weil divisor on X. Recall that (X, D) is *snc* (simple normal crossing) at a point $x \in X$ if X is smooth at x and there are local coordinates $x_1, \cdots, x_n$ such that $\mathrm{Supp}(D) \subset \{x_1 \cdots x_n = 0\}$ near x. We say that (X, D) is snc if its components are smooth and it is snc for every point $x \in X$. Given any pair (X, D), there exists a largest open subset U of X such that $(U, D|_U)$ is snc. This open subset is called the *snc locus* of (X, D).

Let $g\colon X \to Y$ be a fibration with X and Y projective manifolds. Recall that g is called *prepared* if the support of its non-smooth locus is contained in a snc divisor Δ on Y and the preimage $g^{-1}(\Delta)$ is also a snc divisor (see [Cam04, § 4.3]). If $D = \sum_{i \in I} a_i D_i$ is a Weil divisor on X, then the g-vertical part of D is defined as $D^{\mathrm{vert}} = \sum_{i \in I^{\mathrm{vert}}} a_i D_i$, where $I^{\mathrm{vert}} \subset I$ consists of the components of D which are not mapped onto Y. We also denote by D^{hor} the g-horizontal part of D, defined such that $D = D^{\mathrm{vert}} + D^{\mathrm{hor}}$.

We need the following notion of *weak positivity* of torsion free coherent sheaves in the sense of Viehweg.

2.3.1. Definition [Vie83, Definition 1.1]. *Let X be a normal projective variety, and let $\mathcal{E}$ be a non-zero torsion free coherent sheaf on X. Let $i\colon X' \to X$ be the largest open subset of X such that $\mathcal{E}|_{X'}$ is locally free. For all $m \in \mathbb{N}$, we will denote by $S^m(\mathcal{E}|_{X'})$ the m-th symmetric product. Then we define $S^{[m]}\mathcal{E} \colon = i_* S^m(\mathcal{F}|_{X'})$.*

2.3.2. Remark. Since X is normal and $\mathcal{E}$ is torsion free, the subvariety $X \setminus X'$ is of codimension at least two. As a consequence, we have $S^{[m]}\mathcal{E} = (S^m\mathcal{E})^{\vee\vee}$.

2.3.3. Definition [Vie83, Definition 1.2]. *Let X be a normal projective variety, and let $\mathcal{E}$ be a non-zero torsion-free coherent sheaf on X. Let X_* be an open subset of X. We say that the torsion free coherent sheaf $\mathcal{E}$ is weakly positive over X' if for some invertible sheaf $\mathcal{L}$ on X and every $\alpha \in \mathbb{N}$, there exists some $\beta \in \mathbb{N}$ such that $S^{[\alpha\cdot\beta]}\mathcal{E} \otimes \mathcal{L}^{\otimes\beta}$ is globally generated over X' ; that is, the evaluation map*

$$H^0(X, S^{[\alpha\cdot\beta]}\mathcal{E} \otimes \mathcal{L}^{\otimes\beta}) \otimes \mathcal{O}_X \to S^{[\alpha\cdot\beta]}\mathcal{E} \otimes \mathcal{L}^{\otimes\beta}$$

is surjective over X'. The sheaf $\mathcal{E}$ is weakly positive if there exists some open subset X' of X such that $\mathcal{E}$ is weakly positive over X'.

One of the key ingredients in our proof of Theorem 1.1.1 is the following weak positivity theorem due to Campana.

2.3.4. Theorem [Cam04, Theorem 4.13]. *Let $g\colon X \to Y$ be a prepared fibration between projective manifolds. Let $D = \sum_{i\in I} d_i D_i$ be a non-zero effective Weil divisor with D_i distinct on X. We write $D = D^{\mathrm{vert}} + D^{\mathrm{hor}}$. Assume that the support of D^{hor} is a snc divisor. Let m be an positive integer such that $m \geq d_i$ for every $i \in I^{\mathrm{hor}}$, where I^{hor} is the set of coefficients of D^{hor}. Then $g_*(\omega^m_{X/Y} \otimes \mathcal{O}_X(D))$ is weakly positive.*

For a flat family of varieties over a smooth curve, the following semistable reduction theorem is well-known.

2.3.5. Lemma [BLR95, Theorem 2.1]. *Let X be a quasi-projective variety, and $f\colon X \to C$ a surjective flat morphism onto a smooth curve C. Then there exists a finite morphism $C' \to C$ such that $f'\colon X' \to C'$ is flat with reduced fibers, where X' is the normalization of $X \times_C C'$ and f' is the morphism induced by $X \times_C C' \to C'$.*

Let X be a normal projective variety, and let $X \to C$ be a surjective morphism with connected fibers onto a smooth curve. Let Δ be an effective Weil divisor on X such that (X, Δ) is log canonical over the generic point of C. In [AD13, Théorem 5.1], they proved that the divisor $-(K_{X/C} + \Delta)$ cannot be ample. In the next theorem, we give a variant of this result which is the key ingredient in our proof of Theorem 1.1.1.

2.3.6. Theorem. *Let X be a normal projective variety, and $f\colon X \to C$ a surjective morphism with connected fibers onto a smooth curve. Let Δ be a Weil divisor on X such that $K_X + \Delta$ is Cartier and Δ^{hor} is reduced. Assume that there exists an open subset C_0 such that the pair $(X, \Delta^{\mathrm{hor}})$ is snc over $X_0 = f^{-1}(C_0)$. If $X' \subset X$ is an open subset such that no fiber of f is completely contained in $X \setminus X'$ and $X_0 \subset X'$, then the invertible sheaf $\mathcal{O}_X(-K_{X/C} - \Delta)$ is not ample over X'.*

Proof. To prove the theorem, we assume that the invertible sheaf $\mathcal{O}_X(-K_{X/C}-\Delta)$ is ample over X'. Let A be an ample divisor supported on C_0. Then for some $m \gg 1$, the sheaf $\mathcal{O}_X(-m(K_{X/C}+\Delta) - f^*A)$ is very ample over X' (see [Gro61, Corollaire 4.5.11]). It follows that there exists a prime divisor D' on X' such that the pair $(X', \Delta^{\mathrm{hor}}|_{X'} + D')$ is snc over X_0 and

$$D' \sim (-m(K_{X/C} + \Delta) - f^*A)|_{X'}.$$

This implies that there exists a rational function $h \in K(X') = K(X)$ such that the restriction of the Cartier divisor $D = \mathrm{div}(h) - m(K_{X/C} + \Delta) - f^*A$ on X' is D', and D^{hor} is the closure of D' in X. Note that we can write $D = D_+ - D_-$ for some effective divisors D_+ and D_- with no common components. Then we have $\mathrm{Supp}(D_-) \subset X \setminus X'$. In particular, no fiber of f is supported on D_-. By Théorème 0.4.2, there exists a log resolution $\mu\colon \widetilde{X} \to X$ such that the following conditions hold.

(1) Then induced morphism $\widetilde{f} = f \circ \mu\colon \widetilde{X} \to C$ is prepared.
(2) The birational morphism μ is an isomorphism over X_0.
(3) $\mu_*^{-1}\Delta^{\mathrm{hor}} + \mu_*^{-1}D^{\mathrm{hor}}$ is a snc divisor.

Let E be the exceptional divisor of μ. Note that we have $\widetilde{f}_*(E) \neq C$. Moreover, we also have

$$K_{\widetilde{X}} + \mu_*^{-1}\Delta + \frac{1}{m}\mu_*^{-1}D_+ = \mu^*\left(K_X + \Delta + \frac{1}{m}D\right) + \frac{1}{m}\mu_*^{-1}D_- + E_+ - E_-.$$

where E_+ and E_- are effective μ-exceptional divisors with no common components. Set

$$\widetilde{D} = m\mu_*^{-1}\Delta + \mu_*^{-1}D_+ + mE_-.$$

Then $\widetilde{D}^{\mathrm{hor}} = m\mu_*^{-1}\Delta^{\mathrm{hor}} + \mu_*^{-1}D^{\mathrm{hor}}$ is a snc effective divisor with coefficients at most m. Since D is linearly equivalent to $-m(K_{X/C}+\Delta) - f^*A$, we can write

$$K_{\widetilde{X}/C} + \frac{1}{m}\widetilde{D} \sim_{\mathbb{Q}} -\frac{1}{m}\widetilde{f}^*A + \frac{1}{m}\mu_*^{-1}D_- + E_+.$$

After multiplying by some positive l divisible by the denominators of the coefficients of E_+ and E_-, we may assume that lmE_+ and lmE_- are integer coefficients. By replacing $\widetilde{D}$ by $l\widetilde{D}$, the weak positivity theorem 2.3.4 implies that the following direct image sheaf

$$\begin{aligned}\widetilde{f}_*(\omega_{\widetilde{X}/C}^{lm} \otimes \mathcal{O}_{\widetilde{X}}(\widetilde{D})) &\simeq \widetilde{f}_*(\mathcal{O}_{\widetilde{X}}(-l\widetilde{f}^*A + lmE_+ + l\mu_*^{-1}D_-)) \\ &\simeq \mathcal{O}_C(-lA) \otimes \widetilde{f}_*\mathcal{O}_{\widetilde{X}}(lmE_+ + l\mu_*^{-1}D_-)\end{aligned}$$

is weakly positive.

Observe that $\widetilde{f}_*(\mathcal{O}_{\widetilde{X}}(lmE_+ + l\mu_*^{-1}D_-)) = \mathcal{O}_C$. Indeed, E_+ is a μ-exceptional divisor. It follows that we have a natural inclusion $\mu_*(\mathcal{O}_{\widetilde{X}}(lmE_+ + l\mu_*^{-1}D_-)) \to \mathcal{O}_X(lD_-)$. Note that $f_*(\mathcal{O}_X(lD_-))$ is a torsion free coherent sheaf of rank one, it follows that $f_*(\mathcal{O}_X(lD_-)) \cong \mathcal{O}_C(P)$ for some effective divisor P on C such that $\mathrm{Supp}(P) \subset f(\mathrm{Supp}(D_-))$. Let V be an open subset of C, and let $\lambda \in H^0(V, \mathcal{O}_C(P))$. That is, λ is a rational function on C such that $\mathrm{div}(\lambda) + P \geq 0$ over V. It follows that $\mathrm{div}(\lambda \circ f) + lD_- \geq 0$ over $f^{-1}(V)$. Since there is no fiber of f completely supported on D_-, the rational function $\lambda \circ f$ is regular over $f^{-1}(V)$. Consequently, the rational function λ is regular over V. This implies that the natural inclusion $\mathcal{O}_C \to \mathcal{O}_C(P)$ is surjective, which yields $\widetilde{f}_*(\mathcal{O}_{\widetilde{X}}(lmE_+ + l\mu_*^{-1}D_-)) = \mathcal{O}_C$. However, this shows that $\mathcal{O}_C(-lA)$ is weakly positive, a contradiction. Hence, $\mathcal{O}_X(-K_{X/C} - \Delta)$ is not ample over X'. $\square$

2.3.7. Lemma. *Let X be a normal projective variety, and let $f\colon X \to C$ be a surjective morphism with reduced and connected fibers onto a smooth curve C. Let D be a Cartier divisor on X. If there exists a nonzero morphism $\Omega^r_{X/C} \to \mathcal{O}_X(D)$, where r is the relative dimension of f, then there exists an effective Weil divisor Δ on X such that $K_{X/C} + \Delta = D$.*

Proof. Since all the fibers of f are reduced, the sheaf $\Omega^r_{X/C}$ is locally free in codimension one. Hence, the reflexive hull of $\Omega^r_{X/C}$ is $\omega_{X/C} \simeq \mathcal{O}_X(K_{X/C})$. Note that $\mathcal{O}_X(D)$ is reflexive, the nonzero morphism of sheaves $\Omega^r_{X/C} \to \mathcal{O}_X(D)$ induces a nonzero morphism $\omega_{X/C} \to \mathcal{O}_X(D)$. This shows that there exists an effective divisor Δ on X such that $K_{X/C} + \Delta = D$. $\square$

As an application of Theorem 2.3.6, we derive a special property about foliations defined by an ample subsheaf of T_X. A similar result was established for Fano foliations with mild singularities in the work of Araujo and Druel (see [AD13, Proposition 5.3]), and we follow the same strategy.

2.3.8. Proposition. *Let X be a n-dimensional projective manifold. If $\mathcal{F} \subset T_X$ is an ample subsheaf of positive rank $r < n$, then there is a common point in the closure of general leaves of $\overline{\mathcal{F}}$.*

Proof. Since $\mathcal{F}$ is torsion-free and X is smooth, $\mathcal{F}$ is locally free over an open subset $X' \subset X$ such that $\mathrm{codim}(X \setminus X') \geq 2$. In particular, $\mathcal{O}_X(-K_{\mathcal{F}})$ is ample over X'. By Theorem 2.1.9, there exists an open subset $X_0 \subset X$ and a $\mathbb{P}^{d+1}$-bundle $\varphi_0\colon X_0 \to T_0$. Moreover, from the proof of Theorem 2.2.9, the saturation $\overline{\mathcal{F}}$ defines an algebraically integrable foliation on X, and we may assume that $\mathcal{F}$ is locally free over X_0. In particular, we have $X_0 \subset X'$. In view of Lemma 2.2.7, we denote by W the subvariety of $\mathrm{Chow}(X)$ parametrizing the general leaves of $\overline{\mathcal{F}}$ and by V the normalization of the universal cycle over W. Let $p\colon V \to X$ and $\pi\colon V \to W$ be the natural projections. Note that there exists an open subset W_0 of W such that $p(\pi^{-1}(W_0)) \subset X_0$.

To prove our proposition, we assume to the contrary that there is no common point in the general leaves of $\overline{\mathcal{F}}$.

First, we show that there exists a smooth curve C with a finite morphism $n\colon C \to n(C) \subset W$ such that we have the following.

(1) Let U be the normalization of the fiber product $V \times_W C$ with projection $\pi\colon U \to C$. Then the induced morphism $\widetilde{p}\colon U \to X$ is finite onto its image.
(2) There exists an open subset C_0 of C such that the image of U_0 under p is contained in X_0. In particular, $U_0 = \pi^{-1}(C_0)$ is a $\mathbb{P}^r$-bundle over C_0.
(3) For any point $c \in C$, the image of the fiber $\pi^{-1}(c)$ under $\widetilde{p}$ is not contained in $X \setminus X'$.
(4) All the fibers of π are reduced.

Note that we have $X \setminus X' = \mathrm{Sing}\,(\mathcal{F})$ and $\mathrm{codim}(\mathrm{Sing}\,(\mathcal{F})) \geq 2$. We consider the subset

$$Z = \{w \in W \mid \pi^{-1}(w) \subset p^{-1}(\mathrm{Sing}(\mathcal{F}))\}.$$

Since π is equidimensional, it is a surjective universally open morphism (see [Gro66, Théorème 14.4.4]). Therefore, the subset Z is closed. Note that the general fiber of π is disjoint from $p^{-1}(\mathrm{Sing}(\mathcal{F}))$, so $\mathrm{codim}(Z) \geq 1$. Moreover, by the definition of Z, we have $p(\pi^{-1}(Z)) \subset \mathrm{Sing}(\mathcal{F})$ and $\mathrm{codim}(\mathrm{Sing}(\mathcal{F})) \geq 2$. Hence, we can choose some very ample divisors H_i $(1 \leq i \leq n)$ on X such that the curve B defined by complete intersection $\widetilde{p}^*H_1 \cap \cdots \cap \widetilde{p}^*H_n$ satisfies the following conditions.

(1') There is no common point in the closure of the general fibers of π over $\pi(B)$.
(2') $\pi(B) \cap W_0 \neq \emptyset$.
(3') $\pi(B) \subset W \setminus Z$.

Let $B' \to B$ be the normalization, and let $V_{B'}$ be the normalization of the fiber product $V \times_B B'$. The induced morphism $V_{B'} \to V$ is denoted by μ. Then it is easy to check that B' satisfies (1), (2) and (3). By Lemma 2.3.5, there exists a finite morphism $C \to B'$ such that all the fibers of $U \to C$ are reduced, where U is the normalization of $V_{B'} \times_{B'} C$. Then we see at once that C is the desired curve.

The next step is to get a contradiction by applying Theorem 2.3.6. From Remark 2.2.8, we see that the Pfaff field $\Omega^r_X \to \mathcal{O}_X(K_{\overline{\mathcal{F}}})$ extends to a morphism $\Omega^r_{V_{B'}/B'} \to \mu^*p^*\mathcal{O}_X(K_{\overline{\mathcal{F}}})$, and it induces a morphism $\Omega^r_{U/C} \to \widetilde{p}^*\mathcal{O}_X(K_{\overline{\mathcal{F}}})$. The natural inclusion $\mathcal{F} \hookrightarrow \overline{\mathcal{F}}$ induces a morphism $\mathcal{O}_X(K_{\overline{\mathcal{F}}}) \to \mathcal{O}_X(K_{\mathcal{F}})$. This implies that we have a morphism $\Omega^r_{U/C} \to \widetilde{p}^*\mathcal{O}_X(K_{\mathcal{F}})$. By Lemma 2.3.7, there exists an effective Weil divisor Δ on U such that $K_{U/C} + \Delta = \widetilde{p}^*K_{\mathcal{F}}$.

Let Δ^{hor} be the π-horizontal part of Δ. After shrinking C_0, we may assume that $\Delta|_{U_0} = \Delta^{hor}|_{U_0}$. According to the proof of Theorem 2.2.9, for any fiber $F \cong \mathbb{P}^r$ over C_0, we have $(\widetilde{p}^*K_{\mathcal{F}})|_F - K_F = 0$ or H where $H \in |\mathcal{O}_{\mathbb{P}^r}(1)|$. This shows that either Δ^{hor} is zero or Δ^{hor} is a prime divisor such that $\Delta|_{U_0} = \Delta^{\mathrm{hor}}|_{U_0} \in |\mathcal{O}_{U_0}(1)|$. In particular, the pair $(U, \Delta^{\mathrm{hor}})$ is snc over U_0 and Δ^{hor} is reduced. Note that $\widetilde{p}\colon U \to \widetilde{p}(U)$ is a finite morphism, so the invertible sheaf $\widetilde{p}^*\mathcal{O}_X(-K_{\mathcal{F}})$ is ample over $U' = U \cap \widetilde{p}^{-1}(X')$; that is, the sheaf $\mathcal{O}_U(-K_{U/C} - \Delta)$ is ample over U', which contradicts Theorem 2.3.6. □

Now, our main result in this chapter immediately follows.

2.3.9. Theorem. *Let X be a projective manifold of dimension n. Suppose that T_X contains an ample subsheaf $\mathcal{F}$ of positive rank r, then $(X, \mathcal{F})$ is isomorphic to $(\mathbb{P}^n, T_{\mathbb{P}^n})$ or $(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus r})$.*

Proof. Theorem 2.1.9 implies that there exists an open subset $X_0 \subset X$ and a normal variety T_0 such that $X_0 \to T_0$ is a $\mathbb{P}^{d+1}$-bundle and $d + 1 \geq r$. Without loss of generality, we may assume $r < n$. By Theorem 2.2.9 followed by Proposition 2.3.8, $\overline{\mathcal{F}}$ defines an algebraically integrable foliation over X such that there is a common point in the closure of general leaves of $\overline{\mathcal{F}}$. However, this cannot happen if $\dim(T_0) \geq 1$. Hence, we have $\dim T_0 = 0$ and $X \cong \mathbb{P}^n$. □

Recall that a nontrivial quotient of an ample sheaf is again ample (cf. § 2.1.1). Then we obtain the following corollary.

2.3.10. Corollary. *Let X be a projective manifold of dimension n, E an ample vector bundle on X. If there exists a non-zero map $E \to T_X$, then $X \cong \mathbb{P}^n$.*

2.4 Some applications of the main theorem

2.4.1 $\mathbb{P}^r$-bundles as ample divisors

The first application of Theorem 2.3.9 is to classify projective manifolds X containing a $\mathbb{P}^r$-bundle as an ample divisor. This was originally conjectured by Beltrametti and Sommese (see [BS95, Conjecture 5.5.1]). In the remainder of this section, we follow the same notation and assumptions as in Theorem 1.1.3.

The case $r \geq 2$ follows from Sommese's extension theorem [Som76] (see also [BS95, Theorem 5.5.2]). For $r = 1$ and $b = 1$, it is due to Bădescu [Băd84, Theorem D] (see also [BS95, Theorem 5.5.3]). For $r = 1$ and $b = 2$, it is due to the work of several authors (see [BI09, Theorem 7.4]). As mentioned in the introduction, Litt proved the following result, by which we can deduce Theorem 1.1.3 from Corollary 2.3.10.

2.4.1. Proposition [Lit17, Lemma 4]. *Let X be a projective manifold of dimension ≥ 3, and let A be an ample divisor. Assume that $p\colon A \to B$ is a $\mathbb{P}^1$-bundle, then either p extends to a morphism $\widehat{p}\colon X \to B$, or there exists an ample vector bundle E on B and a non-zero map $E \to T_B$.*

For the reader's convenience, we outline the argument of Litt that reduces Theorem 1.1.3 to Corollary 2.3.10.

2.4.2. Theorem. *Let X be a projective manifold of dimension $n \geq 3$, let A be an ample divisor on X. Assume that A is a $\mathbb{P}^r$-bundle, $p\colon A \to B$, over a manifold B of dimension $b > 0$. Then one of the following holds :*

(1) $(X, A) = (\mathbb{P}(E), H)$ for some ample vector bundle E over B such that $H \in |\mathcal{O}_{\mathbb{P}(E)}(1)|$, and p is equal to the restriction to A of the induced projection $\mathbb{P}(E) \to B$.

(2) $(X, A) = (\mathbb{P}(E), H)$ for some ample vector bundle E over $\mathbb{P}^1$ such that $H \in |\mathcal{O}_{\mathbb{P}(E)}(1)|$, $H = \mathbb{P}^1 \times \mathbb{P}^{n-2}$ and p is the projection to the second factor.

(3) $(X, A) = (Q^3, H)$, where Q^3 is a smooth quadric threefold, H is a smooth quadric surface, and p is the projection to one of the factors $H \cong \mathbb{P}^1 \times \mathbb{P}^1$.

(4) $(X, A) = (\mathbb{P}^3, H)$, where H is a smooth quadric surface and p is again a projection to one of the factors of $H \cong \mathbb{P}^1 \times \mathbb{P}^1$.

Proof. Since the case $r \geq 2$ is already known, we can assume that $r = 1$; that is, $p\colon A \to B$ is a $\mathbb{P}^1$-bundle.

If p extends to a morphism $\widehat{p}\colon X \to B$, then the result follows from [BI09, Theorem 5.5] and we are in the case (1) of the theorem.

If p does not extend to a morphism $X \to B$, by Proposition 2.4.1, there exists an ample vector bundle E over B with a non-zero map $E \to T_B$. Due to Corollary 2.3.10, we have $B \cong \mathbb{P}^b$. As the case $b \leq 2$ is also known, we may assume that $b \geq 3$. In this case, by [FSS87, Theorem 2.1], we conclude that X is a $\mathbb{P}^{n-1}$-bundle over $\mathbb{P}^1$ and we are in the case (2) of the theorem. □

2.4.2 Endomorphism of projective varieties

A classic question in algebraic geometry asks for a description of projective manifolds X that admits surjective ramified endomorphisms $f\colon X \to X$ of degree at least two. A long-standing conjecture predicts that if such X is a Fano manifold with $\rho(X) = 1$, then X is isomorphic to some projective space. Note that a counter-example given by Kollár and Xu in [KX09] shows that it is false if X is not smooth.

Given any finite flat morphism $f\colon Y \to X$ with smooth target X, we can associate a natural vector bundle

$$\mathcal{E}_f := (f_*\mathcal{O}_Y/\mathcal{O}_X)^\vee .$$

The trace map give a splitting $f_*\mathcal{O}_Y \cong \mathcal{O}_X \oplus \mathcal{E}_f^\vee$. In [AKP08], it was shown the following theorem.

2.4.3. Theorem [AKP08, Theorem 1.3]. *Let X be a n-dimensional Fano manifold with $\rho(X) = 1$ and let $f: X \to X$ be an endomorphism of degree at least 2. If $\mathcal{E}_{f_k}$ is ample and $h^0(X, f_k^*T_X) > h^0(X, T_X)$ for some iterate f_k of f, then $X \cong \mathbb{P}^n$.*

Own to our main theorem, we can weaken the hypothesis in this theorem. Namely, we have the following result.

2.4.4. Corollary. *Let X be a projective manifold of dimension n with an endomorphism $f: X \to X$ of degree ≥ 2. Then $X \cong \mathbb{P}^n$ if $\mathcal{E}_{f_k}$ is ample and $h^0(X, f_k^*T_X) > h^0(X, T_X)$ for some iterate f_k of f.*

Proof. We follow the argument in [AKP08]. In fact, we consider the splitting

$$\begin{aligned} H^0(X, f_k^*T_X) &= H^0(X, (f_k)_*(f_k^*T_X)) = H^0(X, T_X \otimes (f_k)_*\mathcal{O}_X) \\ &= H^0(X, T_X \oplus (T_X \otimes \mathcal{E}_{f_k}^\vee)) = H^0(X, T_X) \oplus H^0(X, T_X \otimes \mathcal{E}_{f_k}^\vee). \end{aligned}$$

So the hypothesis $h^0(X, f_k^*T_X) > h^0(X, T_X)$ is equivalent to the fact $h^0(X, T_X \otimes \mathcal{E}_{f_k}^\vee) > 0$. Moreover, the latter implies that $Hom(\mathcal{E}_{f_k}, T_X)$ is not zero and hence X is isomorphic to some projective space by Corollary 2.3.10. ☐

2.4.5. Remark. Usually it is difficult to show the ampleness of $\mathcal{E}_{f_k}$. However, in [AKP08, Theorem 3.7], they proved the following criterion : if f_k is a Galois covering of smooth varieties which does not factor through étale covering of X, such that all irreducible components of the ramification divisor R is ample on X, then the bundle $\mathcal{E}_{f_k}$ is ample.

Chapitre 3

Stability of the tangent bundles of complete intersections and effective restriction

This chapter is devoted to study the stability of tangent bundles of Fano manifolds with Picard number one, and most results are included in the paper [Liu17b] except § 3.1. There are two main approaches to this problem : cohomology vanishing and geometry of rational curves (see [PW95, Hwa98]). On the other hand, if $X = G/P$ is a rational homogeneous space with Picard number one, using representation theory, one can prove that every vector bundle E_ρ induced by an irreducible representation ρ of P is stable (see [Ram66]). More general, the tangent bundle T_X is stable if X is a rational homogeneous space of Picard number one (see [AB10]). The main result in this chapter is an improvement of Biswas-Chaput-Mourougane's inequality about vanishing theorems on irreducible Hermitian symmetric spaces of compact type and its applications to the stability problem.

3.1 Vanishing theorems on Hermitian symmetric spaces

This section is devoted to the study of vanishing theorems on irreducible Hermitian symmetric spaces of compact type and the main result in this section is the following improvement of Biswas-Chaput-Mourougane's inequality.

3.1.1. Theorem. *Let M be an irreducible Hermitian symmetric space of compact type. Denote by r_M the index of M. Let ℓ and p be two positive integers such that $H^q(M, \Omega^p_M(\ell)) \neq 0$ for some $q > 0$, then we have*

$$\ell + q - 2 \geq (p-2)\frac{r_M}{\dim(M)}. \tag{3.1}$$

The proof of Theorem 3.1.1 is essentially the same as that in [BCM18]. However, since the inequality (3.1) does not hold for $q = 0$ (see [Sno86, Proposition 3.4]), we need to treat some extremal cases in the inductive argument. We start with the basic concepts about Hermitian symmetric spaces and we refer to [BH58] for further details.

3.1.1 Hermitian symmetric spaces

Let (M, g) be a Riemannian manifold. A non-trivial isometry σ of (M, g) is said to be an *involution* if and only if $\sigma^2 = id$. A Riemannian manifold (M, g) is said to be *Riemannian symmetric* if and only if at each point $x \in M$ there exists an involution σ_x such that x is an isolated fixed point of σ_x.

3.1.2. Definition. *Let (M, g) be a Riemannian symmetric manifold. (M, g) is said to be a Hermitian symmetric manifold if (M, g) is a Hermitian manifold and the involution σ_x at each point $x \in M$ can be chosen to be a holomorphic isometry.*

A Hermitian symmetric space M is called *irreducible* if it cannot be written as the non-trivial product of two Hermitian symmetric spaces. It is well-known that the irreducible Hermitian symmetric spaces of compact type are Fano manifolds of Picard number one. We will denote by $\mathcal{O}_M(1)$ the ample generator of $\mathrm{Pic}(M)$. Moreover, in this case, the index of M is defined to the positive integer r_M such that $\mathcal{O}_M(-K_M) \cong \mathcal{O}_M(r_M)$.

The Hermitian symmetric spaces are homogeneous under their isometry groups. According to Cartan, there are exactly six types of irreducible Hermitian symmetric spaces of compact type : Grassmannians (type A_n), quadric hypersurfaces (type B_n or D_n), Lagrangian Grassmannians (type C_n), spinor Grassmannians (type D_n) and two exceptional cases (type E_6 and E_7).

Let M be a n-dimensional irreducible Hermitian symmetric space of compact type. According to the Kobayashi-Ochiai's theorem [KO73], if $r_M \geq n$, then M is isomorphic to either $\mathbb{P}^n$ or Q^n. The cohomology groups $H^q(M, \Omega_M^p(\ell))$ for projective spaces and smooth quadric hypersurfaces were calculated by Bott and Shiffman-Sommese (cf. [Bot57, SS85]), respectively.

3.1.3. Theorem [Bot57].

$$h^q(\mathbb{P}^n, \Omega_{\mathbb{P}^n}^p(\ell)) = \begin{cases} \binom{n+\ell-p}{\ell}\binom{\ell-1}{p}, & \textit{if } q=0, 0\leq p\leq n, \ell>p; \\ 1, & \textit{if } \ell=0, p=q; \\ \binom{p-\ell}{-\ell}\binom{-\ell-1}{n-p}, & \textit{if } q=n, 0\leq p\leq n, \ell<p-n; \\ 0, & \textit{otherwise.} \end{cases}$$

As a consequence, if ℓ, q and p are positive integers, then we have $H^q(\mathbb{P}^n, \Omega_{\mathbb{P}^n}^p(\ell)) = 0$. In particular, Theorem 3.1.1 holds for projective spaces automatically.

3.1.4. Theorem [Sno86, Theorem 4.1]. *Let X be a n-dimensional smooth quadric hypersurface.*

(1) If $-n+p \leq \ell \leq p$ and $\ell \neq 0, -n+2p$, then $H^q(X, \Omega_X^p(\ell))$ for all q.
(2) $H^q(X, \Omega_X^p) \neq 0$ if and only if $q = p$.
(3) $H^q(X, \Omega_X^p(-n+2p)) \neq 0$ if and only if $p+q = n$.
(4) If $\ell > p$, then $H^q(X, \Omega_X^p(\ell)) \neq 0$ if and only if $q = 0$.
(5) If $\ell < -n+p$, then $H^q(X, \Omega_X^p(\ell)) \neq 0$ if and only if $q = n$.

In particular, if X is a n-dimensional smooth quadric hypersurface and $H^q(X, \Omega_X^p(\ell)) \neq 0$ for some positive integers ℓ, q and p, then we get $q+p = n$ and $\ell = 2p-n$. As a consequence, it follows that we have $q+\ell-2 = p-2$. Note that $r_X = \dim(X)$ for smooth quadric hypersurfaces X, so Theorem 3.1.1 holds with equality for all smooth quadric hypersurfaces. On the other hand, if $\ell \geq r_M$, then the cohomological dimension of $\Omega_M^p(\ell)$ is zero [Sno88, Proposition 1.1]. Equivalently, $H^q(M, \Omega_M^p(\ell)) \neq 0$ if and only if $q = 0$. Therefore, we have $\ell \leq r_M - 1$ in Theorem 3.1.1 since we assume $q > 0$.

In the sequel of this section, we will prove Theorem 3.1.1 case by case. For the details of the combinatorial aspects of the cohomologies of twisted holomorphic forms on irreducible Hermitian symmetric spaces of compact type, we refer the reader to the articles of Snow [Sno86, Sno88].

3.1.2 Exceptional cases (type E_6 and type E_7)

If M is of type E_6, then M is a 16-dimensional Fano manifold of index 12. If M is of type E_7, then M is a 27-dimensional Fano manifold of index 18. For $1 \leq \ell \leq r_M - 1$, the cohomologies of $\Omega_M^p(\ell)$ are given in [Sno88, Table 4.4 and 4.5], respectively. In particular, one can easily derive the following theorem.

3.1.5. Theorem. *Let M be the irreducible compact Hermitian symmetric space of type E_6 or E_7. Let ℓ and p be two positive integers such that $H^q(M, \Omega_M^p(\ell)) \neq 0$ for some $q \geq 0$.*

(1) *If $q \geq 1$, then we have*

$$\ell + q - 2 \geq (p-2)\frac{r_M}{\dim(M)}$$

with equality if and only if M is of type E_6 and $(\ell, q, p) = (9, 2, 14)$.

(2) *If $q = 0$ and $\ell \leq r_M - 1$, then we have*

$$\ell - 1 > p\frac{r_M}{\dim(M)}.$$

3.1.3 Grassmannians (type A_n)

Fix two positive integers $r \geq s \geq 1$ and denote by $Gr(s, r+s)$ the Grassmannian that parametrizes s-dimensional linear subspaces of a fixed $(r+s)$-dimensional $\mathbb{C}$-vector space V. Then $Gr(s, r+s)$ is a (rs)-dimensional homogeneous variety of index $r+s$ and it is isomorphic to the homogeneous space $SU(r+s)/(SU(r+s) \cap U(r) \times U(s))$. Moreover, if $s = 1$, the Grassmann manifold $Gr(1, r+1)$ is just the projective space $\mathbb{P}^r$, and if $r < s$, then $Gr(s, r+s)$ is naturally isomorphic to $Gr(r, r+s)$, so we shall assume that $2 \leq s \leq r$ throughout this subsection.

To explain the algorithm of Snow, we need to settle some definitions and notations regarding partitions and Young diagrams. Let $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_s > 0)$ be a *partition* of size $p(\lambda) = \lambda_1 + \cdots + \lambda_s$.

3.1.6. Definition. *Given a partition λ of size $p(\lambda)$, a Young diagram of shape λ is an array of boxes arranged in rows. There are λ_i boxes in the ith row, the boxes are left justified. We will also denote it by λ.*

Given a Young diagram λ, the *hook length* $h_{i,j}$ of the cell (i,j) is the number of boxes directly to its left, or directly below it, including itself. We denote by λ'_i the number of boxes in the ith column. The transposed Young diagram of λ is the Young diagram of sharp $\lambda' = (\lambda'_1, \lambda'_2, \cdots, \lambda'_r)$, where $r = \lambda_1$.

3.1.7. Definition. *Given a positive integer ℓ, a Young diagram λ of size $p(\lambda)$ is called ℓ-admissible if no hook length is equal to ℓ, and the ℓ-cohomological degree $q(\lambda)$ of λ is defined to be the number of hook lengths which are $> \ell$.*

3.1.8. Example. For the partition $\lambda = (4, 3, 1)$ of 8, the Young diagram of shape λ and its transposed Young diagram are as follows. Moreover, if ℓ is a positive integer, then λ is ℓ-admissible if and only if $\ell = 5$ or $\ell \geq 7$, and the ℓ-cohomological degree is 1 and 0, respectively.

6	4	3	1
4	2	1	
1			

6	4	1
4	2	
3	1	
1		

3.1.9. Proposition [Sno86, Proposition 3.1 and Proposition 2.2]. *Let $M = Gr(s, r+s)$ be a Grassmann manifold. For $1 \leq \ell \leq r_M - 1$. Then $H^q(M, \Omega^p_M(\ell)) \neq 0$ for some $p, q \geq 0$ if and only if there exists an ℓ-admissible partition $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_{s'})$ of size p with ℓ-cohomological degree q such that $\lambda_1 \leq r$ and $s' \leq s$.*

3.1.10. Remark. The "if" part is stated in [Sno86, Proposition 3.1] and the "only if" part is stated in [Sno86, Proposition 2.2]. We remark that the constant c in [Sno86, Proposition 2.2] is 1 and the number $-(\delta, \omega^{-1}\alpha)$ is calculated in [Sno86, p167] and it is exactly the hook number $h_{i,j}$ for some $i, j > 0$.

3.1.11. Proposition. *Let $\lambda = (\lambda_1, \cdots, \lambda_s)$ be an ℓ-admissible Young diagram such that $1 \leq \ell \leq h_{1,1}$. Then we have*

$$\ell + q(\lambda) - 2 \geq (p(\lambda) - 2)\frac{\lambda_1 + s}{s\lambda_1}$$

with equality if and only if one of the following conditions is satisfied.

(1) $\lambda = (2,1)$ *and* $\ell = 2$.
(2) $\lambda = (4,2)$ *and* $\ell = 3$.
(3) $\lambda = (3,1,1)$ *and* $\ell = 3$.

Proof. For convenience, given an ℓ-admissible Young diagram λ, we introduce

$$\Delta(\lambda) = \ell + q(\lambda) - 2 - \frac{p(\lambda)-2}{\lambda_1} - \frac{p(\lambda)-2}{s}.$$

Then our aim is to prove $\Delta(\lambda) \geq 0$. We proceed by repeatedly replacing λ by a combinatorially simpler ℓ-admissible partition $\widehat{\lambda}$ such that $\Delta(\widehat{\lambda}) \leq \Delta(\lambda)$. We denote by $m(\lambda) = \min\{s, \lambda_1\}$, and we proceed by induction on $m(\lambda)$. We start with the simplest case $m(\lambda) = 2$.

Step 1. $m(\lambda) = 2$.

After replacing λ by its transposition λ' if necessary, we can suppose that $s = 2$. Since λ is ℓ-admissible and $1 \leq \ell \leq \lambda_1 + 1$, we get $\ell = \lambda_1 - \lambda_2 + 1$ and $\lambda_2 \leq \lambda_1 - \lambda_2$.

λ_1+1	$\dots$	$\lambda_1-\lambda_2+2$	$\lambda_1-\lambda_2$	$\dots$	1
λ_2	$\dots$	1			

By definition, $q(\lambda) = \lambda_2$ and $p(\lambda) = \lambda_1 + \lambda_2$, so we get

$$\begin{aligned}\Delta(\lambda) &= (\lambda_1 - \lambda_2 + 1) + \lambda_2 - 2 - (\lambda_1 + \lambda_2 - 2)\left(\frac{1}{\lambda_1} + \frac{1}{2}\right) \\ &= \frac{\lambda_1}{2} - \frac{\lambda_2 - 2}{\lambda_1} - \frac{\lambda_2 + 2}{2}.\end{aligned}$$

If $\lambda_2 - 2 < 0$, then $\lambda_2 = 1$. As $\lambda_1 \geq 2$, then we get

$$\Delta(\lambda) = \frac{\lambda_1}{2} + \frac{1}{\lambda_1} - \frac{3}{2} \geq 0$$

with equality if and only if $\lambda_1 = 2$. If $\lambda_1 - 2 \geq 0$, since $\lambda_1 \geq 2\lambda_2$, one can derive

$$\Delta(\lambda) \geq \frac{2\lambda_2}{2} - \frac{\lambda_2 - 2}{2\lambda_2} - \frac{\lambda_2 + 2}{2} = \frac{\lambda_2}{2} + \frac{1}{\lambda_2} - \frac{3}{2} \geq 0$$

with equality if and only if $\lambda_2 = 2$ and $\lambda_1 = 4$.

Step 2. $h_{2,1} < \ell < h_{1,1}$ ***and*** $m(\lambda) \geq 3$.

For convenience, we set $q = q(\lambda)$ and $p' = \lambda'_{q+1}$. Since λ is ℓ-admissible and $h_{2,1} < \ell$, it follows that $h_{1,q} > \ell > h_{1,q+1}$. Then λ can be embedded into a new Young diagram $\widehat{\lambda} = (\widehat{\lambda}_1, \widehat{\lambda}_2, \cdots, \widehat{\lambda}_s)$ defined as follows (see Figure 3.1) :

$$\begin{cases} \widehat{\lambda}_1 = \lambda_1, & \\ \widehat{\lambda}_i = \lambda_2, & 2 \leq i \leq p' \\ \widehat{\lambda}_i = q, & p' + 1 \leq i \leq s. \end{cases}$$

Then $\widehat{\lambda}$ is also an ℓ-admissible Young diagram such that $q(\widehat{\lambda}) = q(\lambda) = q$ and $p(\widehat{\lambda}) \geq p(\lambda)$. In particular, we have $\Delta(\lambda) \geq \Delta(\widehat{\lambda})$. Therefore, to prove $\Delta(\lambda) \geq 0$, it is enough to show $\Delta(\widehat{\lambda}) \geq 0$. We set $p = p(\widehat{\lambda})$.

Case 2.a) $\ell = q + s - 1$.

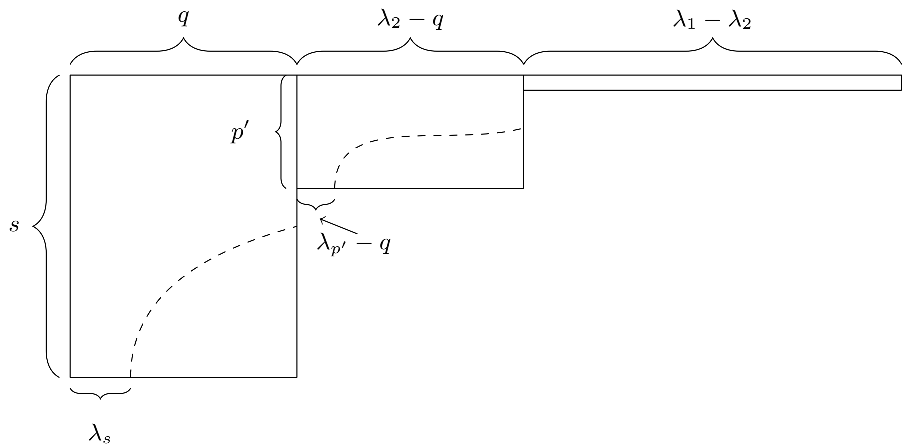


FIGURE 3.1 – Young diagrams λ and $\widehat{\lambda}$

In this case, we have $h_{2,1} = \ell - 1$ and $\lambda_2 = q$. By definition, we get $p = sq + \lambda_1 - q$. Note that we have

$$h_{1,q} \leq \lambda_1 - \lambda_2 + s = \lambda_1 - q + s \text{ and } h_{1,q+1} = \lambda_1 - \lambda_2 + p' - 1 = \lambda_1 - q.$$

Since $h_{1,q+1} < \ell < h_{1,q}$ and $\ell = q + s - 1$, we obtain

$$2q \leq \lambda_1 \leq 2q + s - 2.$$

This implies

$$\begin{aligned} \Delta(\widehat{\lambda}) &= 2q + s - 3 - \frac{sq + \lambda_1 - q - 2}{\lambda_1} - \frac{sq + \lambda_1 - q - 2}{s} \\ &= q + s - 4 + \frac{q+2}{s} - \frac{sq - q - 2}{\lambda_1} - \frac{\lambda_1}{s}. \end{aligned}$$

As $s \geq 3$ and $q \geq 1$, we have $sq - q - 2 \geq 0$. In particular, for fixed s and q, the function $\Delta(\widehat{\lambda})$ is a concave function in the variable λ_1. As $2q \leq \lambda_1 \leq 2q + s - 2$, if we fix s and q, the function $\Delta(\widehat{\lambda})$ attains the minimum value at $\lambda_1 = 2q$ or $\lambda_1 = 2q + s - 2$. By a straightforward computation, we obtain

$$\Delta(\widehat{\lambda}) = \begin{cases} \dfrac{s}{2} + \dfrac{2}{s} + \dfrac{s-1}{s} q + \dfrac{1}{q} - \dfrac{7}{2}, & \text{if } \lambda_1 = 2q; \\[2ex] \dfrac{sq - q - 2 + (s-2)(s-3)}{s} - \dfrac{sq - q - 2}{2q + s - 2}, & \text{if } \lambda_1 = 2q + s - 2. \end{cases}$$

As $s \geq 3$, $q \geq 1$ and $s \leq 2q + s - 2$, one can easily see that we have $\Delta(\widehat{\lambda}) \geq 0$ in each case. Moreover, the equality holds if and only if $\lambda_1 = 2q + s - 2$, $s = 3$ and $q = 1$. As a consequence, we conclude $\Delta(\lambda) \geq 0$ with equality if and only if $\lambda = \widehat{\lambda}$ and $(\ell, \lambda_1, s, q) = (3, 3, 3, 1)$, i.e., $\lambda = (3, 1, 1)$ and $\ell = 3$.

Case 2.b) $\ell \geq q + s$.

In this case, one observes that the Young diagram $\widehat{\lambda}$ can be embedded into the Young diagram ν defined as follows (see Figure 3.2) :

$$\begin{cases} \nu_1 = \widehat{\lambda}_1 = \lambda_1, \\ \nu_i = \ell + 1 - s, & 2 \leq i \leq \ell + q - \widehat{\lambda}_1 \\ \nu_i = q, & i \geq \ell + q - \widehat{\lambda}_1. \end{cases}$$

In particular, the hook lengths $h_{2,1}(\nu)$ and $h_{1,q+1}(\nu)$ of ν are both equal to $\ell - 1$. Note that the Young

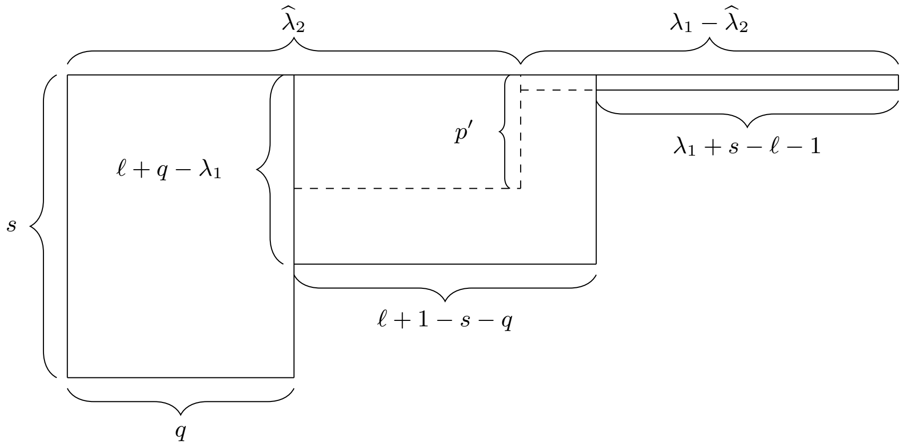


FIGURE 3.2 – Young diagrams $\widehat{\lambda}$ and ν

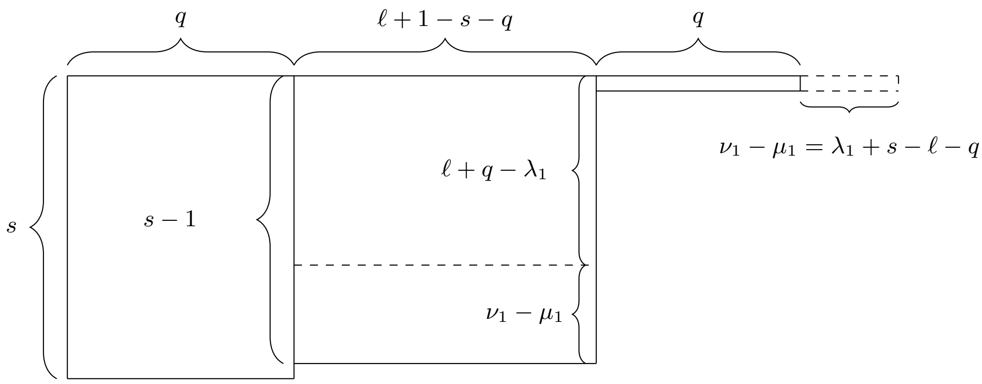


FIGURE 3.3 – Young diagrams ν and μ

diagram ν is ℓ-admissible with $q(\nu) = q(\widehat{\lambda})$ and $p(\nu) \geq p(\widehat{\lambda})$. Thus we have $\Delta(\nu) \leq \Delta(\widehat{\lambda})$. Moreover, as $h_{1,q}(\widehat{\lambda}) = \lambda_1 - q + s \geq \ell + 1$, we have $\lambda_1 \geq \ell + q + 1 - s$. Now we consider the Young diagram μ defined as follows (see Figure 3.3) :

$$\begin{cases} \mu_1 = \ell + q + 1 - s, & \\ \mu_i = \ell + 1 - s, & 2 \leq i \leq s - 1 \\ \mu_s = q. & \end{cases}$$

Then μ is an ℓ-admissible Young diagram such that $q(\mu) = q(\nu)$ and

$$p(\mu) = p(\nu) + (\nu_2 - q - 1)(\nu_1 - \mu_1).$$

As $\nu_2 - q - 1 = \ell - q - s \geq 0$ by our assumption and $\nu_1 = \lambda_1 \geq \mu_1$, we obtain $\Delta(\nu) \geq \Delta(\mu)$. Moreover, observe that we have

$$p(\mu) = s\mu_1 - (\mu_1 - q) - (s - 2)q.$$

As $\ell = \mu_1 + s - q - 1$, we obtain

$$\begin{aligned}\Delta(\mu) &= (\mu_1 + sq - 3q + 2)\left(\frac{1}{\mu_1} + \frac{1}{s}\right) - 3 \\ &= \frac{(s-3)q+2}{\mu_1} + \frac{\mu_1 - 2q + 2}{s} + \frac{s-1}{s}q - 2.\end{aligned}$$

As $\mu_1 = \ell + q + 1 - s \geq 2q + 1$ and $s \geq 3$, we get

$$\Delta(\mu) > \frac{3}{s} + \frac{s-1}{s}q - 2 \geq 1 + \frac{2}{3}q - 2.$$

In particular, we have $\Delta(\mu) > 0$ if $q \geq 2$. So it remains to consider the case $q = 1$. If $q = 1$, as $s \geq 3$ and $\mu_1 \geq 2q + 1 = 3$, we get

$$\Delta(\mu) = \frac{s-1}{\mu_1} + \frac{\mu_1 - 1}{s} - 1 > 0.$$

Hence, we conclude $\Delta(\lambda) \geq \Delta(\nu) \geq \Delta(\mu) > 0$.

Step 3. $h_{2,1} \geq \ell + 1$ ***and*** $m(\lambda) \geq 3$.

In this case, we may assume that $h_{1,2} \geq \ell + 1$ and $\lambda_1 \geq s$. Otherwise we can proceed by considering the transposed Young diagram λ'. Moreover, we assume that $\Delta(\lambda) \geq 0$ for $m(\lambda) \leq s - 1$. Now we consider the Young diagram $\widehat{\lambda}$ obtained by removing the first column in λ, i.e., $\widehat{\lambda}_i = \max\{\lambda_i - 1, 0\}$. Then $\widehat{\lambda}$ is ℓ-admissible with $q(\widehat{\lambda}) \leq q(\lambda) - 2$ and $p(\widehat{\lambda}) = p(\lambda) - s$. Moreover, as $\lambda_1 \geq s$, we have

$$\begin{aligned}\Delta(\lambda) &= \ell + q(\lambda) - 2 - \frac{p(\lambda) - 2}{\lambda_1} - \frac{p(\lambda) - 2}{s} \\ &\geq \ell + q(\widehat{\lambda}) - \frac{p(\widehat{\lambda}) + s - 2}{\lambda_1} - \frac{p(\widehat{\lambda}) + s - 2}{s} \\ &\geq \ell + q(\widehat{\lambda}) - 2 - \frac{p(\widehat{\lambda}) - 2}{\widehat{\lambda}_1 + 1} - \frac{p(\widehat{\lambda}) - 2}{s} \\ &> \Delta(\widehat{\lambda}).\end{aligned}$$

If $\lambda_1 = s$, then $m(\widehat{\lambda}) = s - 1$ and $\Delta(\widehat{\lambda}) \geq 0$ by our inductive assumption. Thus, by an inductive argument on λ_1, one can derive $\Delta(\lambda) > 0$. □

3.1.12. Remark. Our result is sharp in the following sense : if $r \geq 3$, the inequality

$$\ell + q(\lambda) - r - \frac{p(\lambda) - r}{\lambda_1} - \frac{p(\lambda) - r}{s} \geq 0$$

does not hold in general. To see this, consider the partition $\lambda = (r, 1, , \cdots, 1)$ with $s = r$, $q(\lambda) = 1$ and $\ell = r$.

3.1.13. Theorem. *Let $M = Gr(s, r+s)$ be a Grassmann manifold such that $r \geq s \geq 2$. Let ℓ and p be two positive integers such that $H^q(M, \Omega^p_M(\ell)) \neq 0$ for some $q \geq 0$.*

(1) If $q \geq 1$, then we have

$$\ell + q - 2 \geq (p-2)\frac{r_M}{\dim(M)}. \tag{3.2}$$

(2) If $q = 0$ and $\ell < r_M - 1$, then we have

$$\ell - 1 > (p-1)\frac{r_M - 1}{\dim(M) - 1} \tag{3.3}$$

Proof. By Proposition 3.1.9, there exists an ℓ-admissible partition $\lambda = (\lambda_1, \cdots, \lambda_{s'})$ of p whose ℓ-

cohomological degree is q. If $q \geq 1$, we get $\ell \leq \lambda_1 + s' - 1$ and $p \geq 3$. Then Proposition 3.1.11 implies

$$\ell + q - 2 \geq (p-2)\left(\frac{1}{\lambda_1} + \frac{1}{s'}\right) \geq (p-2)\left(\frac{1}{r} + \frac{1}{s}\right) = (p-2)\frac{r_M - 1}{\dim(M) - 1}.$$

If $q = 0$, then we get $\lambda_1 + s' \leq \ell$ and $p = p(\lambda) \leq s'\lambda_1$. If $p(\lambda) = 1$, then we get $\ell \geq 2$ and we are done. Thus we may assume $p(\lambda) > 1$ and we see that it is enough to prove

$$\frac{\lambda_1 + s' - 1}{\lambda_1 s' - 1} > \frac{r + s - 1}{rs - 1}.$$

As $1 \leq \lambda_1 \leq r$ and $1 \leq s' \leq s$, there exist two nonnegative integers a and b such that $s = s' + a$ and $r = \lambda_1 + b$. Then we get

$$\begin{aligned}(s' + \lambda_1 - 1)(rs - 1) - (s'\lambda_1 - 1)(r + s - 1) &= a(\lambda_1^2 - \lambda_1 + 1) + b(s'^2 - s' + 1) + ab(s' + \lambda_1 - 1) \\ &\geq a + b + ab > 0.\end{aligned}$$

The last inequality follows from our assumption $\ell \leq r_M - 1$ and $\lambda_1 + s' \leq \ell$, i.e., $a + b > 0$. □

3.1.4 Lagrangian Grassmannians (type C_n)

In this case, M parametrizes n-dimensional Lagrangian subspaces of $\mathbb{C}^{2n}$ equipped with the standard symplectic form. It is the $n(n+1)/2$-dimensional homogeneous space $Sp(2n)/U(n)$ with index $n+1$.

3.1.14. Definition. *Let $\ell, n \in \mathbb{N}$ be two fixed positive integers. A n-tuple of integers $\boldsymbol{a}_n = (a_i)_{1 \leq i \leq n}$ is called an ℓ-admissible C_n-sequence if $|a_i| = i$ and $a_i + a_j \neq 2\ell$ for all $i \leq j$. Its weight is defined to be $p(\boldsymbol{a}_n) = \sum_{a_i > 0} a_i$ and its ℓ-cohomological degree is defined to be*

$$q(\boldsymbol{a}_n) = \#\{\,(i,j) \mid i \leq j \text{ and } a_i + a_j > 2\ell\,\}.$$

3.1.15. Example. We consider an ℓ-admissible C_3-sequence $\mathbf{a}_3 = (a_1, a_2, a_3)$ such that such that $q(\mathbf{a}_3) \geq 1$.

(1) If $\ell = 2$, then $a_2 = -2$ and $a_3 = 3$. As $a_1 + a_3 \neq 4$, we get $\mathbf{a}_3 = (-1, -2, 3)$.
(2) If $\ell = 1$, then $a_1 = -1$ and $a_3 = -3$. This implies $\mathbf{a}_3 = (-1, 2, -3)$.

3.1.16. Proposition [Sno88]. *Let $M = Sp(2n)/U(n)$ be a type C_n irreducible Hermitian symmetric space of compact type. Then $H^q(M, \Omega_M^p(\ell)) \neq 0$ implies that there exists an ℓ-admissible C_n-sequence such that its weight is p and its ℓ-cohomological degree is q.*

3.1.17. Proposition. *Let $\boldsymbol{a}_n = (a_i)_{1 \leq i \leq n}$ be an ℓ-admissible C_n-sequence such that $n \geq 3$. If $q(\boldsymbol{a}_n) > 0$ and $\ell > 0$, then we have*

$$\ell + q(\boldsymbol{a}_n) - 2 \geq (p(\boldsymbol{a}_n) - 2)\frac{2}{n} \tag{3.4}$$

with equality if and only if $\boldsymbol{a}_n = (-1, 2, -3, \cdots, -n)$ and $\ell = 1$.

Proof. We proceed by induction on n. For convenience, for an ℓ-admissible C_n-sequence $\mathbf{a}_n$, we introduce

$$\Delta(\mathbf{a}_n) = \ell + q(\mathbf{a}_n) - 2 - (p(\mathbf{a}_n) - 2)\frac{2}{n}.$$

As $\ell > 0$ and $q(\mathbf{a}_n) > 0$, we see $p(\mathbf{a}_n) \geq 2$. In view of Example 3.1.15, we can assume that $\Delta(\mathbf{a}_n) \geq 0$ for $n \leq k$ with equality if and only if $\mathbf{a}_n = (-1, 2, -3, \cdots, -n)$. Now we consider the case $n = k+1$.

Let $\mathbf{a}_{k+1} = (a_1, \cdots, a_{k+1})$ be an ℓ-admissible C_{k+1}-sequence with $q(\mathbf{a}_{k+1}) > 0$. Then $\mathbf{a}_k = (a_1, \cdots, a_k)$ is an ℓ-admissible C_k-sequence. If $a_{k+1} = -(k+1)$, then $q(\mathbf{a}_k) = q(\mathbf{a}_{k+1}) > 0$. Moreover, note that $p(\mathbf{a}_k) = p(\mathbf{a}_{k+1}) \geq 2$ in this case, we obtain $\Delta(\mathbf{a}_{k+1}) \geq \Delta(\mathbf{a}_k)$ with equality if and only if $p(\mathbf{a}_k) = 2$. By our inductive assumption, we conclude $\Delta(\mathbf{a}_{k+1}) \geq 0$ with equality if and only if $\Delta(\mathbf{a}_k) = 0$ and $a_{k+1} = -(k+1)$. Therefore, we can suppose that $a_{k+1} = k+1$ from now on and we set $m = \max\{\, a_i \mid 1 \leq i \leq k \,\}$.

Case 1. $a_{k+1} = k+1$ ***and*** $q(\boldsymbol{a}_k) = 0$.

If $m \leq 0$, then $p(\mathbf{a}_{k+1}) = k+1$, $q(\mathbf{a}_{k+1}) = 1$. Since $\mathbf{a}_{k+1}$ is ℓ-admissible and $\ell \geq 1$, we get $\ell \geq k/2+1$ and $q(\mathbf{a}_{k+1}) = 1$. In particular, as $k \geq 3$, we have

$$\Delta(\mathbf{a}_{k+1}) \geq \frac{k}{2} + 1 + 1 - 2 - (k+1-2)\frac{2}{k} = \frac{k-4}{2} + \frac{2}{k} > 0.$$

If $m > 0$, then the assumption $q(\mathbf{a}_k) = 0$ implies $m \leq \ell - 1$. First we consider the case $q(\mathbf{a}_{k+1}) = 1$. In this case, we have

$$m + k + 1 \leq 2\ell - 1.$$

In particular, we see $2\ell - 1 \geq k+2$. As $q(\mathbf{a}_{k+1}) > 0$, we get $k+3 \leq 2\ell \leq 2k$. Moreover, by the definition of m, we obtain

$$p(\mathbf{a}_{k+1}) \leq \frac{m(m+1)}{2} + k + 1 \leq \frac{4\ell^2 - (4k+6)\ell + (k+4)(k+1)}{2}.$$

This implies

$$\begin{aligned}
\Delta(\mathbf{a}_{k+1}) &= \ell + 1 - 2 + (p(\mathbf{a}_{k+1}) - 2)\frac{2}{k+1} \\
&\geq \frac{(\ell-1)(k+1) - \left[4\ell^2 - (4k+6)\ell + (k+4)(k+1) - 4\right]}{k+1} \\
&\geq \frac{-4\ell^2 + (5k+7)\ell - (k^2+6k+1)}{k+1}.
\end{aligned}$$

For a fixed k, the function

$$F(\ell) = \frac{-4\ell^2 + (5k+7)\ell - (k^2+6k+1)}{k+1}$$

is a concave function in the variable ℓ. As $k+3 \leq 2\ell \leq 2k$, the function $F(\ell)$ attains the minimum value at the point $\ell = (k+3)/2$ or $\ell = k$. By a straightforward computation, we get

$$F(\ell) = \begin{cases} \dfrac{k-1}{k+1}, & \text{if } \ell = k; \\[2ex] \dfrac{(k-1)^2}{2(k+1)}, & \text{if } \ell = \dfrac{k+3}{2}. \end{cases}$$

In particular, we obtain $\Delta(\mathbf{a}_{k+1}) \geq F(\ell) > 0$ since $k \geq 3$. Now we consider the case $q(\mathbf{a}_{k+1}) \geq 2$. As $m \leq \ell - 1$, it follows

$$p(\mathbf{a}_{k+1}) = p(\mathbf{a}_k) + k + 1 \leq \frac{m(m+1)}{2} + k + 1 \leq \frac{(\ell-1)\ell}{2} + k + 1,$$

and this implies

$$\begin{aligned}\Delta(\mathbf{a}_{k+1}) &\geq \ell + q(\mathbf{a}_{k+1}) - 2 - \frac{\ell(\ell-1)+2k-2}{k+1}\\ &\geq \frac{-\ell^2+(k+2)\ell-2k+2}{k+1}.\end{aligned}$$

For a fixed k, the function

$$G(\ell) = \frac{-\ell^2+(k+2)\ell-2k+2}{k+1}$$

is a concave function in the variable ℓ. Moreover, note that we have

$$2 \leq m+1 \leq \ell \leq k,$$

so the function $G(\ell)$ attains the minimum value at $\ell = 2$ or $\ell = k$. Also a straightforward computation shows

$$G(\ell) = \begin{cases} \dfrac{2}{k+1}, & \text{if } \ell = 2;\\[2ex] \dfrac{2}{k+1}, & \text{if } \ell = k. \end{cases}$$

Hence we get $\Delta(\mathbf{a}_{k+1}) \geq G(\ell) > 0$.

Case 2. *$a_{k+1} = n+1$ **and** $q(\mathbf{a}_k) > 0$.*

As $q(\mathbf{a}_k) > 0$, then $m \geq \ell + 1 \geq 2$. This implies $p(\mathbf{a}_k) \geq 2$ and $q(\mathbf{a}_{k+1}) - q(\mathbf{a}_k) \geq 2$. Moreover, by definition, we have $p(\mathbf{a}_{k+1}) = p(\mathbf{a}_k) + k + 1$, and it follows

$$\begin{aligned}\Delta(\mathbf{a}_{k+1}) &= \Delta(\mathbf{a}_k) + q(\mathbf{a}_{k+1}) - q(\mathbf{a}_k) - 2 + (p(\mathbf{a}_k) - 2)\frac{2}{k(k+1)}\\ &\geq \Delta(\mathbf{a}_k) + q(\mathbf{a}_{k+1}) - q(\mathbf{a}_k) - 2.\end{aligned}$$

As a consequence we conclude $\Delta(\mathbf{a}_{k+1}) \geq 0$ by our inductive assumption, and the equality holds if and only if $p(\mathbf{a}_k) = 2$, $\Delta(\mathbf{a}_k) = 0$ and $q(\mathbf{a}_{k+1}) = q(\mathbf{a}_k) + 2$. As $p(\mathbf{a}_k) = 2$ and $q(\mathbf{a}_k) > 0$, we get $m = 2$ and $\ell = 1$. This implies $\mathbf{a}_{k+1} = (-1, 2, -3, \cdots, -k, k+1)$. However, note that $\mathbf{a}_{k+1}$ can not be 1-admissible if $k \geq 4$. On the other hand, if $k = 3$, then $\mathbf{a}_4 = (-1, 2, -3, 4)$. Then we see $q(\mathbf{a}_4) = 4$ and $\Delta(\mathbf{a}_4) > 0$ in this case. Hence, $\Delta(\mathbf{a}_{k+1}) \geq 0$ is actually a strict inequality. $\square$

3.1.18. Remark. We consider the 2-admissible C_4 sequence $\mathbf{a}_4 = (-1, -2, -3, 4)$. Then $q(\mathbf{a}_4) = 1$ and $p(\mathbf{a}_4) = 4$. In particular, we have

$$\ell + q(\mathbf{a}_4) - 3 - \frac{2}{4}(p(\mathbf{a}_4) - 3) = -\frac{1}{2}.$$

Recall that if $n = 1$ and 2, then $Sp(2n)/U(n)$ is isomorphic to $\mathbb{P}^1$ and Q^3, respectively. Thus we shall assume that $n \geq 3$ in the following theorem.

3.1.19. Theorem. *Let M be a type C_n irreducible Hermitian symmetric space of compact type such that $n \geq 3$. Let ℓ and p be two positive integers such that $H^q(M, \Omega_M^p(\ell)) \neq 0$ for some $q \geq 0$.*

(1) If $q \geq 1$, then we have

$$\ell + q - 2 \geq (p-2)\frac{r_M}{\dim(M)}. \tag{3.5}$$

(2) If $q = 0$ and $\ell \leq r_M - 1$, then we have

$$\ell - 1 \geq p\frac{r_M}{\dim(M)} \tag{3.6}$$

with equality if and only if $(\ell, p) = (n, n(n-1)/2)$.

Proof. As $M \cong Sp(2n)/U(n)$, we have $r_M/\dim(M) = 2/n$. If $q \geq 1$, the inequality (3.5) follows from Proposition 3.1.16 and Proposition 3.1.17.

If $q = 0$, by Proposition 3.1.16, there exists an ℓ-admissible C_n-sequence $\mathbf{a}_n$ whose ℓ-cohomological degree is 0. Moreover, as $p \geq 0$, we have $\ell \geq 2$. On the other hand, by the definition, we also have

$$p = p(\mathbf{a}_n) \leq \frac{(\ell-1)\ell}{2}.$$

As $\ell \leq r_M - 1 = n$, we obtain

$$\frac{\ell-1}{p} \geq \frac{2(\ell-1)}{(\ell-1)\ell} = \frac{2}{\ell} \geq \frac{2}{n} = \frac{r_M}{\dim(M)}.$$

The equality holds if and only if $\ell = n$ and $p = n(n-1)/2$. □

3.1.5 Spinor Grassmannians (type D_n)

In this case, M parametrizes one of the two families of n-dimensional isotropic subspaces of $\mathbb{C}^{2n}$ equipped with a non-degenerate quadratic form. It is the $n(n-1)/2$-dimensional homogeneous space $SO(2n)/U(n)$ with index $2n-2$.

3.1.20. Definition. *Let $\ell, n \in \mathbb{N}$ be two fixed positive integers. A n-tuple of integers $(a_i)_{0\leq i\leq n-1}$ is called an ℓ-admissible D_n-sequence if $|a_i| = i$ and $a_i + a_j \neq \ell$ for all $i < j$. Its weight is defined to be $p = \sum_{a_i>0} a_i$ and its ℓ-cohomological degree is defined to be*

$$q = \#\{\,(i,j) \mid i < j \text{ and } a_i + a_j > \ell\,\}.$$

3.1.21. Example. Let $\mathbf{a}_5 = (0, a_1, a_2, a_3, a_4)$ be an ℓ-admissible D_5-sequence such that $q(\mathbf{a}_5) \geq 1$.

(1) If $\ell = 6$, then $q(\mathbf{a}_5) = 1$, $a_3 = 3$ and $a_4 = 4$. Then we get $a_2 = -2$ and $p(\mathbf{a}_5) \leq 8$.
(2) If $\ell = 5$, then $a_4 = 4$, $a_1 = -1$ and $q(\mathbf{a}_5) \leq 2$. However, if $q(\mathbf{a}_5) = 2$, then we must have $a_2 = 2$ and $a_3 = 3$, which is absurd. Therefore, we get $q(\mathbf{a}_5) = 1$ and $p(\mathbf{a}_5) \leq 7$.
(3) If $\ell = 4$, then $a_4 = -4$ and $q(\mathbf{a}_5) = 1$.Then we obtain $a_2 = 2$ and $a_3 = 3$. Then we must have $\mathbf{a}_5 = (0, -1, 2, 3, -4)$ and $p(\mathbf{a}_5) = 5$.
(4) If $\ell = 3$, then $a_3 = -3$ and $\mathbf{a}_4 = 4$. Then we have $a_1 = 1$ and $a_2 = -2$. Hence we obtain $\mathbf{a}_5 = (0, 1, -2, -3, 4)$, $q(\mathbf{a}_5) = 2$ and $p(\mathbf{a}_5) = 5$.
(5) If $\ell = 2$, then $a_2 = -2$ and $a_4 = -4$. As $a_1 + a_3 \neq 2$ and $q(\mathbf{a}_5) \geq 1$, we get $a_1 = 1$ and $a_3 = 3$; that is $\mathbf{a}_5 = (0, 1, -2, 3, -4)$. Moreover, we obtain $q(\mathbf{a}_5) = 2$ and $p(\mathbf{a}_5) = 4$.
(6) If $\ell = 1$, then it is easy to see $\mathbf{a}_5 = (0, -1, -2, -3, -4)$. This is impossible as we assume $q(\mathbf{a}_5) > 0$.

3.1.22. Proposition [Sno88]. *Let $M = SO(2n)/U(n)$ be a type D_n irreducible Hermitian symmetric space of compact type. Then $H^q(M, \Omega^p_M(\ell)) \neq 0$ implies that there exists an ℓ-admissible D_n-sequence such that its weight is p and its ℓ-cohomological degree is q.*

3.1.23. Proposition. *Let $\boldsymbol{a}_n = (0, a_1, \cdots, a_{n-1})$ be an ℓ-admissible D_n-sequence such that $n \geq 5$, $q(\boldsymbol{a}_n) > 0$ and $\ell > 0$. Then we have*

$$\ell + q(\boldsymbol{a}_n) - 2 \geq (p(\boldsymbol{a}_n) - 2)\frac{4}{n}. \tag{3.7}$$

with equality if and only if $(\ell, q(\boldsymbol{a}_n), p(\boldsymbol{a}_n), n) = (5, 1, 7, 5)$.

Proof. We will prove the proposition by induction on n. For convenience, for an ℓ-admissible D_n-sequence $\mathbf{a}_n$, we introduce

$$\Delta(\mathbf{a}_n) = \ell + q(\mathbf{a}_n) - 2 - (p(\mathbf{a}_n) - 2)\frac{4}{n}.$$

In view of Example 3.1.21, we may assume that we have $\Delta(\mathbf{a}_n) \geq 0$ for all $n \leq k$ and we will prove it for $n = k+1$. Let $\mathbf{a}_{k+1} = (0, a_1, \cdots, a_{k-1}, a_k)$ be an ℓ-admissible D_{k+1}-sequence such that $q(\mathbf{a}_{k+1}) > 0$. We set $\mathbf{a}_k = (0, a_1, \cdots, a_{k-1})$. If $a_k = -k$, then $q(\mathbf{a}_k) = q(\mathbf{a}_{k+1}) > 0$ and $p(\mathbf{a}_k) = p(\mathbf{a}_{k+1})$. Then we get $\Delta(\mathbf{a}_{k+1}) \geq \Delta(\mathbf{a}_k) \geq 0$ with equality if and only if $p(\mathbf{a}_{k+1}) = 2$ and $\Delta(\mathbf{a}_k) = 0$, which is impossible in view of Example 3.1.21. From now on, we assume that $a_k = k$ and we set

$$m = \max\{\, 0,\ a_i \mid 1 \leq i \leq k-1 \,\}.$$

As $q(\mathbf{a}_{k+1}) > 0$, it follows that we have $m > \max\{\, 0,\ \ell - k \,\}$.

Case 1. *$\ell > k$.*

We denote by u the number of a_i's such that $a_i > 0$ and $i > \ell - k$. Then, by definition, we have $q(\mathbf{a}_{k+1}) = q(\mathbf{a}_k) + (u-1)$, and we can derive

$$\begin{aligned}\Delta(\mathbf{a}_{k+1}) &= \ell + (q(\mathbf{a}_k) + u - 1) - 2 - (p(\mathbf{a}_k) + k - 2)\frac{4}{k+1} \\ &> \left(\ell + q(\mathbf{a}_k) - \frac{4p(\mathbf{a}_k)}{k}\right) + (u-3) - \frac{4(k-2)}{k+1}.\end{aligned}$$

The last inequality holds since $p(\mathbf{a}_k) \geq m \geq \ell - k + 1$. Then, by [BCM18, Proposition 2.10], we obtain $\Delta(\mathbf{a}_{k+1}) > 0$ if $u \geq 7$. So we can assume that $u \leq 6$. On the other hand, as $q(\mathbf{a}_{k+1}) > 0$, we see $m > \ell - k$ and $u \geq 2$. By definition, we have

$$p(\mathbf{a}_{k+1}) \leq \sum_{i=0}^{\ell-k-1} |a_i| + \sum_{j=0}^{u-1}(k-j) = \frac{(\ell-k)(\ell-k-1)}{2} + uk - \frac{(u-1)u}{2}.$$

This implies

$$\begin{aligned}F(\ell)\colon &= (\ell + (u-1) - 2)\,(k+1) - 4(p(\mathbf{a}_{k+1}) - 2) \\ &\geq -2\ell^2 + (5k+3)\ell + 2u^2 - (3k+1)u - 2k^2 - 5k + 5\end{aligned}$$

For fixed k and u, then function $F(\ell)$is concave. As $\ell \geq k+1$ and $2u \leq 2k - \ell + 2$, by a straightforward computation, we obtain

$$F(\ell) = \begin{cases} 2u^2 - (3k+1)u + k^2 - k + 6, & \text{if } \ell = k+1; \\ -6u^2 + (3k+9)u - 5k + 3, & \text{if } \ell = 2k - 2u + 2. \end{cases} \tag{3.8}$$

In particular, if $u = 2$, then we have

$$F(\ell) \geq \min\{(k-2)(k-5) + 2, k-3\}.$$

As $k \geq 5$ by our assumption, we obtain $\Delta(\mathbf{a}_{k+1}) \geq F(\ell)/(k+1) > 0$ if $u = 2$. Hence we can assume also that $u \geq 3$.

Case 1.a) $q(\boldsymbol{a}_k) > 0$.

By inductive assumption, we have $\Delta(\mathbf{a}_k) > 0$. On the other hand, note that we have

$$\Delta(\mathbf{a}_{k+1}) > \Delta(\mathbf{a}_k) + (u-1) - \frac{4k}{k+1},$$

since $p(\mathbf{a}_k) \geq 2m - 1 > 2$. In particular, if $u \geq 5$, then we get $\Delta(\mathbf{a}_{k+1}) > 0$. Thus it remains to consider the cases $u = 3$ and $u = 4$. Moreover, as $q(\mathbf{a}_k) > 0$, we have $q(\mathbf{a}_{k+1}) \geq q(\mathbf{a}_k) + (u-1) \geq u$. If $u = 3$, then (3.8) implies

$$F(\ell) \geq \min\{k^2 - 10k + 21, 4k - 24\}.$$

Then we have

$$\Delta(\mathbf{a}_{k+1}) \geq 1 + \frac{F(\ell)}{k+1} > 0.$$

If $u = 4$, then (3.8) implies

$$F(\ell) \geq \min\{k^2 - 13k + 34, 7k - 57\}.$$

Moreover, as $k + 1 \leq \ell \leq 2k - 2u + 2$, we get $k \geq 7$. Then we see

$$\Delta(\mathbf{a}_{k+1}) \geq 1 + \frac{F(\ell)}{k+1} \geq 0,$$

and the last equality holds if and only if $k = 7$, $\ell = 2k - 6 = 8$ and $q(\mathbf{a}_{k+1}) = 4$. Then it is easy to see that there are only two possibilities of $\mathbf{a}_{k+1}$:

$$(0, -1, 2, -3, 4, 5, -6, 7) \qquad \text{and} \qquad (0, -1, -2, 3, 4, -5, 6, 7).$$

Then we obtain $\Delta(\mathbf{a}_{k+1}) = 1/8$ and $1/18$, respectively. Hence we have $\Delta(\mathbf{a}_{k+1}) > 0$.

Case 1.b) $q(\boldsymbol{a}_k) = 0$.

First we consider the case $m \geq \ell - 1 - m$. we can refine the upper bound of $p(\mathbf{a}_{k+1})$. More precisely, we have

$$\begin{aligned} p(\mathbf{a}_{k+1}) &\leq \sum_{i=0}^{\ell-k-1} |a_i| + \sum_{j=0}^{u-3} (\ell - 1 - m - j) + m + k \\ &\leq \frac{\ell^2 - (2k - u + 2)\ell + k^2 + 3k - u^2 + 4u - 5}{2}. \end{aligned}$$

This implies

$$\begin{aligned} (k+1)\Delta(\mathbf{a}_{k+1}) &\geq \ell + (u-1) - 2 - 4(p(\mathbf{a}_{k+1}) - 2) \\ &\geq -2\ell^2 + (5k - 2u + 5)\ell - 2k^2 - 9k + 2u^2 + (k-7)u + 15. \end{aligned}$$

As before, for fixed k and u, the function

$$G(\ell) = -2\ell^2 + (5k - 2u + 5)\ell - 2k^2 - 9k + 2u^2 + (k-7)u + 15$$

is concave. Moreover, as $k + 1 \leq \ell \leq 2k - 2u + 2$, we have

$$G(\ell) \geq \min\{G(k+1), G(2k - 2u + 2)\}.$$

By a straightforward computation, we get

$$G(\ell) = \begin{cases} 2u^2 - (k+9)u + k^2 - 3k + 18, & \text{if } \ell = k + 1; \\ -2u^2 + (3k-5)u - 5k + 17, & \text{if } \ell = 2k - 2u + 2. \end{cases}$$

Note that we have

$$G'(u) \colon = 2u^2 - (k+9)u + k^2 - 3k + 18 \geq \frac{7(k-3)^2}{8}.$$

Moreover, as $3 \leq u \leq 6$, we obtain

$$\begin{aligned} G''(u) \colon &= -2u^2 + (3k-5)u - 5k + 17 \\ &\geq \min\{G''(3), G''(6)\} \\ &\geq \min\{4k - 16, 13k - 85\}. \end{aligned}$$

Therefore, if $k \geq 7$, then we have $G(\ell) > 0$. On the other hand, if $5 \leq k \leq 6$, we have $u = 3$ since $2u \leq k+1$. Thus we have also $G(\ell) > 0$ for $k \leq 6$.

If $\ell - 1 - m > m$, we consider the following sequence $\mathbf{a}'_{k+1}$

$$\begin{cases} a'_i = a_i, & \text{if } i \neq m, \ell-1-m; \\ a_i = -m, & \text{if } i = m; \\ a_i = \ell - 1 - m, & \text{if } i = \ell - 1 - m. \end{cases}$$

Then we see $\Delta(\mathbf{a}_{k+1}) > \Delta(\mathbf{a}'_{k+1})$ and we have $\Delta(\mathbf{a}'_{k+1}) \geq 0$ by our argument above.

Case 2. $\ell < k$.

We denote by u the number of a_i's such that $a_i > 0$ and $i > k - \ell$. Then by definition, we have

$$q(\mathbf{a}_{k+1}) = q(\mathbf{a}_k) + (k-\ell) + 1 + u - 1 = q(\mathbf{a}_k) + k - \ell + u.$$

Then we get

$$\Delta(\mathbf{a}_{k+1}) \geq \ell + q(\mathbf{a}_k) + k - \ell + u - 2 - (p(\mathbf{a}_k) + k - 2)\frac{4}{k+1} \tag{3.9}$$

Case 2.a) $q(\boldsymbol{a}_k) > 0$.

By our inductive assumption, we have $\Delta(\mathbf{a}_k) \geq 0$. Moreover, in view of (3.9), we have

$$\Delta(\mathbf{a}_{k+1}) > \Delta(\mathbf{a}_k) + k - \ell + u - \frac{4(k-2)}{k+1}.$$

Therefore, if $k - \ell + u \geq 4$, then we obtain $\Delta(\mathbf{a}_{k+1})$. So we assume that $k - \ell + u \leq 3$. As $u \geq 1$, we see $\ell \geq k - 2$. However, if $u = 1$, since $q(\mathbf{a}_k) > 0$, it yields

$$3 = k - \ell + (k - \ell - 1) \geq \ell + 1 \geq k - 1.$$

This is impossible. Therefore, we have $u = 2$. Then we get $\ell = k - 1$. Since the sequence $\mathbf{a}_{k+1}$ is ℓ-admissible, then we have $a_1 = 1$. As $u = 2$ and $q(\mathbf{a}_k) > 0$, we get $m + 1 > \ell$. This implies $m = k - 1$. This contradicts the assumption that $\mathbf{a}_{k+1}$ is ℓ-admissible.

Case 2.b) $q(\boldsymbol{a}_k) = 0$.

According to [BCM18, Proposition 2.10] and (3.9), we have

$$\begin{aligned} \Delta(\mathbf{a}_{k+1}) &> \ell - \frac{4p(\mathbf{a}_k)}{k} + k - \ell + u - 2 - \frac{4(k-2)}{k+1} \\ &\geq k - \ell + u - 2 - \frac{4(k-2)}{k+1}. \end{aligned}$$

In particular, if $k - \ell + u \geq 6$, then we have $\Delta(\mathbf{a}_{k+1}) > 0$. So it remains to consider the case $k - \ell + u \leq 5$.

If $u = 1$, then we have

$$p(\mathbf{a}_{k+1}) \leq \sum_{i=0}^{k-\ell} |a_i| + k = \frac{\ell^2 - 2(k+1)\ell + k^2 + 3k}{2}.$$

Then we obtain

$$\begin{aligned} \Delta(\mathbf{a}_{k+1}) &\geq \ell + (k - \ell + 1) - 2 - (p(\mathbf{a}_{k+1}) - 2)\frac{4}{k+1} \\ &\geq \frac{-2\ell^2 + (4k+2)\ell - k^2 - 6k + 7}{k+1}. \end{aligned}$$

As $k-4 \le \ell \le k-1$, we get

$$\Delta(\mathbf{a}_{k+1})(k+1) \ge \begin{cases} k^2-4k+3, & \text{if } \ell = k-1; \\ k^2-4k-5, & \text{if } \ell = k-2; \\ k^2-4k-17, & \text{if } \ell = k-3; \\ k^2-4k-33, & \text{if } \ell = k-4. \end{cases}$$

Note that we have $\ell - 1 \ge k - \ell$, then we get

$$k \ge 7 \ \text{ if } \ell = k-3 \qquad \text{and} \qquad k \ge 9 \ \text{ if } \ell = k-4.$$

Then we obtain $\Delta(\mathbf{a}_{k+1}) \ge 0$. Moreover, if the equality holds, then we have $k = 5$ and $\ell = k - 2 = 3$. As $u = 1$, there is only one possibility of $\mathbf{a}_{k+1}$:

$$(0, -1, 2, -3, -4, 5).$$

Then we see $\Delta(\mathbf{a}_{k+1}) = 2/3$. Hence, we get $\Delta(\mathbf{a}_{k+1}) > 0$ if $u = 1$.

If $u = 2$, since $q(\mathbf{a}_k) = 0$, we obtain

$$p(\mathbf{a}_{k+1}) \le \sum_{i=0}^{k-\ell-1} |a_i| + (\ell - 1) + k = \frac{\ell^2 - (2k-3)\ell + k^2 + k - 2}{2}.$$

This implies

$$\begin{aligned} \Delta(\mathbf{a}_{k+1}) &\ge \ell + (k-\ell+2) - 2 - (p(\mathbf{a}_{k+1}) - 2)\frac{4}{k+1} \\ &\ge \frac{-2\ell^2 + (4k-6)\ell - k^2 - k + 12}{k+1}. \end{aligned}$$

We set

$$H(\ell) = -2\ell^2 + (4k-6)\ell - k^2 - k + 12.$$

Then $H(\ell)$ is concave. As $k-3 \le \ell \le k-1$, we get

$$\begin{aligned} (k+1)\Delta(\mathbf{a}_{k+1}) \ge H(\ell) &\ge \min\{H(k-1), H(k-3)\} \\ &= \min\{k^2 - 7k + 16, k^2 - 7k + 12\}. \end{aligned}$$

As $k \ge 5$, then we conclude $\Delta(\mathbf{a}_{k+1}) > 0$.

If $u = 3$, since $q(\mathbf{a}_{k+1}) = 0$, we get

$$p(\mathbf{a}_{k+1}) \le \sum_{i=0}^{k-\ell} |a_i| + (\ell - 1) + k = \frac{\ell^2 - (2k-1)\ell + k^2 + 3k - 2}{2}.$$

Then we obtain

$$\begin{aligned} \Delta(\mathbf{a}_{k+1}) &\ge \ell + (k-\ell+3) - 2 - (p(\mathbf{a}_{k+1}) - 2)\frac{4}{k+1} \\ &\ge \frac{-2\ell^2 + (4k-2)\ell - k^2 - 4k + 13}{k+1}. \end{aligned}$$

As $k-2 \le \ell \le k-1$, then we get

$$(k+1)\Delta(\mathbf{a}_{k+1}) \ge \begin{cases} k^2-6k+13, & \text{if } \ell = k-1; \\ k^2-6k+9, & \text{if } \ell = k-2. \end{cases}$$

As $k \geq 5$, we obtain $\Delta(\mathbf{a}_{k+1}) > 0$.

If $u = 4$, then $\ell = k - 1$ and we have

$$p(\mathbf{a}_{k+1}) \leq (\ell - 5) + (\ell - 1) + k = 3k - 8.$$

This implies

$$\Delta(\mathbf{a}_{k+1}) \geq k + 2 - (3k - 10)\frac{4}{k+1} = \frac{k^2 - 9k + 42}{k+1}.$$

As $k \geq 5$, we get $\Delta(\mathbf{a}_{k+1}) > 0$. This finishes the proof. $\square$

3.1.24. Theorem. *Let M be a type D_n irreducible Hermitian symmetric space of compact type. Let ℓ and p be two positive integers such that $H^q(M, \Omega_M^p(\ell)) \neq 0$ for some $q \geq 0$.*

(1) If $q \geq 1$, then we have

$$\ell + q - 2 \geq (p-2)\frac{r_M}{\dim(M)}. \tag{3.10}$$

(2) If $q = 0$ and $\ell \leq r_M - 1$, then

$$\ell - 1 > p\frac{r_M}{\dim(M)}. \tag{3.11}$$

Proof. If $n \leq 4$, then $SO(2n)/U(n)$ is isomorphic to either a projective space or a quadric hypersurface. Thus we shall assume that $n \geq 5$

If $q > 0$, then the inequality (3.10) follows from Proposition 3.1.22 and Proposition 3.1.23. If $q = 0$, according to Proposition 3.1.22, there exists an ℓ-admissible D_n-sequence $\mathbf{a}_n$ whose ℓ-cohomological degree is 0. We denote by m the maximum value of a_i in $\mathbf{a}_n$. After replacing $a_{\ell-m-1}$ by $|a_{\ell-m-1}|$, we may assume that we have $m \geq \ell - m - 1$. Then we get

$$p(\mathbf{a}_n) \leq \sum_{i=0}^{\ell-m-1} |a_i| + m = \frac{m^2 - (2\ell - 3)m + \ell^2 - \ell}{2}.$$

As $\ell - 1 \leq 2m \leq 2(\ell - 1)$, if $\ell \geq 3$, then we get

$$p(\mathbf{a}_n) \geq \frac{(\ell-1)^2 - 2(2\ell-3)(\ell-1) + 4(\ell^2 - \ell)}{8} \geq \frac{\ell^2 + 4\ell - 5}{8}.$$

This implies

$$\ell - 1 - \frac{4p}{n} \geq \frac{2n(\ell-1) - (\ell^2 + 4\ell - 5)}{2n}$$

Then we are done if $2n \geq \ell + 6$. Moreover, if $\ell \leq r_M - 1 \leq 2n - 3$, thus it remains to consider the case $2n - 5 \leq \ell \leq 2n - 3$.

If $\ell = 2n - 5$ and $\ell - 1 = 2m$, then we get $\ell - m - 1 = m = n - 3$. Thus we have

$$\ell - 1 - \frac{4p}{n} > \frac{2n(\ell-1) - (\ell^2 + 4\ell - 5)}{2n} = 0.$$

If $\ell = 2n - 4$, then we get

$$p(\mathbf{a}_n) \leq \frac{(n-2)(n-1)}{2}.$$

This implies

$$\ell - 1 - \frac{4p}{n} \geq \frac{n-4}{n} > 0.$$

If $\ell = 2n-3$, then we get

$$p(\mathbf{a}_n) \leq \frac{(n-3)(n-2)}{2} + n - 1 = \frac{n^2-3n+4}{2}.$$

Then we obtain

$$\ell - 1 - \frac{4p}{n} \geq \frac{2n-8}{n} > 0.$$

If $\ell \leq 2$, then we have $p = 1$ since $q = 0$ and $p > 0$. In particular, the inequality (3.11) holds. $\square$

3.1.6 Twisted $(n-1)$-forms and special cohomologies

In this subsection, we determine the cohomologies of twisted $(n-1)$-forms of n-dimensional irreducible Hermitian symmetric spaces of compact type. In particular, we introduce the notion of special cohomology and we prove that all irreducible Hermitian symmetric spaces of compact type have special cohomologies. This is useful to study the twisted vector fields over complete intersections in Hermitian symmetric spaces in the next section.

3.1.25. Example. Denote by M the Lagrangian Grassmannian $Sp(8)/U(4)$. Then M is a 10-dimensional Fano manifold with index 5. Moreover, if ℓ is an integer such that $1 \leq \ell \leq 4$, then $H^q(M, \Omega_M^9(\ell)) = 0$ for any $q \geq 0$. In fact, if $H^q(M, \Omega_M^9(\ell)) \neq 0$, by Proposition 3.1.16, there exists an ℓ-admissible C_4-sequence $\mathbf{a}$ with ℓ-cohomological degree q and weight 9. This implies

$$\mathbf{a} = (-1, 2, 3, 4).$$

As $1 \leq \ell \leq 4$, then one can easily see that $\mathbf{a}$ cannot be ℓ-admissible.

3.1.26. Proposition. *Let M be a n-dimensional irreducible Hermitian symmetric space of compact type such that $n \geq 2$. Let $\ell \in \mathbb{Z}$ be an integer. Then $H^q(M, \Omega_M^{n-1}(\ell)) \neq 0$ if and only if one of the following conditions is satisfied.*

(1) $q = 0$ and $\ell \geq \min\{n, r_M\}$.
(2) $q = n-1$ and $\ell = 0$.
(3) $q = n$ and $\ell \leq -2$.
(4) $M \cong Q^n$, $q = 1$ and $\ell = n-2$.

Proof. If $n \leq r_M$ or $n \leq 2$, then X is isomorphic to $\mathbb{P}^n$ or Q^n and we can conclude by Bott's formula and [Sno86, Theorem 4.1]. On the other hand, it is well-known that $H^q(M, \Omega_M^p) \neq 0$ if and only if $q = p$. Moreover, If $\ell \geq r_M$, by [Sno88, Proposition 1.1], the cohomological degree of $\Omega_M^{n-1}(\ell)$ is 0. As a consequence, $H^q(M, \Omega_M^{n-1}(\ell)) \neq 0$ if and only if $q = 0$. So we shall assume that $n-1 \geq r_M \geq \ell+1$, $n \geq 3$ and $\ell \neq 0$. In particular, M is not of type B_n.

If $\ell \leq -2$, by Serre duality, $H^q(M, \Omega_M^{n-1}(\ell)) \neq 0$ if and only if $H^{n-q}(M, \Omega_M^1(-\ell)) \neq 0$. Recall that the cohomological degree of the sheaf $\Omega_M^1(-\ell)$ is 0 if $-\ell \geq 2$ by [Sno88, Proposition 1.1]. So $H^q(M, \Omega_M^{n-1}(\ell)) \neq 0$ if and only if $q = n$ if $\ell \leq -2$.

If $\ell = -1$, by Serre duality again, $H^q(M, \Omega_M^{n-1}(-1)) \neq 0$ if and only if $H^{n-q}(M, \Omega_M^1(1)) \neq 0$. Thanks to [Sno86, Theorem 2.3], we have $H^{n-q}(M, \Omega_M^1(1)) = 0$ for all $q \geq 0$ if M is not of type C_n. The vanishing of $H^q(M, \Omega_M^1(1))$ follows from [Sno88, Theorem 2.3] if M is of type C_n.

If $1 \leq \ell \leq r_M - 1$, we can prove the result case by case. If M is of type E_6 or E_7, from [Sno88, Table 4.4 and Table 4.5], we have $H^q(M, \Omega_M^{n-1}(\ell)) = 0$ for any $q \geq 0$. If M is of type A_n, as M is not isomorphic to $\mathbb{P}^n$ or Q^n, we get $H^q(M, \Omega_M^{n-1}(\ell)) = 0$ for all $q \geq 0$ by [Sno86, Theorem 3.4 (3)]. We remark that $Gr(2,4)$ is isomorphic to Q^4. If M is of type C_n and $n \neq 4$, we have $H^q(M, \Omega_M^{n-1}(\ell)) = 0$ for all $q \geq 0$ by [Sno88, Theorem 2.4 (3)]. If M is of type C_4, then M is isomorphic to the 10-dimensional homogeneous space $Sp(8)/U(4)$, and we get $H^q(M, \Omega_M^9(\ell)) = 0$ for all $q \geq 0$ according to Example 3.1.25. If M is of type D_n, it follows from [Sno88, Theorem 3.4 (3)] that $H^q(M, \Omega_M^{n-1}(\ell)) = 0$ for all

$q \geq 0$ if $n \geq 5$. If M is of type D_n and $n \leq 4$, then M is isomorphic to either $\mathbb{P}^n$ or Q^n. This is impossible by our assumption. □

As a direct application, we get the following result which is useful to describe the twisted vector fields over complete intersections.

3.1.27. Corollary. *Let M be a n-dimensional irreducible Hermitian symmetric space of compact type such that $n \geq 3$. Then $H^{n-1}(M, \Omega^1_M(\ell)) \neq 0$ if and only if $\ell = -n+2$ and M is isomorphic to a smooth quadric hypersurface Q^n.*

Proof. As $H^{n-1}(M, \Omega^1_M(\ell)) \neq 0$ if and only if $H^1(M, \Omega^{n-1}_M(-\ell)) \neq 0$ by Serre duality, then the result follows from Proposition 3.1.26. □

Moreover, one can easily derive the following result for smooth hypersurfaces in projective spaces by Bott's formula.

3.1.28. Lemma. *Let $Y \subset \mathbb{P}^{n+1}$ be a smooth hypersurface of degree d such that $n \geq 3$. Then we have $H^{n-1}(Y, \Omega^1_Y(-r_Y + t)) = 0$ for $t > d$, where $r_Y = n + 2 - d$.*

Proof. According to Theorem 3.1.3 and the following exact sequence of sheaves

$$0 \to \Omega^1_{\mathbb{P}^{n+1}}(-r_Y + t - d) \to \Omega^1_{\mathbb{P}^{n+1}}(-r_Y + t) \to \Omega^1_{\mathbb{P}^{n+1}}(-r_Y + t)|_Y \to 0,$$

we see that $H^{n-1}(Y, \Omega^1_{\mathbb{P}^{n+1}}(-r_Y + t)|_Y) = 0$ for any $t \in \mathbb{Z}$. Therefore, the following exact sequence of $\mathcal{O}_Y$-sheaves

$$0 \to \mathcal{O}_Y(-r_Y + t - d) \to \Omega^1_{\mathbb{P}^{n+1}}(-r_Y + t)|_Y \to \Omega^1_Y(-r_Y + t) \to 0$$

induces an injective map of groups

$$H^{n-1}(Y, \Omega^1_Y(-r_Y + t)) \to H^n(Y, \mathcal{O}_Y(-r_Y + t - d)).$$

Then we can conclude by Kodaira's vanishing theorem. □

3.1.29. Definition. *Let $(Z, \mathcal{O}_Z(1))$ be a polarized projective manifold of dimension ≥ 4. We say that the manifold Z has special cohomologies if $H^q(Z, \Omega^1_Z(\ell)) = 0$ for $2 \leq q \leq \dim(Z) - 2$ and $\ell \in \mathbb{Z}$.*

We remark that our definition of special cohomology is much weaker than that given in [PW95].

3.1.30. Example. By [Nar78, Corollary 2.3.1], a n-dimensional smooth complete intersection Y in a projective space has special cohomologies if $n \geq 4$. Moreover, if $\widetilde{Y}$ is a cyclic covering of Y, then Y has special cohomologies (see [PW95, Theorem 1.6]).

3.1.31. Example [Fle81, Satz 8.11]. Let Y be a smooth weighted complete intersection of dimension n in a weighted projective space, and let $\mathcal{O}_Y(1)$ be the restriction to Y of the universal $\mathcal{O}(1)$-sheaf from the weighted projective space. Then $(Y, \mathcal{O}_Y(1))$ has special cohomologies.

3.1.32. Proposition. *Let $(M, \mathcal{O}_M(1))$ be a n-dimensional irreducible Hermitian symmetric space of compact type. If $n \geq 4$, then $(M, \mathcal{O}_M(1))$ has special cohomologies.*

Proof. By Serre's duality, it suffices to consider the group $H^{n-q}(M, \Omega^{n-1}_M(-\ell))$. As $2 \leq q \leq n-2$, we get $2 \leq n - q \leq n - 2$. Then the result follows from Proposition 3.1.26. □

3.2 Extension of twisted vector fields

This section is devoted to study various global twisted vector fields over a complete intersection in an irreducible Hermitian symmetric space of compact type. The main aim is to show that the global twisted vector fields over complete intersections always come from the global twisted vector fields over the ambient space (cf. Theorem 3.2.5).

3.2.1 Twisted vector fields over complete intersections

Let $(Z, \mathcal{O}_Z(1))$ be a polarized manifold, and let $Y \subset Z$ be a submanifold. Then we have a natural restriction map

$$\rho_t\colon H^0(Z, T_Z(t)) \longrightarrow H^0(Y, T_Z(t)|_Y)$$

for any $t \in \mathbb{Z}$. This subsection is devoted to investigate the surjectivity of ρ_t in some special cases.

3.2.1. Notation. *Let Z be a projective manifold, and let $\{H_1, \cdots, H_r\}$ be a collection of hypersurfaces. We denote by Y_j $(1 \leq j \leq r)$ the scheme-theoretic complete intersection $H_1 \cap \cdots \cap H_j$. Moreover, for convenience, we will also denote Z by Y_0.*

3.2.2. Lemma. *Let $(Z, \mathcal{O}_Z(1))$ be a polarized projective manifold. Let $H_i \in |\mathcal{O}_Z(d_i)|$ $(1 \leq i \leq r)$ be a collection of hypersurface such that the complete intersections Y_j are smooth for all $1 \leq j \leq r$. Assume that $(Z, \mathcal{O}_Z(1))$ has special cohomologies and $\dim(Y_r) \geq 4$. Then, for any $2 \leq q \leq \dim(Y_r) - 2$ and any $\ell \in \mathbb{Z}$, we have*

$$H^q(Y_r, \Omega^1_Z(\ell)|_{Y_r}) = 0.$$

Proof. We prove the lemma by induction on r. If $r = 0$, the result follows from the definition of special cohomology. Now we assume that the lemma holds for $r - 1$. Then the exact sequence of sheaves

$$0 \to \Omega^1_Z(\ell - d_r)|_{Y_{r-1}} \to \Omega^1_Z(\ell)|_{Y_{r-1}} \to \Omega^1_Z(\ell)|_{Y_r} \to 0.$$

induces an exact sequence of groups

$$\to H^q(Y_{r-1}, \Omega^1_Z(\ell)|_{Y_{r-1}}) \to H^q(Y_r, \Omega^1_Z(\ell)|_{Y_r}) \to H^{q+1}(Y_{r-1}, \Omega^1_Z(\ell - d_r)|_{Y_{r-1}}) \to .$$

As $2 \leq q \leq \dim(Y_r) - 2$, our inductive assumption implies immediately that we have

$$H^q(Y_{r-1}, \Omega^1_Z(\ell)|_{Y_{r-1}}) = H^{q+1}(Y_{r-1}, \Omega^1_Z(\ell - d_r)|_{Y_{r-1}}) = 0$$

for any $\ell \in \mathbb{Z}$. It follows that $H^q(Y_r, \Omega^1_Z(\ell)|_{Y_r})$ for all $\ell \in \mathbb{Z}$. □

3.2.3. Lemma. *Let $(Z, \mathcal{O}_Z(1))$ be a polarized projective manifold. Let $H_i \in |\mathcal{O}_Z(d_i)|$ $(1 \leq i \leq r)$ be a collection of hypersurfaces such that the complete intersections Y_j are smooth for all $1 \leq j \leq r$. If $(Z, \mathcal{O}_Z(1))$ has special cohomologies and $\dim(Y_r) \geq 3$, then the induced map*

$$\alpha_\ell\colon H^{\dim(Y_r)-1}(Y, \Omega^1_Z(\ell)|_{Y_r}) \longrightarrow H^{\dim(Z)-1}(Z, \Omega^1_Z(\ell - d_1 - \cdots - d_r))$$

is injective for every $\ell \in \mathbb{Z}$.

Proof. Set $n = \dim(Y_r)$. For any $0 \leq j \leq r - 1$ and any $k \in \mathbb{Z}$, the following exact sequence

$$0 \to \Omega^1_Z(k - d_{r-j})|_{Y_{r-j-1}} \to \Omega^1_Z(k)|_{Y_{Y_{r-j-1}}} \to \Omega^1_Z(k)|_{Y_{r-j}} \to 0$$

induces an exact sequence of groups

$$\begin{aligned} \to H^{n+j-1}(Y_{r-j-1}, \Omega^1_Z(k)|_{Y_{r-j-1}}) &\to H^{n+j-1}(Y_{r-j}, \Omega^1_Z(k)|_{Y_{r-j}}) \\ &\to H^{n+j}(Y_{r-j-1}, \Omega^1_Z(k - d_{r-j})|_{Y_{r-j-1}}) \to . \end{aligned}$$

Note that $2 \leq n + j - 1 \leq (n + j + 1) - 2 = \dim(Y_{r-j-1}) - 2$, so Lemma 3.2.2 implies that we have

$$H^{n+j-1}(Y_{r-j-1}, \Omega^1_Z(k)|_{Y_{r-j-1}}) = 0$$

for any $k \in \mathbb{Z}$ and any $0 \leq j \leq r - 1$. Thus, the map

$$H^{n+j-1}(Y_{r-j}, \Omega^1_Z(k)|_{Y_{r-j}}) \to H^{n+j}(Y_{r-j-1}, \Omega^1_Z(k - d_{r-j})|_{Y_{r-j-1}})$$

is injective for any $0 \leq j \leq r-1$ and any $k \in \mathbb{Z}$. As $Y_0 = Z$ and $\dim(Y_{r-j}) = n+j$, we obtain that the composite map

$$H^{n-1}(Y_r, \Omega^1_Z(\ell)|_{Y_r}) \to H^n(Y_{r-1}, \Omega^1_Z(\ell - d_r)|_{Y_{r-1}}) \to \cdots \to H^{n+r-1}(Z, \Omega^1_Z(\ell - d_1 - \cdots - d_r))$$

is injective for any $\ell \in \mathbb{Z}$. $\square$

Now we can prove the following result related to the surjectivity of ρ_t.

3.2.4. **Proposition.** *Let $(Z, \mathcal{O}_Z(1))$ be a polarized projective manifold. Let $H_i \in |\mathcal{O}_Z(d_i)|$ $(1 \leq i \leq r)$ be a collection of hypersurfaces such that the complete intersections Y_j are smooth for all $1 \leq j \leq r$. Assume moreover that there exists an integer $r_Z \in \mathbb{Z}$ such that $\mathcal{O}_Z(-K_Z) \cong \mathcal{O}_Z(r_Z)$. Let $t \in \mathbb{Z}$ be an integer. If $(Z, \mathcal{O}_Z(1))$ has special cohomologies, $\dim(Y_r) \geq 2$ and $H^{\dim(Z)-1}(Z, \Omega^1_Z(-r_Z + d_i - t)) = 0$ for all $1 \leq i \leq r$, then the natural restriction*

$$\rho_t \colon H^0(Z, T_Z(t)) \longrightarrow H^0(Y_r, T_Z(t)|_{Y_r})$$

is surjective.

Proof. If $r = 1$, the result follows directly from the following exact sequence of sheaves

$$0 \to T_Z(t - d_1) \to T_Z(t) \to T_Z(t)|_{Y_1} \to 0$$

and Serre duality

$$H^1(Z, T_Z(t - d_1)) \cong H^{\dim(Z)-1}(Z, \Omega^1_Z(-r_Z + d_1 - t))^*.$$

Now we assume that the theorem holds for $r-1$. Consider the following exact sequence of sheaves

$$0 \to T_Z(t - d_r)|_{Y_{r-1}} \to T_Z(t)|_{Y_{r-1}} \to T_Z(t)|_{Y_r} \to 0.$$

By our assumption, to prove the surjectivity of ρ_t, it is enough to show $H^1(Y_{r-1}, T_Z(t-d_r)|_{Y_{r-1}}) = 0$. Thanks to Serre duality, we have

$$H^1(Y_{r-1}, T_Z(t - d_r)|_{Y_{r-1}}) \cong H^{\dim(Y_{r-1})-1}\left(Y_{r-1}, \Omega^1_Z(-r_Z + d_1 \cdots + d_{r-1} + d_r - t)|_{Y_{r-1}}\right)^*.$$

Note that we have $\dim(Y_{r-1}) = \dim(Y_r) + 1 \geq 3$ by our assumption, then Lemma 3.2.3 implies that we have $H^1(Y_{r-1}, T_Z(t - d_r)|_{Y_{r-1}}) = 0$ if $H^{\dim(Z)-1}(Z, \Omega^1_Z(-r_Z + d_r - t)) = 0$. Hence, the restriction map

$$H^0(Y_{r-1}, T_Z(t)|_{Y_{r-1}}) \longrightarrow H^0(Y_r, T_Z(t)|_{Y_r})$$

is surjective by our assumption, and we can conclude by our inductive assumption. $\square$

As an immediate application, we derive the following theorem which will play a key role in the proof of Theorem 1.1.11.

3.2.5. **Theorem.** *Let M be a $(n+r)$-dimensional irreducible Hermitian symmetric space of compact type which is not isomorphic to a smooth quadric hypersurface Q^{n+r}. Let $H_i \in |\mathcal{O}_M(d_i)|$ $(1 \leq i \leq r)$ be a collection of hypersurfaces such that the complete intersections Y_j are smooth for all $1 \leq j \leq r$, $\dim(Y_r) = n \geq 2$ and $n + r \geq 4$. Then the natural restriction*

$$\rho_t \colon H^0(M, T_M(t)) \longrightarrow H^0(Y_r, T_M(t)|_{Y_r}).$$

is surjective for any $t \in \mathbb{Z}$.

Proof. This follows from Proposition 3.1.27, Proposition 3.1.32 and Proposition 3.2.4. $\square$

If M is a smooth quadric hypersurface, then we can also regard Y_r as a complete intersection of degree $(2, d_1, \cdots, d_r)$ in the projective space $\mathbb{P}^{n+r+1}$. In particular, we have the following result.

3.2.6. Theorem. *Let $H_i \in |\mathcal{O}_{\mathbb{P}^{n+r}}(d_i)|$ $(1 \leq i \leq r)$ be a collection of hypersurfaces such that the complete intersections Y_j are smooth for all $1 \leq j \leq r$, $\dim(Y_r) = n \geq 2$ and $n+r \geq 5$. If $d_i \geq 2$ for all $1 \leq i \leq r$ and t is an integer such that $d_i - t > d_1$ for any $2 \leq i \leq r$, then the natural restriction*

$$\rho_t \colon H^0(Y_1, T_{Y_1}(t)) \to H^0(Y_r, T_{Y_1}(t)|_{Y_r})$$

is surjective.

Proof. By the definition and [Nar78, Corollary 2.3.1], the hypersurface Y_1 has special cohomologies (cf. Example 3.1.30). Thanks to Proposition 3.2.4, it suffices to verify that we have

$$H^{\dim(Y_1)-1}(Y_1, \Omega^1_{Y_1}(-r_{Y_1} + d_i - t)) = 0$$

for every $2 \leq i \leq r$. Since $t \in \mathbb{Z}$ is an integer such that $d_i - t > d_1$ for any $2 \leq i \leq r$, then our result follows from Lemma 3.1.28. $\square$

3.2.2 Lefschetz properties and Fröberg's conjecture

In this subsection, we collect basic materials of Lefschetz properties of Milnor algebras of hypersurfaces in projective spaces. Let $R = \mathbb{C}[x_1, \cdots, x_r]$ be the graded polynomial ring in r variables over $\mathbb{C}$. Let

$$A = R/I = \bigoplus_{i=0}^{n} A_i$$

be a graded Artinian algebra. Then, by definition, A is finite dimensional over $\mathbb{C}$.

3.2.7. Definition. *Let A be a graded Artinian algebra.*

(1) We say that A has the weak Lefschetz property (WLP) if there exists a linear form ℓ such that the homomorphism induced by multiplication by ℓ

$$\times \ell \colon A_i \longrightarrow A_{i+1}$$

has maximal rank for all i (i.e., is injective or sujective).

(2) We say that A has the maximal rank property (MRP) if for any d there exists a form f of degree d such that the homomorphism induced by multiplication by f

$$\times f \colon A_i \longrightarrow A_{i+d}$$

has maximal rank for all i (i.e., is injective or surjective).

(3) We say that A has the strong Lefschetz property (SLP) if there exists a linear form ℓ such that the homomorphism induced by multiplication by ℓ

$$\times \ell^d \colon A_i \longrightarrow A_{i+d}$$

has maximal rank for all i and all d (i.e., is injective or surjective).

3.2.8. Remark. Both the weak and strong Lefschetz properties have been extensively investigated in the literature (see for instance [MMR03, MMRN12, MN13] and the references therein), the maximal rank property has only been introduced in [MMR03] by Migliore and Miró-Roig. SLP implies MRP by semicontinuity and MRP is clearly stronger than WLP. It is known that none of the opposite implications hold true (see [MMR03, MN13]). Moreover, if A has WLP (resp. SLP and MRP), then the homomorphism induced by multiplication by a general linear form ℓ (resp. ℓ^d for a general linear form ℓ and a general form f of degree d) has maximal rank by semicontinuity.

Both these two concepts are motivated by the following theorem which was proved by Stanley in [Sta80] using algebraic topology, by Watanabe in [Wat87] using representation theory, by Reid, Roberts and Roitman in [RRR91] with algebraic methods.

3.2.9. Theorem. *Let $R = \mathbb{C}[x_1, \cdots, x_r]$. Let I be the Artinian complete intersection $\langle x_1^{d_1}, \cdots, x_r^{d_r}\rangle$. Then R/I has the SLP.*

Let $\mathbb{P}^{n+1}$ be the $(n+1)$-dimensional complex projective space, and let $Y \subset \mathbb{P}^{n+1}$ be a hypersurface defined by a homogeneous polynomial h of degree d. We denote by

$$J(Y) = \langle \partial h/\partial x_0, \cdots, \partial h/\partial x_{n+1}\rangle$$

the *Jacobian ideal* of Y, where $[x_0 : \cdots : x_{n+1}]$ are the coordinates of $\mathbb{P}^{n+1}$. Then the *Milnor algebra* of Y is defined to be the graded $\mathbb{C}$-algebra

$$M(Y) :\, = \mathbb{C}[x_0, \cdots, x_{n+1}]/\langle \partial h/\partial x_0, \cdots, \partial h/\partial x_{n+1}\rangle.$$

3.2.10. Remark [Dim87, p. 109]. One observe that the Hilbert series of the Milnor algebra $M(Y)$ of a general degree d hypersurface Y in $\mathbb{P}^{n+1}$ is

$$H(M(Y))(t) = (1 + t + t^2 + \cdots + t^{d-2})^{n+2},$$

where $\rho = (d-2)(n+2)$ is the top degree of $M(Y)$. The famous Macaulay's theorem says that the multiplication map

$$\mu_{i,j} :\; M(Y)_i \times M(Y)_j \longrightarrow M(Y)_{i+j}$$

is non-degenerated for $i + j \le \rho$. Using the perfect pairing

$$M(Y)_i \times M(Y)_{\rho-i} \to M(Y)_\rho \cong \mathbb{C},$$

we see that the dimension of $M(Y)_i$ is symmetric. Recall that an element $f \in M(Y)$ of degree j is called faithful if the multiplication $\times f : M(Y)_i \to M(Y)_{i+j}$ has maximal rank for all i. Since the dimension of $M(Y)_i$ is strictly increasing over the interval $[0, \rho/2]$, an element f of degree j is faithful if and only if it induces injections $M(Y)_i \to M(Y)_{i+j}$ for $i \le (\rho - j)/2$, equivalently it induces surjections $M(Y)_i \to M(Y)_{i+j}$ for $i \ge (\rho - j)/2$.

The proof of Theorem 1.1.12 relies on nonexistence of certain twisted vector fields over X. To prove this, we reduce the problem to the nonexistence of certain twisted vector fields over Y by proving an extension result (cf. Theorem 3.2.13). The main ingredient of the proof of Theorem 3.2.13 is the SLP of the Milnor algebra $M(Y)$ which is well-known to experts. Recall that the Fermat hypersurface of degree d in $\mathbb{P}^{n+1}$ is defined by the equation $x_0^d + \cdots + x_{n+1}^d = 0$.

3.2.11. Proposition. *Let $Y \subset \mathbb{P}^{n+1}$ be a general hypersurface of degree d. Then the Milnor algebra $M(Y)$ of Y has the strong Lefschetz property (SLP). In particular, $M(Y)$ has the maximal rank property.*

Proof. Thanks to Theorem 3.2.9, the Milnor algebra of the Fermat hypersurface of degree d in $\mathbb{P}^{n+1}$ has SLP. Then we conclude by semi-continuity. □

In general, the Hilbert series of a graded commutative algebra is an important invariant in commutative algebra and algebraic geometry. A difficult problem is : if I is generated by forms $f_1, \cdots, f_r$ of degrees $d_1, \cdots, d_r$, what can the Hilbert series of the algebra $\mathbb{C}[x_1, \cdots, x_n]/I$ be ? It was shown by Fröberg-Löfwall that there is only a finite number of Hilbert series for fixed $d_1, \cdots, d_r$, and that there is an open nonempty subspace of the space of coordinates for the f_i's on which the Hilbert series is constant (see [FL91]). There is a longstanding conjecture due to Fröberg for this Hilbert series.

3.2.12. Conjecture [Frö85]. *Let $S = \mathbb{C}[x_1, \cdots, x_n]$ be the polynomial ring in n variables. Let $f_1, \cdots, f_r$ be general forms of degrees $d_1, \cdots, d_r$ respectively. Set $I = \langle f_1, \cdots, f_r\rangle$. Then Hilbert series of S/I is* *given by*

$$H(t) = \left[\frac{\prod_{i=1}^{r}(1-t^{d_i})}{(1-t)^n} \right],$$

where $[\cdot]$ *means that we truncate a real formal power series at its first negative term.*

In [Frö85], Fröberg proved the conjecture for $n = 2$, and noticed that the left-hand side is bigger or equal than the right-hand side in the lexicographic sense. Later in [Ani86], Anick proved the conjecture for $n = 3$. On the other hand, it is easy to prove the following equality

$$\left[(1-t^{d_{r+1}}) \left[\frac{\prod_{i=1}^{r}(1-t^{d_i})}{(1-t)^n} \right] \right] = \left[\frac{\prod_{i=1}^{r+1}(1-t^{d_i})}{(1-t)^n} \right].$$

Consequently, in Conjecture 3.2.12, S/I has the expected Hilbert series if and only if $S/\langle f_1, \cdots, f_s \rangle$ has the MRP for $s \leq r-1$. Therefore, thanks to Theorem 3.2.9, Conjecture 3.2.12 is true for $r \leq n+1$. We refer to [FL17] and the references therein for more discussions around this conjecture.

3.2.3 Twisted vector fields over hypersurfaces in projective spaces

Given a global section σ of $T_{\mathbb{P}^{n+1}}(t)$, we can express σ in homogeneous polynomials of degree $t+1$. To see this, tensoring the Euler sequence of $\mathbb{P}^{n+1}$ by $\mathcal{O}_{\mathbb{P}^{n+1}}(t)$, we obtain an exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^{n+1}}(t) \to \mathcal{O}_{\mathbb{P}^{n+1}}(t+1)^{\oplus(n+2)} \to T_{\mathbb{P}^{n+1}}(t) \to 0.$$

Taking global sections and using that $H^1(\mathbb{P}^{n+1}, \mathcal{O}_{\mathbb{P}^{n+1}}(t)) = 0$, we have

$$0 \to S_t \to S_{t+1}^{\oplus(n+2)} \to H^0(\mathbb{P}^{n+1}, T_{\mathbb{P}^{n+1}}(t)) \to 0$$

where S_t is the set of all homogeneous polynomials of degree t over $\mathbb{P}^{n+1}$. From this we deduce that a global section of $T_{\mathbb{P}^{n+1}}(t)$ is given in homogeneous coordinates by a vector field

$$\sigma = f_0 \frac{\partial}{\partial x_0} + \cdots + f_{n+1} \frac{\partial}{\partial x_{n+1}}, \tag{3.12}$$

where f_i are homogeneous polynomials of degree $t+1$, modulo multiples of the radial vector field

$$R\colon = x_0 \frac{\partial}{\partial x_0} + \cdots + x_{n+1} \frac{\partial}{\partial x_{n+1}}.$$

Let Y be a smooth hypersurface of $\mathbb{P}^{n+1}$ defined by a homogeneous polynomial h of degree d_h. Then $T_Y(t)$ is a subbundle of $T_{\mathbb{P}^{n+1}}(t)|_Y$. In particular, we have the following exact sequence of vector bundles

$$0 \to T_Y(t) \to T_{\mathbb{P}_{n+1}}(t)|_Y \to \mathcal{O}_Y(d_h + t) \to 0.$$

It induces a map $\beta_t\colon H^0(Y, T_{\mathbb{P}^{n+1}}(t)|_Y) \to H^0(Y, \mathcal{O}(d_h + t))$. By Theorem 3.2.5, a global section of $T_{\mathbb{P}^{n+1}}(t)|_Y$ can be extended to be a global section of $T_{\mathbb{P}^{n+1}}(t)$. Let σ be a global section of $T_{\mathbb{P}^{n+1}}(t)$ of the form (3.12). Then the image $\beta_t(\sigma)$ coincides with the restriction of the following homogeneous polynomial over Y

$$f_0 \frac{\partial h}{\partial x_0} + \cdots + f_{n+1} \frac{\partial h}{\partial x_{n+1}}.$$

In particular, given a point $y \in Y$, the vector $\sigma(y)$ lies in the vector subspace $T_{Y,y}(t) \subset T_{\mathbb{P}^{n+1},y}(t)$ if and only if we have

$$f_0(y) \frac{\partial h}{\partial x_0}(y) + \cdots + f_{n+1}(y) \frac{\partial h}{\partial x_{n+1}}(y) = 0.$$

Moreover, the set-theoretic zero locus of σ coincides with the subset defined by the ideal of 2×2-minors, $x_i f_j - x_j f_i$ of the matrix

$$\begin{pmatrix} x_0 & x_1 & \cdots & x_{n+1} \\ f_0 & f_1 & \cdots & f_{n+1} \end{pmatrix}.$$

Let Y be a general complete intersection in a $(n+r)$-dimensional irreducible Hermitian symmetric space M of compact type such that $n+r \geq 4$, and let $X \in |\mathcal{O}_Y(d)|$ be a general hypersurface of Y such that $\dim(X) \geq 2$. By [Wah83] (cf. Theorem 2.3.9), $H^0(M, T_M(t)) \neq 0$ for some $t < 0$ if and only if $M \cong \mathbb{P}^{n+r}$ and $t = -1$. According to Theorem 3.2.5, we have

$$H^0(Y, T_Y(t)) = H^0(X, T_Y(t)|_X) = H^0(M, T_M(t)) = 0$$

for any $t \leq -2$. In the following theorem, we generalize this result to show that if Y is a general hypersurface of $\mathbb{P}^{n+1}$ and $X \in |\mathcal{O}_Y(d)|$ is a general divisor, then the natural restriction

$$\alpha_t \colon H^0(Y, T_Y(t)) \longrightarrow H^0(X, T_Y(t)|_X)$$

is surjective for $t \leq t_0$ where t_0 is a positive number large enough depending only on the degrees and the dimensions of X and Y. This theorem is a key ingredient of the proof of Theorem 1.1.12.

3.2.13. Theorem. *Let $Y \subset \mathbb{P}^{n+1}$ be a general smooth hypersurface defined by the homogeneous polynomial h of degree $d_h \geq 2$ and let $X \in |\mathcal{O}_Y(d)|$ be a general smooth divisor. Assume $n \geq 3$, then the restriction map*

$$H^0(Y, T_Y(t)) \longrightarrow H^0(X, T_Y(t)|_X)$$

is surjective for $t \leq (\rho + d)/2 - d_h$, where ρ is the top degree of the Milnor algebra of Y.

Proof. Since the natural restriction $H^0(\mathbb{P}^{n+1}, \mathcal{O}_{\mathbb{P}^{n+1}}(d)) \to H^0(Y, \mathcal{O}_Y(d))$ is surjective, there exists a general homogeneous polynomial f of degree d such that $X = \{f = h = 0\}$. We denote by $M(Y)$ and $J(Y)$ the Milnor algebra and Jacobian ideal of Y, respectively. Since $H^0(X, T_Y(t)|_X)$ is a subset of $H^0(X, T_{\mathbb{P}^{n+1}}(t)|_X)$ and $H^0(\mathbb{P}^{n+1}, T_{\mathbb{P}^{n+1}}(t)) = 0$ for $t \leq -2$, we may assume $t \geq -1$ in the sequel of our proof.

Let $s \in H^0(X, T_Y(t)|_X)$ be a global section. By Theorem 3.2.5, the section s is the restriction of some global section $\sigma \in H^0(\mathbb{P}^{n+1}, T_{\mathbb{P}^{n+1}}(t))$. Then there exist some polynomials f_i of degree $t+1$ such that

$$s = \sigma|_X = f_0 \frac{\partial}{\partial x_0} + \cdots + f_{n+1} \frac{\partial}{\partial x_{n+1}} \bigg|_X$$

and

$$f_0 \frac{\partial h}{\partial x_0} + \cdots + f_{n+1} \frac{\partial h}{\partial x_{n+1}} \bigg|_X = 0.$$

As a consequence, there exist two homogeneous polynomials g and p (maybe zero) such that

$$f_0 \frac{\partial h}{\partial x_0} + \cdots + f_{n+1} \frac{\partial h}{\partial x_{n+1}} = gf + ph.$$

We show that g is contained in the Jacobian ideal $J(Y)$ of Y. In fact, by Euler's homogeneous function theorem, it follows

$$\left(f_0 - \frac{1}{d_h} p x_0\right) \frac{\partial h}{\partial x_0} + \cdots + \left(f_{n+1} - \frac{1}{d_h} p x_{n+1}\right) \frac{\partial h}{\partial x_{n+1}} = gf.$$

Thanks to Theorem 3.2.11, the Milnor algebra $M(Y)$ has maximal rank property, hence, by the generic assumption of X, the multiplication map

$$(\times f) \colon M(Y)_{t+d_h-d} \longrightarrow M(Y)_{t+d_h}$$

has maximal rank. Moreover, by the assumption, we have

$$t + d_h - d \leq \frac{\rho - d}{2},$$

so the multiplication map $(\times f)$ is injective (cf. Remark 3.2.10). It follows that $g = 0$ in $M(Y)$, or equivalently, the polynomial g is contained in the Jacobian ideal of Y. Then there exist some homogeneous polynomials g_i's of degree $t - d + 1$ such that

$$g = g_0 \frac{\partial h}{\partial x_0} + \cdots + g_{n+1} \frac{\partial h}{\partial x_{n+1}}.$$

This yields

$$\left(f_0 \frac{\partial h}{\partial x_0} + \cdots + f_{n+1} \frac{\partial h}{\partial x_{n+1}} \right) - \left(g_0 f \frac{\partial h}{\partial x_0} + \cdots + g_{n+1} f \frac{\partial h}{\partial x_{n+1}} \right) = ph.$$

We denote by $\sigma' \in H^0(\mathbb{P}^{n+1}, T_{\mathbb{P}^{n+1}}(t))$ the global section defined by

$$g_0 f \frac{\partial}{\partial x_0} + \cdots + g_{n+1} f \frac{\partial}{\partial x_{n+1}}.$$

Then $(\sigma - \sigma')|_Y \in H^0(Y, T_Y(t))$. Moreover, note $\sigma'|_X \equiv 0$, it follows that we have $(\sigma - \sigma')|_X = \sigma|_X = s$, hence the restriction map

$$H^0(Y, T_Y(t)) \to H^0(X, T_Y(t)|_X)$$

is surjective. □

3.2.14. Remark. For a general complete intersections in projective spaces, the Fröberg's conjecture will imply an extension theorem of the same type. However, so far I do not know whether the upper bound given by the Fröberg's conjecture is large enough to prove Theorem 1.1.12 for complete intersections.

As an immediate simple application of our extension theorem, we consider the problem of the splitting of the tangent sequences of hypersurfaces over smooth divisors. To be more precise, we prove the following theorem.

3.2.15. Corollary. *Let Y be a general smooth hypersurface of a $(n+1)$-dimensional irreducible Hermitian symmetric space M of compact type such that $n \geq 3$. Let $X \in |\mathcal{O}_Y(d)|$ be a general smooth divisor. Then the tangent sequence*

$$0 \to T_X \to T_Y|_X \to N_{X/Y} \to 0$$

splits if and only if $d = 1$ and Y is a projective space or a quadric hypersurface.

Proof. As $N_{X/Y} = \mathcal{O}_X(d)$, the splitting of the tangent sequence implies $H^0(X, T_Y(-d)|_X) \neq 0$. If Y is a projective space, then we get $d = 1$ since the restriction

$$H^0(Y, T_Y(t)) \longrightarrow H^0(X, T_Y(t)|_X)$$

is surjective for all $t \in \mathbb{Z}$ (cf. [vdV59]). Now assume that Y is a hypersurface which is not isomorphic to $\mathbb{P}^n$. If M is not isomorphic to $\mathbb{P}^{n+1}$ or Q^{n+1}, then Theorem 3.2.5, the restriction map

$$H^0(M, T_M(-d)) \to H^0(X, T_M(-d)|_X)$$

is surjective. It follows that $H^0(X, T_M(-d)|_X) = 0$ since M is not isomorphic to $\mathbb{P}^{n+1}$. As a consequence, we obtain $H^0(X, T_Y(-d)|_X) = 0$. Now we assume that M is isomorphic to $\mathbb{P}^{n+1}$ or Q^{n+1} and $d_Y \geq 2$. As $d \geq 1$, it follows by Theorem 3.2.5 and Theorem 3.2.6, the restriction

$$H^0(M, T_M(-d)) \to H^0(X, T_M(-d)|_X)$$

is surjective as $d_Y + d > d_Y$. Thus $H^0(X, T_M(-d)|_X) \neq 0$ implies $M \cong \mathbb{P}^{n+1}$ and $d = 1$. In particular, Y is a general hypersurface of degree $d_Y \geq 2$ in $M \cong \mathbb{P}^{n+1}$. As $H^0(Y, T_Y(-1)) = 0$, Theorem 3.2.13 implies $-d > (\rho + d)/2 - d_Y$. Equivalently we have

$$-1 > \frac{(n+2)(d_Y - 2) + 1}{2} - d_Y = \frac{n(d_Y - 2) - 3}{2}.$$

As $n \geq 3$, the only possibility is $d_Y = 2$, i.e., Y is a general quadric hypersurface. $\square$

3.2.16. **Remark**. In general, the splitting of tangent sequence gives a strong restriction on the geometry of the submanifold and the ambient space. We refer to [Jah05] for further discussion on this problem.

3.3 Stability and effective restrictions with invariant Picard group

This section is devoted to study the stability of tangent bundles of complete intersections in Hermitian symmetric spaces. As mentioned in the introduction, this problem was studied by Peternell and Wiśniewski in [PW95] in the projective spaces case. Moreover, we also consider the effective restriction problems for tangent bundles.

3.3.1 Stability of the tangent bundles of complete intersections

We start with a simple but useful observation. It allows us to prove Theorem 1.1.5 by induction on codimension. It is very useful when we consider the cohomologies of hypersurfaces in some projective manifolds with many cohomology vanishings.

3.3.1. **Lemma**. *Let $(Z, \mathcal{O}_Z(1))$ be a polarized projective manifold of dimension $n+1 \geq 3$. Let $Y \in |\mathcal{O}_Z(d)|$ be a smooth hypersurface of degree d. If $H^q(Y, \Omega_Y^p(\ell)) \neq 0$ and set $m = \min\{p, n-q\}$, then one of the following holds.*

(1) There exists $0 \leq j \leq m-1$ such that $H^{q+j}(Z, \Omega_Z^{p-j}(\ell - jd)) \neq 0$.
(2) There exists $0 \leq j \leq m-1$ such that $H^{q+j+1}(Z, \Omega_Z^{p-j}(\ell - jd - d)) \neq 0$.
(3) $H^{q+m}(Y, \Omega_Y^{p-m}(\ell - md)) \neq 0$.

Proof. To prove the lemma, we assume that both (1) and (2) in the Lemma do not hold. Consider the natural exact sequence

$$0 \to \Omega_Z^{p-j}(\ell - jd - d) \to \Omega_Z^{p-j}(\ell - jd) \to \Omega_Z^{p-j}(\ell - jd)|_Y \to 0.$$

Then our assumption implies that we have $H^{q+j}(Z, \Omega_Z^{p-j}(\ell - jd)|_Y) = 0$ for any $0 \leq j \leq m-1$. On the other hand, the following conormal sequence of Y

$$0 \to \mathcal{O}_Y(-d) \to \Omega_Z^1|_Y \to \Omega_Y^1 \to 0$$

induces the following exact sequence of vector bundles

$$0 \to \Omega_Y^{p-j-1}(-d) \to \Omega_Z^{p-j}|_Y \to \Omega_Y^{p-j} \to 0.$$

Tensoring it by $\mathcal{O}_Y(\ell - jd)$, we obtain the following exact sequence over Y

$$0 \to \Omega_Y^{p-j-1}(\ell - jd - d) \to \Omega_Z^{p-j}(\ell - jd)|_Y \to \Omega_Y^{p-j}(\ell - jd) \to 0.$$

Then the natural map

$$H^{q+j}(Y, \Omega_Y^{p-j}(\ell - jd)) \longrightarrow H^{q+j+1}(Y, \Omega_Y^{p-j-1}(\ell - jd - d))$$

is injective for all $0 \leq j \leq m-1$. Therefore, the assumption $H^q(Y, \Omega_Y^p(\ell)) \neq 0$ implies immediately that we have $H^{q+m}(Y, \Omega_Y^{p-m}(\ell - md)) \neq 0$. $\square$

Using Lemma 3.3.1 together with Akizuki-Nakano vanishing theorem, one can easily derive the following result.

3.3.2. **Lemma.** *Let $(Z, \mathcal{O}_Z(1))$ be a $(n+1)$-dimensional polarized projective manifold such that $n \geq 2$. Assume that here exists an integer r_Z such that $\mathcal{O}_Z(K_Z) \cong \mathcal{O}_Z(-r_Z)$ for some integer r_Z. Assume moreover that for any $q \geq 0$, $p \geq 1$ and $\ell \in \mathbb{Z}$ such that $q+p \leq n$, the following conditions hold.*

(a) If $H^q(Z, \Omega_Z^p) \neq 0$, then $p = q$.
(b) If $H^q(Z, \Omega_Z^p(\ell)) \neq 0$ for some $\ell \neq 0$, then $(n+1)(\ell + q) \geq p r_Z$.

If $Y \in |\mathcal{O}_Z(d)|$ is a smooth hypersurface of degree $d \geq 2$, then for $q \geq 0$, $p \geq 1$ and $\ell \in \mathbb{Z}$ such that $q + p \leq n-1$, the following results hold.

(1) If $H^q(Y, \Omega_Y^p) \neq 0$, then $p = q$.
(2) If $H^q(Y, \Omega_Y^p(\ell)) \neq 0$ for some $\ell \neq 0$, then $n(\ell + q) > p r_Y$, where $r_Y = r_Z - d$.

Proof. First we consider the case $Z \cong \mathbb{P}^{n+1}$. Thanks to [Nar78, Corollary 2.3.1], under our assumption, we have $H^q(Y, \Omega_Y^p(\ell)) \neq 0$ if and only if $q = p$ and $\ell = 0$. Hence the result holds if $r_Z = n+2$. From now on, we shall assume that $r_Z \leq n+1$. As a consequence, we have $r_Y \leq n-1$ since $d \geq 2$.

Proof of (1). By Kodaira's vanishing theorem, we have $H^{q+p}(Y, \mathcal{O}_Y(-pd)) = 0$ since $q + p \leq n-1$. Thus, by Lemma 3.3.1, there exists an integer $0 \leq j \leq p-1$ such that

$$H^{q+j}(Z, \Omega_Z^{p-j}(-jd)) \neq 0 \;\text{ or }\; H^{q+j+1}(Z, \Omega_Z^{p-j}(-jd-d)) \neq 0.$$

Note that we have $q+p+1 \leq n < \dim(Z)$ and $jd + d > 0$, so Akizuki-Nakano vanishing theorem implies $H^{q+j+1}(Z, \Omega_Z^{p-j}(-jd-d)) = 0$. It follows that $H^{q+j}(Z, \Omega_Z^{p-j}(-jd)) \neq 0$. By Akizuki-Nakano vanishing theorem again, we obtain $-jd \geq 0$. As a consequence, we get $j = 0$. Then the assumption (a) implies $p = q$.

Proof of (2). Since $q + p \leq n-1$, if $H^{q+p}(Y, \mathcal{O}_Y(\ell - pd)) \neq 0$, by Kodaira's vanishing theorem, we get $\ell - pd \geq 0$. As a a consequence, we have

$$\ell + q \geq pd + q \geq 2p > p\frac{r_Y}{n}.$$

Thus, we may assume that $H^{q+p}(Y, \mathcal{O}_Y(\ell - pd)) = 0$. According to Lemma 3.3.1, there exists an integer $0 \leq j \leq p-1$ such that

$$H^{q+j}(Z, \Omega_Z^{p-j}(\ell - jd)) \neq 0 \;\text{ or }\; H^{q+j+1}(Z, \Omega_Z^{p-j}(\ell - jd - d)) \neq 0.$$

If $H^{q+j}(Z, \Omega_Z^{p-j}(\ell - jd)) \neq 0$, by Akizuki-Nakano vanishing theorem, we have $\ell - jd \geq 0$. If $\ell = jd$, then the assumption (a) implies $q + j = p - j$. As a consequence, we obtain

$$\ell + q = jd + p - 2j \geq p > p\frac{r_Y}{n}.$$

If $\ell > jd$, then assumption (b) implies

$$\ell - jd + q + j \geq (p-j)\frac{r_Z}{n+1}.$$

As a consequence, we get

$$\ell + q \geq \left(d - 1 - \frac{r_Z}{n+1}\right) j + p\frac{r_Z}{n+1} \geq p\frac{r_Z}{n+1} > p\frac{r_Y}{n}.$$

If $H^{q+j+1}(Z, \Omega_Z^{p-j}(\ell - jd - d)) \neq 0$, then $\ell \geq jd + d$ by Akizuki-Nakano vanishing theorem. If

$\ell = jd + d$, then $q + j + 1 = p - j$ by assumption (a). As a consequence, we get

$$\ell + q = jd + d + p - 2j - 1 \geq p + 1 > p\frac{r_Y}{n}.$$

If $\ell > jd + d$, then by assumption (b), we obtain

$$\ell - jd - d + q + j + 1 \geq (p - j)\frac{r_Z}{n+1}.$$

Equivalently, we have

$$\ell + q \geq \left(d - 1 - \frac{r_Z}{n+1}\right) j + d - 1 + p\frac{r_Z}{n+1} > p\frac{r_Y}{n}.$$

This completes the proof. □

3.3.3. **Remark**. From the proof above, if $q + p \leq n - 1$, we see that the assumption (a) implies (1) for any $d \geq 1$.

Now we are in the position to prove the main technical result in this section. The idea is to use repeatedly the Lemma 3.3.2.

3.3.4. **Theorem**. *Let M be an irreducible compact Hermitian symmetric space of dimension $n + r$. Let $H_i \in |\mathcal{O}_M(d_i)|$ $(1 \leq i \leq r)$ be a collection of hypersurfaces such that $d_i \geq 2$ for all $1 \leq i \leq r$. Then, for any $q \geq 0$, $p \geq 1$, $\ell \in \mathbb{Z}$ and $q + p \leq n - 1$, the following hold.*

(1) If $H^q(Y_r, \Omega^p_{Y_r}) \neq 0$, then $p = q$.
(2) If $H^q(Y_r, \Omega^p_{Y_r}(\ell)) \neq 0$ for some $\ell \neq 0$, then $n(\ell + q) > pr_{Y_r}$, where $r_{Y_r} = r_M - d_1 - \cdots - d_r$.

Proof. It is enough to verify that M satisfies the assumptions in Lemma 3.3.2. As M is an irreducible Hermitian symmetric space of compact type, it is well-known that $H^q(M, \Omega^p_M) \neq 0$ if and only if $p = q$. If $\ell \neq 0$, by Akizuki-Nakano vanishing theorem, $H^q(M, \Omega^p_M(\ell)) \neq 0$ implies $\ell > 0$. Then by [BCM18, Theorem C] and Bott's formula, we conclude that M satisfies the assumptions in Lemma 3.3.2. □

The following theorem is a direct consequence of Theorem 3.3.4.

3.3.5. **Theorem**. *Let M be a n-dimensional irreducible Hermitian symmetric space of compact type, and denote by $\mathcal{O}_M(1)$ the ample generator of* $\mathrm{Pic}(M)$. *Let Y be a submanifold of M such that the restriction* $\mathrm{Pic}(M) \to \mathrm{Pic}(Y)$ *is surjective. Then the tangent bundle T_Y is stable if one of the following conditions holds.*

(1) There exist hypersurfaces $H_i \in |\mathcal{O}_M(d_i)|$ with $d_i \geq 2$ and $1 \leq i \leq r \leq n - 2$ such that the complete intersections $H_1 \cap \cdots \cap H_j$ are smooth for all $1 \leq j \leq r$ and $Y = H_1 \cap \cdots \cap H_r$.
(2) Y is a smooth hypersurface.

Proof. To prove the stability of T_Y, it is equivalent to prove the stability of Ω^1_Y. Let $\mathcal{F} \subset \Omega^1_Y$ be a nontrivial saturated proper subsheaf of rank p $(1 \leq p \leq n - 1)$. We denote by ℓ the unique integer such that $\det(\mathcal{F}) \cong \mathcal{O}_Y(-\ell)$. Then we have $H^0(Y, \Omega^p_Y(\ell)) \neq 0$. Since $p \leq n - 1$, the Akizuki-Nakano theorem implies $\ell \geq 0$. As $p \geq 1$, Theorem 3.3.4 (1) implies $\ell > 0$. Then (1) follows from Theorem 3.3.4 (2) directly.

Now we prove (2). By (1), it remains to consider the case $d = 1$. If M is isomorphic to a quadric hypersurface or a projective space, then Y is again a quadric hypersurface or a projective space. Moreover, as $\rho(Y) = 1$ by our assumption, Ω^1_Y is also stable. So we shall assume that M is not isomorphic to either a quadric hypersurface or a projective space. Considering the following sequence

$$0 \to \Omega^p_Y(\ell) \to \Omega^{p+1}_M(\ell + 1)|_Y \to \Omega^{p+1}_Y(\ell + 1) \to 0,$$

one observe that the map

$$H^0(Y, \Omega^p_Y(\ell)) \to H^0(Y, \Omega^{p+1}_M(\ell + 1)|_Y)$$

is injective. We consider the natural exact sequence

$$0 \to \Omega_M^{p+1}(\ell) \to \Omega_M^{p+1}(\ell+1) \to \Omega_M^{p+1}(\ell+1)|_Y \to 0.$$

Then we get $H^0(M, \Omega_M^{p+1}(\ell+1)) \neq 0$ or $H^1(M, \Omega_M^{p+1}(\ell)) \neq 0$.

First we consider the case $H^0(M, \Omega_M^{p+1}(\ell+1)) \neq 0$. If $\ell \geq r_M$, since $p \leq \dim(M) - 2$, it is clear that we have $\mu(\mathcal{F}) < \mu(\Omega_Y^1)$. If $\ell \leq r_M - 1$, by the results proved in the last section (cf. Theorem 3.1.5, 3.1.13, 3.1.19 and 3.1.24), we have

$$(\ell+1) - 1 > ((p+1) - 1)\frac{r_M - 1}{\dim(M) - 1}.$$

Here we recall that we have $r_M < \dim(M)$ by our assumption. As a consequence, it follows that we have $\mu(\mathcal{F}) < \mu(\Omega_Y^1)$.

Next we consider the case $H^1(M, \Omega_M^{p+1}(\ell)) \neq 0$. According to Theorem 3.1.1, we get

$$(\ell+1) - 2 \geq ((p+1) - 2)\frac{r_M}{\dim(M)} \geq (p-1)\frac{r_M - 1}{\dim(M) - 1} = (p-1)\frac{r_Y}{\dim(Y)}.$$

Since $r_M < \dim(M)$, we have also $r_Y < \dim(Y)$. This implies

$$\frac{\ell}{p} > \frac{r_Y}{\dim(Y)}.$$

As a consequence, we obtain $\mu(\mathcal{F}) < \mu(\Omega_Y^1)$. Hence, the cotangent sheaf Ω_Y^1 is stable. $\square$

3.3.6. **Remark.** If M is of type E_6 or E_7, for $d = 1$, the stability of Ω_Y^1 can be derived also from [Hwa01, Theorem 2.11].

3.3.2 Effective restriction of tangent bundles

In this subsection, we proceed to prove various effective restriction theorems for the tangent bundles of complete intersections in irreducible Hermitian symmetric spaces of compact type. We use some standard cohomological arguments to reduce the problem to the existence of twisted vector fields.

3.3.7. **Proposition.** *Let M be a $(n+r)$-dimensional irreducible Hermitian symmetric space of compact type. Let $H_i \in |\mathcal{O}_M(d_i)|$ $(1 \leq i \leq r)$ be a collection of hypersurfaces such that Y_j is smooth for any $1 \leq j \leq r$. Set $Y = Y_r$. Let $X \in |\mathcal{O}_Y(d)|$ be a smooth divisor of dimension at least two. Assume that the composite of restrictions*

$$\mathrm{Pic}(M) \to \mathrm{Pic}(Y) \to \mathrm{Pic}(X)$$

is surjective. Moreover, if $Y \cong Q^n$, we assume $d \geq 2$. Then the vector bundle $T_Y|_X$ is stable if and only if $H^0(X, T_Y(t)|_X) = 0$ for any $t \leq -r_Y/n$, where $r_Y = r_M - d_1 - \cdots - d_r$.

Proof. One implication is clear. Now we assume that we have $H^0(X, T_Y(t)|_X) = 0$ for any $t \leq -r_Y/n$. Note that $T_Y|_X$ is stable if and only if $\Omega_Y^1|_X$ is stable. Let $\mathcal{F}$ be a proper saturated subsheaf of $\Omega_Y^1|_X$ of rank p. We denote by ℓ the unique integer such that $\det(\mathcal{F}) = \mathcal{O}_X(-\ell)$. Then, by assumption, we get $H^0(X, \Omega_Y^p(\ell)|_X) \neq 0$. To prove the stability of $\Omega_Y^1|_X$, it suffices to show that the following inequality

$$\mu(\mathcal{F}) = \frac{-\ell}{p}\mathcal{O}_X(1)^{n-1} < \mu(\Omega_Y^1|_X) = \frac{-r_Y}{n}\mathcal{O}_X(1)^{n-1}$$

holds for all pairs of integers (ℓ, p) such that $H^0(X, \Omega_Y^p(\ell)|_X) \neq 0$ and $1 \leq p \leq n-1$. We consider the exact sequence

$$0 \to \Omega_Y^p(\ell - d) \to \Omega_Y^p(\ell) \to \Omega_Y^p(\ell)|_X \to 0.$$

Then the fact $H^0(X, \Omega_Y^p(\ell)|_X) \neq 0$ implies that we have

$$H^0(Y, \Omega_Y^p(\ell)) \neq 0 \text{ or } H^1(Y, \Omega_Y^p(\ell - d)) \neq 0.$$

Case 1. $p \leq n-2$. As $p \geq 1$, if $\ell \neq d$ or $H^0(Y, \Omega_Y^p(\ell)) \neq 0$, by Theorem 3.3.4, we have

$$n\ell \geq n(\ell - qd + q) > pr_Y$$

for $q = 0$ and 1. If $\ell = d$ and $H^0(Y, \Omega_Y(\ell)) = 0$, then we must have $p = 1$. As a consequence, we have

$$n\ell = nd > pr_Y = r_Y$$

unless $d = 1$ and $r_Y \geq n$. If $r_Y \geq n$, by Kobayashi-Ochiai theorem, Y is isomorphic to either $\mathbb{P}^n$ or Q^n. As $d_i \geq 2$, then Y must be Q^n. However, by our assumption, if $Y \cong Q^n$, then we have $d \geq 2$. As a consequence, we get $n\ell > r_Y$.

Case 2. $p = n-1$. We denote by $\mathcal{Q}$ the quotient $\left(\Omega_Y^1|_X\right)/\mathcal{F}$. Then $\mathcal{Q}$ is a torsion-free coherent sheaf of rank one such that $\mathcal{Q}^{\vee\vee} \cong \mathcal{O}_X(-r_Y + \ell)$. Since $\mathcal{Q}^{\vee\vee}$ is a subsheaf of $T_Y|_X$, we get

$$H^0(X, T_Y(\ell - r_Y)|_X) \neq 0.$$

By our assumption, we get $\ell - r_Y > -r_Y/n$. As a consequence, we get

$$\frac{\ell}{p} = \frac{\ell}{n-1} > \frac{r_Y}{n}.$$

This completes the proof. □

As an application of Proposition 3.3.7, we derive the following effective restriction result by the nonexistence of global twisted vector fields.

3.3.8. Theorem. *Let M be a $(n+r)$-dimensional irreducible Hermitian symmetric space of compact type such that $n \geq 3$ and $r \geq 1$. Let $H_i \in |\mathcal{O}_M(d_i)|$ $(1 \leq i \leq r)$ be a collection of hypersurfaces such that $d_i \geq 2$, $d_1 \leq \cdots \leq d_r$ and the complete intersections $H_1 \cap \cdots \cap H_j$ are smooth for all $1 \leq j \leq r$. Denote $H_1 \cap \cdots \cap H_r$ by Y. Let $X \in |\mathcal{O}_Y(d)|$ be a general smooth hypersurface. Assume moreover that we have $d_1 \geq 2$ and the restrictions*

$$\mathrm{Pic}(M) \to \mathrm{Pic}(Y) \to \mathrm{Pic}(X)$$

are surjective. Then the restriction $T_Y|_X$ is stable if one of the following conditions holds.

(1) Y is a Fano manifold and M is isomorphic to neither the projective space $\mathbb{P}^{n+r}$ nor a smooth quadric hypersurface Q^{n+r}.
(2) Y is a Fano manifold, M is isomorphic to the projective space $\mathbb{P}^{n+r}$ with $n+r \geq 5$ and $d \geq d_1$.
(3) Y is a Fano manifold, M is isomorphic to a smooth quadric hypersurface Q^{n+r} and $d \geq 2$.
(4) $d > d_r - r_Y/n$, where r_Y is the unique integer such that $\omega_Y \cong \mathcal{O}_Y(-r_Y)$.

Proof. Let X be a projective manifold of dimension $N \geq 2$, and let L be an ample line bundle. Recall that $H^0(X, T_X \otimes L^{-1}) \neq 0$ if and only if $X \cong \mathbb{P}^N$ and $L \cong \mathcal{O}_{\mathbb{P}^N}(1)$ (cf. [Wah83]). In particular, if M is not isomorphic to a projective space, then we have $H^0(M, T_M(t)) = 0$ for any $t < 0$.

Proof of (1). Under our assumption, by Theorem 3.2.5, the natural restriction map

$$\rho_t \colon H^0(M, T_M(t)) \to H^0(X, T_M(t)|_X)$$

is surjective for all $t \in \mathbb{Z}$. In particular, we have $H^0(X, T_M(t)|_X) = 0$ for all $t < 0$. This implies $H^0(X, T_Y(t)|_X) = 0$ for all $t < 0$ since $H^0(X, T_Y(t)|_X)$ is a subgroup of $H^0(X, T_M(t)|_X)$. As Y is Fano, we have $r_Y > 0$. Then we conclude by Proposition 3.3.7.

Proof of (2). By Theorem 3.2.6, the natural restriction map $\rho_t \colon H^0(Y_1, T_{Y_1}(t)) \to H^0(X, T_{Y_1}(t)|_X)$ is surjective for all $t \leq -1$ if $d \geq d_1$. In particular, it follows that we have $H^0(X, T_{Y_1}(t)|_X) = 0$ for all

$t \le -1$ if $d \ge d_1$. Again, since Y is Fano, we have $r_Y > 0$ and we can conclude by Proposition 3.3.7.

Proof of (3). If M is isomorphic to a smooth quadric hypersurface Q^{n+r} and $Y \subset M$ is a complete intersection of degree $(d_1, \cdots, d_r)$ such that $d_i \ge 2$ for all $1 \le i \le r$. Then Y is also a complete intersection in $\mathbb{P}^{n+r+1}$ of degree $(2, d_1, \cdots, d_r)$ and we conclude by (2).

Proof of (4). Note that we have $d > d_1 - r_Y/n \ge 1$ as $d_1 \ge 2$ and $r_Y \le n$. Thus, by Proposition 3.3.7, it suffices to show that

$$H^0(X, T_Y(t)|_X) = 0 \qquad \text{for} \qquad t \le -\frac{r_Y}{n}.$$

Since M is not isomorphic to a quadric hypersurface, by Theorem 3.2.5, the natural restriction map

$$H^0(M, T_M(t)) \to H^0(Y, T_M(t)|_Y) \to H^0(X, T_M(t)|_X)$$

is surjective for all $t \in \mathbb{Z}$. Let $\sigma \in H^0(X, T_Y(t)|_X)$ be a global section. Then σ is also a global section of $T_M(t)|_X$. Thus there exists a global twisted vector field $\widetilde{\sigma} \in H^0(M, T_M(t))$ such that $\widetilde{\sigma}|_X = \sigma$. In particular, $\widetilde{\sigma}|_X = \sigma$ is a global section of $T_{Y_j}(t)|_X$ for all $1 \le j \le r$. Consider the following exact sequence

$$0 \to T_{Y_j}(t)|_Y \to T_{Y_{j-1}}(t)|_Y \xrightarrow{\beta_j(t)} \mathcal{O}_Y(d_j + t) \to 0.$$

Then $\sigma \in H^0(X, T_Y(t)|_X)$ implies that the image $\widehat{\beta}_j(t)(\widetilde{\sigma}|_Y)$ vanishes over X, where $\widehat{\beta}_j(t)$ is the induced map

$$H^0(Y, T_M(t)|_Y) \longrightarrow H^0(Y, \mathcal{O}_Y(d_j + t)).$$

However, note that we have $d > d_j + t$ for any $1 \le j \le r$ by our assumption. This implies that we have $\widehat{\beta}_j(t)(\widetilde{\sigma}|_Y) = 0$ for any $1 \le j \le r$ since X is general. It follows that $\widetilde{\sigma}|_Y \in H^0(Y, T_{Y_j}(t)|_Y)$ for any $1 \le j \le r$, i.e., $\widetilde{\sigma}|_Y \in H^0(Y, T_Y(t))$. On the other hand, since T_Y is stable (cf. Theorem 3.3.5), we have

$$H^0(Y, T_Y(t)) = 0 \qquad \text{for} \qquad t \le -\frac{r_Y}{n}.$$

Then we obtain $\widetilde{\sigma}|_Y = 0$ and consequently $\sigma = 0$. This completes the proof. $\square$

Though the estimates in (2), (3) and (4) of the theorem are not optimal, they have the advantage to give a lower bound which is quite easy to compute. In the following, we will consider the case where Y is a general smooth hypersurface of $\mathbb{P}^{n+1}$ and we give a complete answer to the effective restriction problem for T_Y as an application of Theorem 3.2.13 and Proposition 3.3.7.

3.3.9. Theorem. *Let Y be a general smooth hypersurface in the projective space $\mathbb{P}^{n+1}$ of dimension $n \ge 3$. Let $X \in |\mathcal{O}_Y(d)|$ be a general smooth hypersurface of degree d on Y. Assume furthermore that the restriction homomorphism* $\mathrm{Pic}(Y) \to \mathrm{Pic}(X)$ *is surjective, then $T_Y|_X$ is stable unless $d = 1$ and Y is isomorphic to either $\mathbb{P}^n$ or Q^n.*

Proof. If Y is isomorphic to either $\mathbb{P}^n$ or Q^n, this follows from [BCM18, Theorem A]. So we shall assume that Y is a general smooth hypersurface defined by a homogeneous polynomial h of degree $d_h \ge 3$. By Proposition 3.3.7, it is enough to prove that $H^0(X, T_Y(t)|_X) = 0$ for $t \le -r_Y/n$. As $n \ge 3$, $d_h \ge 3$ and $r_Y = n + 2 - d_h$, we have

$$\begin{aligned}
\left(\frac{\rho + d}{2} - d_h\right) - \left(-\frac{r_Y}{n}\right) &\ge \frac{(d_h - 2)(n+2) + d}{2} - d_h - \left(\frac{d_h}{n} - \frac{n+2}{n}\right) \\
&\ge \left(\frac{n}{2} - \frac{1}{n}\right) d_h - n - \frac{1}{2} + \frac{2}{n} \\
&\ge \frac{3n}{2} - \frac{3}{n} - n - \frac{1}{2} + \frac{2}{n} \\
&> 0.
\end{aligned}$$

This implies $-r_Y/n \leq (\rho+d)/2 - d_h$. According to Theorem 3.2.13, we see that the map

$$H^0(Y, T_Y(t)) \to H^0(X, T_Y(t)|_X)$$

is surjective for $t \leq -r_Y/n$. Then we conclude by the stability of T_Y. □

3.3.10. **Remark**. In Theorem 3.3.9, if Y is only a smooth hypersurface, then the argument above does not work, since the strong Lefschetz property (SLP) of Milnor algebras of smooth hypersurfaces is still open.

3.4 Hyperplane of cubic threefolds

In this section, we consider the case where the map $\mathrm{Pic}(Y) \to \mathrm{Pic}(X)$ is not surjective. By Noether-Lefschetz theorem mentioned in the introduction, this happens if X is a quadric section of a quadric threefold Q^3, or X is a quadric surface in $\mathbb{P}^3$, or X is a cubic surface in $\mathbb{P}^3$. In these cases, X is always a del Pezzo surface, i.e., the anti-canonical divisor $-K_X$ is ample.

3.4.1 Projective one forms

We denote by $\pi\colon S_r \to \mathbb{P}^2$ the surface obtained by blowing-up $\mathbb{P}^2$ in $9-r(\leq 8)$ points $p_1, \cdots, p_{9-r}$ in general position and denote by E_j the exceptional divisor over p_j. Then S_r is a Del Pezzo surface with degree $K_{S_r}^2 = r$. It is well-known that the cotangent bundle $\Omega^1_{S_r}$ is stable with respect to the anti-canonical polarization $-K_{S_r}$ for $r \leq 7$ [Fah89].

We consider the saturated subsheaves L of the cotangent bundle of $\Omega^1_{S_r}$. Let E_0 be the pull-back of a line over $\mathbb{P}^2$. Since the Picard group of S_r are generated by E_0 and the exceptional divisors E_j, we can write

$$L \sim \mathcal{O}_{S_r}\left(-aE_0 - \sum_{j=1}^{9-r} b_j E_j\right)$$

for some integers $a, b_j \in \mathbb{Z}$. Then we have an injective map

$$\begin{aligned} H^0(S_r, \Omega^1_{S_r} \otimes L^{-1}) &\hookrightarrow H^0\left(S_r \setminus \cup E_i, \Omega^1_{S_r} \otimes L^{-1}|_{S_r \setminus \cup E_i}\right) \\ &\cong H^0\left(\mathbb{P}^2 \setminus \cup\{p_i\}, \Omega^1_{\mathbb{P}^2}(a)|_{\mathbb{P}^2\setminus\cup\{p_i\}}\right) \cong H^0(\mathbb{P}^2, \Omega^1_{\mathbb{P}^2}(a)). \end{aligned}$$

On the other hand, a global section of $\Omega^1_{\mathbb{P}^2}(a)$ also induces a saturated subsheaf of $\Omega^1_{S_r}$: indeed, let $\omega \in H^0(\mathbb{P}^2, \Omega^1_{\mathbb{P}^2}(a))$ be a global section and let $\pi\colon S_r \to \mathbb{P}^2$ be the blowing-up. Then the pull back $\pi^*\omega$ is a global section of $\pi^*\Omega^1_{\mathbb{P}^2}(a)$. Note that $\pi^*\Omega^1_{\mathbb{P}^2}(a)$ is a subsheaf of $\Omega^1_{S_r} \otimes \mathcal{O}_{S_r}(aE_0)$, it follows that $\pi^*\omega$ is also a global section of $\Omega^1_{S_r} \otimes \mathcal{O}_{S_r}(aE_0)$. Let $\mathrm{div}(\pi^*\omega)$ be the divisor defined by the zeros of $\pi^*\omega$. Then $\pi^*\omega$ induces an injective morphism of sheaves $\mathcal{O}_{S_r}(-aE_0 + \mathrm{div}(\pi^*\omega)) \to \Omega^1_{S_r}$.

The correspondence above is actually one-to-one. Let σ be a global section of $H^0(S_r, \Omega^1_{S_r} \otimes L^{-1})$ corresponding to the inclusion $L \to \Omega^1_{S_r}$ and let ω be the induced global section of $\Omega^1_{\mathbb{P}^2}(a)$. Let L' be the saturated subsheaf of $\Omega^1_{S_r}$ induced by $\pi^*\omega$. Since L is saturated, the quotient $\mathcal{G} = \Omega^1_{S_r}/L$ is a torsion-free sheaf. Moreover, the composite morphism $L' \to \mathcal{G}$ vanishes over a Zariski open subset of S_r, so it vanishes identically. This implies that the sheaf L' is a subsheaf of L. Now the saturation of L' shows that L' and L are isomorphic.

A global section of $\Omega^1_{\mathbb{P}^2}(a)$ is called a *projective one form* on $\mathbb{P}^2$ and $a-2$ is called the degree. Moreover, if $[X : Y : T]$ are the coordinates of $\mathbb{P}^2$, then a global section $\omega \in H^0(\mathbb{P}^2, \Omega^1_{\mathbb{P}^2}(a))$ can be written in homogeneous coordinates as

$$\omega = AdX + BdY + CdT,$$

where A, B and C are homogeneous polynomials of degree $a-1$ satisfying $AX + BY + CT = 0$. Moreover, the set-theoretic zero locus of ω consists of the common zeros of A, B and C.

3.4.1. Example. We recall several examples given in [Fah89].

(1) The form $\omega = x_0 dx_1 - x_1 dx_0 \in H^0(\mathbb{P}^2, \Omega^1_{\mathbb{P}^2}(2))$ defines a subsheaf of $\Omega^1_{S_r}$ which is isomorphic to $\mathcal{O}_{S_r}(-2E_0 + 2E_j)$, where E_j is the exceptional divisor above $[0:0:1]$. Moreover, the only saturated subsheaves of rank one of $\Omega^1_{S_r}$ with $a = 2$ are $\mathcal{O}_{S_r}(-2E_0 + 2E_j)$ and $\mathcal{O}_{S_r}(-2E_0)$.

(2) We choose four points $p_1 = [1:0:0]$, $p_2 = [0:1:0]$, $p_3 = [0:0:1]$ and $p_4 = [1:1:1]$ in $\mathbb{P}^2$. Then the form defined by

$$\omega = (x_1^2 x_2 - x_2^2 x_1)dx_0 + (x_2^2 x_0 - x_0^2 x_2)dx_1 + (x_0^2 x_1 - x_1^2 x_0)dx_2$$

induces a subsheaf $\mathcal{O}_{S_r}(-4E_0 + 2\sum_{j=1}^4 E_j)$ of $\Omega^1_{S_r}$, and there does not exist a subsheaf of $\Omega^1_{S_r}$ of the form $L' = \mathcal{O}^1_{S_r}(-4E_0 + 2\sum_{j=1}^4 E_j + E_i)$ for $5 \leq i \leq r$. In fact, let ω' be the corresponding projective one form of L'. Then ω' is proportional to ω. Nevertheless, the zeros of ω are $p_1, \cdots, p_4$ and the points $[0:1:1]$, $[1:0:1]$ and $[1:1:0]$. Since there are at most four points of these points which are in general position, we get a contradiction.

We will use the following lemma in the proof of Theorem 1.1.13.

3.4.2. Lemma [Fah89, Lemme 1]. *Let L, N be two proper saturated subsheaves of $\Omega^1_{S_r}$. If L is not isomorphic to N, then we have $h^0(S_r, \omega_{S_r} \otimes L^{-1} \otimes N^{-1}) \geq 1$.*

3.4.2 Subsheaves of cotangent bundle of cubic surfaces

A cubic surface $S \subset \mathbb{P}^3$ is a blow-up $\pi\colon S \to \mathbb{P}^2$ of six points p_j on $\mathbb{P}^2$ in general position. The exceptional divisor $\pi^{-1}(p_j)$ is denoted by E_j. Let K_S be the canonical divisor of S and E_0 the pull-back of a line in $\mathbb{P}^2$. Then we have

$$-K_S = 3E_0 - \sum_{j=1}^{6} E_j \sim H|_S,$$

where $H \in |\mathcal{O}_{\mathbb{P}^3}(1)|$ is a hyperplane in $\mathbb{P}^3$. Let us recall the following well-known classical result of cubic surfaces.

- There are exactly 27 lines lying over a cubic surface : the exceptional divisors E_j above the six blown up points p_j, the proper transforms of the fifteen lines in $\mathbb{P}^2$ which join two of the blown up points p_j, and the proper transforms of the six conics in $\mathbb{P}^2$ which contain all but one of the blown up points.

The following result gives an upper bound for the degree of the saturated subsheaves of Ω^1_S. In particular, it implies that there is no foliation $\mathcal{F} \subset T_S$ on smooth cubics S such that $c_1(\mathcal{F}) \cdot (-K_S) > 0$.

3.4.3. Proposition. *Let S be a cubic surface and let $L \subset \Omega^1_S$ be a saturated invertible subsheaf. Then*

$$c_1(L) \cdot (-K_S) \leq -3.$$

Proof. Note that $\mu(\Omega^1_S) = -3/2$ and Ω^1_S is stable, we get $c_1(L) \cdot (-K_S) \leq -2$. Thus it suffices to prove $c_1(L) \cdot K_S \neq 2$. To prove this, we assume to the contrary $c_1(L) \cdot K_S = 2$. Note that we have

$$c_1(L) = -aE_0 - \sum_{j=1}^{6} b_j E_j$$

for some $a, b_j \in \mathbb{Z}$ with $a \geq 2$. If $a = 2$, then L is isomorphic to $\mathcal{O}_S(-2E_0)$ or some $\mathcal{O}_S(-2E_0 + 2E_i)$. In the former case we have $c_1(L) \cdot K_S = 6$ and in the latter case we have $c_1(L) \cdot K_S = 4$. So we may assume $a \geq 3$ in the sequel. By Lemma 3.4.2, for fixed i, there exist some effective divisors C_i such that

$$C_i \sim K_S - L - (-2E_0 + 2E_i) = (a-1)E_0 + (b_i - 1)E_i + \sum_{j \neq i} (b_j + 1)E_j.$$

Denote by $d = -K_S \cdot C_i = 3a + \sum_{j=1}^6 b_j + 1$ the degree of C_i. The hypothesis $c_1(L) \cdot K_S = 2$ is equivalent to $3a + \sum b_j = 2$, so $d = 3$. Moreover, as $a \geq 3$, we have $\sum_{j=1}^6 b_j \leq -7$. As a consequence, there is at least one $b_j \leq -2$.

Step 1. We will show $b_j \geq -2$ for all $1 \leq j \leq 6$. There exist some π-exceptional effective divisors $\sum_{j=1}^6 c_{ij} E_j$ such that the effective divisors C_i' defined as

$$C_i' = C_i - \sum_{j=1}^{6} c_{ij} E_j \sim (a-1)E_0 + (b_i - c_{ii} - 1)E_i + \sum_{j \neq i} (b_j - c_{ij} + 1) E_j$$

don't contain π-exceptional components. We denote the integer $b_j - c_{ij}$ by b_{ij} and denote the degree $-K_S \cdot C_i'$ of C_i' by d_i', then we have

$$b_{ij} \leq b_j \text{ and } d_i' \leq d. \tag{3.13}$$

Since the exceptional divisor E_i is a line on S and $-K_S \sim H|_X$ for some hyperplane $H \subset \mathbb{P}^3$, we get $\mathrm{Bs}\,|-K_S - E_i| \subset E_i$. Moreover, since C_i' does not contain E_i, we obtain

$$(-K_S - E_i) \cdot C_i' \geq 0 \text{ and } -b_{ii} + 1 = C_i' \cdot E_i \leq -K_S \cdot C_i' = d_i'. \tag{3.14}$$

Combining (3.13) and (3.14) gives

$$-b_i \leq -b_{ii} \leq d_i' - 1 \leq d - 1 = 2. \tag{3.15}$$

Since i is arbitrary, we conclude $b_j \geq -2$ for $j = 1, \cdots, 6$.

Step 2. We show $b_j \leq -1$ for all $1 \leq j \leq 6$ and $\sum_{j=1}^6 b_j \leq -8$. Since there is at least one $b_j \leq -2$ and $b_i \geq -2$ for all i, without loss of generality we assume $b_1 = -2$. As a consequence of inequality (3.15), we have

$$b_{11} = -2 \text{ and } d_1' = d = 3.$$

This shows $C_1' = C_1$ and

$$-K_S \cdot C_1 = E_1 \cdot C_1 = 3. \tag{3.16}$$

Moreover, since C_1 does not contain E_j, we have $-b_j - 1 = C_1 \cdot E_j \geq 0$ for $j \geq 2$, this yields $b_j \leq -1$ for $j \geq 2$. As a consequence, we get

$$-12 \leq \sum_{j=1}^{6} b_j \leq -7 \text{ and } 3 \leq a \leq 4.$$

Let $C_{1\ell}$ be a component of C_1. Since $\mathrm{Bs}|-K_S - E_1| \subset E_1$ and C_1 does not contain E_1, we have $(-K_S - E_1) \cdot C_{1\ell} \geq 0$. Then the equality (3.16) implies $(-K_S - E_1) \cdot C_{1\ell} = 0$, this means that $C_{1\ell}$ is a plane curve and there exists a plane $H_\ell \subset \mathbb{P}^3$ such that $C_{1\ell} + E_1 \leq H_\ell|_S$. In particular, we have

$$-K_S \cdot C_{1\ell} = H_\ell|_S \cdot C_{1\ell} \leq 2.$$

since $-K_S \cdot C_1 = 3$, there exists at least one component of C_1, denoted by C_{11}, such that $-K_S \cdot C_{11} = 1$, i.e., C_{11} is a line over S. However, C_{11} is not π-exceptional, so the line C_{11} passes at least two π-exceptional divisors. This shows that there exists some j (≥ 2) such that

$$-2 \leq b_j = -1 - C_1 \cdot E_j \leq -2.$$

Hence we obtain $\sum_{j=1}^6 b_j \leq -8$.

Step 3. We exclude the case $c_1(L)\cdot K_S = 2$. By our argument above, if $c_1(L)\cdot K_S = 2$, then we have

$$a \geq 3,\quad -2 \leq b_j \leq -1 \ \text{ and } \ -12 \leq \sum_{j=1}^{6} b_j \leq -8.$$

Then the equality $3a + \sum_{j=1}^{6} b_j = 2$ shows $a = 4$ and $\sum_{j=1}^{6} b_j = -10$, this forces that L is a line bundle of the form

$$-4E_0 + 2E_1 + 2E_2 + 2E_3 + 2E_4 + E_5 + E_6.$$

Nevertheless, we have seen that such a line bundle cannot be a saturated subsheaf of Ω^1_S (cf. Example 3.4.1), a contradiction. □

3.4.3 Stability of restriction of tangent bundle of cubic threefolds

In this subsection, we will prove Theorem 1.1.13. First we consider the saturated subsheaves of $\Omega^1_Y|_X$ of rank two and we give an upper bound for the degree of $c_1(\mathcal{F})$ with respect to $-K_X$.

3.4.4. Lemma. *Let Y be a general smooth cubic threefold and let $X \in |\mathcal{O}_Y(1)|$ be a general smooth divisor. If $\mathcal{F} \subset \Omega^1_Y|_X$ is a saturated subsheaf of rank two, then we have*

$$c_1(\mathcal{F})\cdot(-K_X) \leq -5.$$

Proof. The natural inclusion $\mathcal{F} \subset \Omega^1_Y|_X$ implies $h^0(X, \Omega^2_Y|\otimes \det(\mathcal{F})^\vee) \geq 1$. Moreover, using the short exact sequence

$$0 \to \Omega^1_X(-1)\otimes \det(\mathcal{F})^\vee \to \Omega^2_Y|_X \otimes \det(\mathcal{F})^\vee \to \omega_X \otimes \det(\mathcal{F})^\vee \to 0,$$

we have either $h^0(X, \Omega^1_X(-1)\otimes\det(\mathcal{F})^\vee) \geq 1$ or $h^0(X, \omega_X \otimes \det(\mathcal{F})^\vee) \geq 1$. In the former case, the stability of Ω^1_X implies

$$(c_1(\mathcal{F}) + c_1(\mathcal{O}_X(1)))\cdot(-K_X) < \frac{K_X\cdot(-K_X)}{2} = -\frac{3}{2}.$$

This yields

$$c_1(\mathcal{F})\cdot(-K_X) < -c_1(\mathcal{O}_X(1))\cdot(-K_X) - \frac{3}{2} = -\frac{9}{2} < -4.$$

In the latter case, we have $c_1(\mathcal{F})\cdot(-K_X) \leq K_X\cdot(-K_X) = -3$ with equality if and only if $c_1(\mathcal{F}) = -K_X$, and the quotient $\mathcal{G} := \left(\Omega^1_Y|_X\right)/\mathcal{F}$ is a torsion-free sheaf of rank one.

If $c_1(\mathcal{F})\cdot(-K_X) = -3$, then $\det(\mathcal{F}) \cong \omega_X \cong \mathcal{O}_X(-1)$ and we have $\det(\mathcal{G}) = \mathcal{O}_X(-1)$. Since $\mathcal{G}^\vee$ is a subsheaf of $T_Y|_X$, we obtain

$$h^0(X, T_Y|_X \otimes \det(\mathcal{G})) = h^0(X, T_Y(-1)|_X) \geq 1.$$

Since $T_Y(-1)|_X$ is a subsheaf of $T_Y|_X$, we get $H^0(X, T_Y|_X) \neq 0$. Then, by Theorem 3.2.13, it follows $H^0(Y, T_Y) \neq 0$. Nevertheless, it is well-known that there are no global holomorphic vector fields over a cubic threefold (cf. [KS99, Theorem 11.5.2]), we get a contradiction.

If $c_1(\mathcal{F})\cdot(-K_X) = -4$, then $\det(\mathcal{F}) \cong \mathcal{O}_X(-1)\otimes\mathcal{O}_X(-\ell)$ for some line $\ell \subset X$. As a consequence, we have $\det(\mathcal{G}) = \mathcal{O}_X(-1)\otimes\mathcal{O}(\ell)$. Since $\mathcal{G}^\vee$ is a subsheaf of $T_Y|_X$, we get

$$H^0(X, T_Y(-C)|_X) \neq 0,$$

where C is a conic such that $\mathcal{O}_X(C) \cong \mathcal{O}_X(1)\otimes\mathcal{O}_X(-\ell)$. Note that the sheaf $T_Y(-C)|_X$ is a subsheaf of $T_Y|_X$, it follows $H^0(X, T_Y|_X) \neq 0$. Similarly, Theorem 3.2.13 implies $H^0(Y, T_Y) \neq 0$, which is impossible. □

Now we are in the position to prove the main theorem in this section.

3.4.5. Theorem. *Let $Y \subset \mathbb{P}^4$ be a general cubic threefold and $X \in |\mathcal{O}_Y(1)|$ a general smooth linear section. Then the restriction $T_Y|_X$ is stable with respect to $\mathcal{O}_X(1)$.*

Proof. It is enough to prove that $\Omega^1_Y|_X$ is stable with respect to $\mathcal{O}_X(1)$. Note that we have $\mu(\Omega^1_Y|_X) = -2$, so it suffices to prove that the following inequality holds for any proper saturated subsheaf $\mathcal{F}$ of $\Omega^1_Y|_X$.

$$\mu(\mathcal{F}) = \frac{c_1(\mathcal{F}) \cdot -K_X}{\mathrm{rk}(\mathcal{F})} < -2$$

Case 1. Let $\mathcal{F} \subset \Omega^1_Y|_X$ be a saturated subsheaf of rank one. Since $\mathcal{F}$ is a reflexive sheaf of rank one and X is smooth, $\mathcal{F}$ is actually an invertible sheaf. Then the exact sequence

$$0 \to \mathcal{O}_X(-1) \otimes \mathcal{F}^\vee \to \Omega^1_Y|_X \otimes \mathcal{F}^\vee \to \Omega^1_X \otimes \mathcal{F}^\vee \to 0$$

implies that we have either $h^0(X, \mathcal{O}_X(-1) \otimes \mathcal{F}^\vee) \geq 1$ or $h^0(X, \Omega^1_X \otimes \mathcal{F}^\vee) \geq 1$. In the former case, we have

$$\mu(\mathcal{F}) = c_1(\mathcal{F}) \cdot (-K_X) \leq c_1(\mathcal{O}_X(-1)) \cdot (-K_X) = -3 < -2.$$

In the latter case, let $\overline{\mathcal{F}}$ be the saturation of $\mathcal{F}$ in Ω^1_X, then Proposition 3.4.3 implies

$$\mu(\mathcal{F}) \leq \mu(\overline{\mathcal{F}}) = c_1(\overline{\mathcal{F}}) \cdot (-K_X) \leq -3.$$

Case 2. Let $\mathcal{F} \subset \Omega^1_Y|_X$ be a saturated subsheaf of rank two. In this case, by Lemma 3.4.4, we have

$$\mu(\mathcal{F}) = \frac{c_1(\mathcal{F}) \cdot (-K_X)}{2} \leq \frac{-5}{2} < -2.$$

This completes the proof. □

3.5 Smooth surfaces in $\mathbb{P}^3$

In this section, we consider the surfaces in $\mathbb{P}^3$. Let $S \subset \mathbb{P}^3$ be a smooth surface. Then Ω^1_S is stable if $S \cong \mathbb{P}^2$ or S is a cubic surface, and Ω^1_S is semi-stable if S is a quadric surface [Fah89, Théorème]. We have the following result for higher degree surfaces.

3.5.1. Proposition. *Let $S \subset \mathbb{P}^3$ be a smooth surface of degree $d \geq 4$. Then the tangent bundle T_S is stable with respect to $\mathcal{O}_S(1)$.*

Proof. First we calculate the total Chern class of S. The Chern class of $\mathbb{P}^3$ is known : the total class of $T_{\mathbb{P}^3}$ is

$$(1 + H)^4 = 1 + 4H + 6H^2 + 4H^3 + H^4.$$

Then the tangent sequence

$$0 \to T_S \to T_{\mathbb{P}^3}|_S \to \mathcal{O}_S(d) \to 0$$

implies

$$(1 + c_1(S) + c_2(S))(1 + dH|_S) = (1 + 4H|_S + 6(H|_S)^2).$$

It follows that

$$c_1(S) = (4 - d)H|_S \text{ and } c_2(S) = (d^2 - 4d + 6)(H|_S)^2.$$

To prove the stability of Ω^1_S, equivalently, it is enough to prove the stability of T_S. As $K_S \geq 0$, by Calabi-Yau Theorem and Aubin-Yau Theorem, the tangent bundle T_S carries a Kähler-Einstein metric, so the tangent bundle T_S is a polystable vector bundle by Donaldson-Uhlenbeck-Yau Theorem. Thus either $T_S = L_1 \oplus L_2$ with the same slope $\mu(T_S) = \mu(L_i)$, $i = 1, 2$ or T_S is stable.

We assume that $T_S = L_1 \oplus L_2$. Then L_1 and L_2 are both integrable. By Baum-Bott's formula [BB70, Theorem 1], we have

$$0 = c_2(S) - c_1^2(S) + c_1(L_i) \cdot c_1(S).$$

On the other hand, as $\mu(L_i) = \mu(T_S)$, we obtain

$$c_1(L_i) \cdot c_1(S) = (4-d)c_1(L_i) \cdot H|_S = (4-d)\mu(L_i) = \frac{(4-d)^2 d}{2} \geq 0.$$

Nevertheless, as $c_2(S) = d(d^2 - 4d + 6)$, we get

$$c_2(S) - c_1^2(S) = d(d^2 - 4d + 6) - (4-d)^2 d = d(4d - 10) > 0.$$

This is a contradiction. Hence the tangent bundle T_S is stable. □

3.5.2. **Theorem**. *Let $S \subset \mathbb{P}^3$ be a smooth surface of degree $d \geq 3$. Let $C \in |kH|_S|$ be a general smooth curve. If*

$$k > (d-2)^2 d + \frac{d^4 + 1}{2d},$$

then the restriction $T_S|_C$ is stable with respect to $\mathcal{O}_{\mathbb{P}^3}(1)|_S$.

Proof. By [Fah89, Théorème] and Proposition 3.5.1, the tangent bundle T_S is stable. Then the theorem follows directly from [Lan04, Theorem 5.2]. □

Deuxième partie

Geometry of fundamental divisors

Chapitre 4

Introduction to Part II

4.1 Main results

Birational geometry aims to classify algebraic varieties up to birational isomorphism by identifying "nice" elements in each birational class and then classifying such elements. The minimal model program (MMP for short) provides a way to pick up a more special representative in each birational class. To be more precise we want to establish the following conjecture which is the core of MMP.

4.1.1. Conjecture (Minimal model and abundance). *Let X be an irreducible projective variety. Then X is birational to a projective variety Y with terminal singularities such that either*

(1) Y is canonically polarized, or
(2) Y admits a Fano fibration, or
(3) Y admits a Calabi-Yau fibration.

As predicted by this conjecture, Fano varieties, Calabi-Yau varieties and canonically polarized varieties should be the building blocks of algebraic varieties. Since Fano varieties have a natural ample line bundle, namely $-K_X$, there is hope to classify these varieties. For instance, the Fano manifolds with coindex at most three are completely classified (see the book of Iskovskikh and Prokhorov [IP99]). In general, Kollár, Miyaoka and Mori proved in [KMM92] that n-dimensional Fano manifolds form a bounded family. However, this is no longer true in the singular case if we do not give restriction on the singularities. Recently Birkar proved the so-called Borisov-Alexeev-Borisov or BAB conjecture in [Bir16b] : for any real number $\epsilon > 0$, ϵ-log terminal Fano varieties of a given dimension form a bounded family. All these existing works highlight the importance of studying the singularties of divisors in the $\mathbb{Q}$-anticanonical systems. In general, we have the following related problem.

4.1.2. Question (Good Divisor Problem). *Construct a regular ladder for a nef and big Cartier divisor H on a normal projective variety X. This means to find a "good" member S in $|H|$ and then repeat for the pair $(S, H|_S)$.*

A "good" member here means an irreducible reduced normal projective variety having singularities close to its ambient spaces. We are mainly interested in the case where X is a Fano variety with mild singularities and H is the fundamental divisor of X (see §5.1 for the definitions). In this case, there are various partial answers in the literature. Existence of good divisors for the fundamental divisor H of a Fano variety X was known for : Fano manifolds with coindex at most three ([Fuj77b, Mel99]), klt log weak $\mathbb{Q}$-Fano varieties (X, Δ) with coindex < 4 ([Amb99]), canonical Gorenstein Fano varieties with coindex at most three ([Ale91, Mel99]), canonical Gorenstein Fano fourfolds ([Kaw00]), smooth Fano fourfolds ([Heu15]) etc. A general member S has the same singularities as the ambient space in all the cases mentioned above except in the last case where S has only terminal singularities. In fact, one can not expect the existence of smooth member in the last case as showed by the following example due to Höring and Voisin.

4.1.3. Example [HV11, Example 2.12]. Let S be the blow-up of $\mathbb{P}^2$ at eight points in general position. Then S is a del Pezzo surface whose anticanonical system has exactly one base point which is denoted by P. Set $X = S \times S$ and $S_i := p_i^{-1}(P)$ where p_i is the projection of the i-th factor. Then X is a smooth Fano fourfold and

$$\mathrm{Bs}\,|-K_X| = S_1 \cup S_2.$$

Let $Y \in |-K_X|$ be a general member, then $\mathrm{Bs}\,|-K_X| \subset Y$, so the surfaces S_1 and S_2 are Weil divisors in Y. If they are $\mathbb{Q}$-Cartier, their intersection $S_1 \cap S_2$ would have dimension at least one, but we have $S_1 \cap S_2 = (P, P)$. Thus Y is not $\mathbb{Q}$-factorial.

If (X, Δ) is a klt log weak $\mathbb{Q}$-Fano variety and $-(K_X + \Delta) \sim_{\mathbb{Q}} rH$ for some nef and big Cartier divisor H and $r > 0$, then the good divisor problem can split into two pieces.

4.1.4. Problem.

(A) (Effective nonvanishing) The invertible sheaf $\mathcal{O}_X(H)$ has a nonzero global section.
(B) For a general element $S \in |H|$, the pair $(X, S + \Delta)$ has at worst plt singularties.

We remark that (B) implies that the pair $(S, \Delta|_S)$ has at worst klt singularities by adjunction formula. On the other hand, in [Kaw98], Kawamata shows that (B) is implied by subadjunction formula ([Kaw98, FG12]) and the following more general version of effective nonvanishing theorem in lower dimension.

4.1.5. Conjecture [Kaw00, Conjecture 2.1]. *Let (X, Δ) be a klt pair and let D be a nef Cartier divisor such that $D - (K_X + \Delta)$ is nef and big. Then $H^0(X, D) \neq 0$.*

It is easy to see that (A) is just a special case of Conjecture 4.1.5. Thus effective nonvanishing implies the existence of good divisors (cf. Theorem 5.4.4). The effective nonvanishing conjecture is confirmed only for curves and surfaces ([Kaw00]) and there are some partial results in dimension three. Inspired by the work of Kawamata on Gorenstein canonical Fano fourfolds, Floris solved Question (B) for Gorenstein canonical Fano varieties with coindex four in [Flo13]. Moreover, for Fano manifolds with coindex four, using Riemann-Roch formula, Floris gave an affirmative answer to Question (A) under the assumption that $\dim(X) \neq 6$ and 7. In the following result, we complete Floris's result.

4.1.6. Theorem (= Theorem 5.3.6). *Let X be a Fano manifold of dimension $n \geq 4$ and index $n - 3$. Let H be the fundamental divisor. Then $h^0(X, H) \geq n - 2$.*

As an immediately application, combining this result with [HV11, Theorem 1.4] will yield the following theorem.

4.1.7. Corollary (= Corollary 5.3.7). *Let X be a n-dimensional Fano manifold with index $n - 3$. Then the group $H^{2n-2}(X, \mathbb{Z})$ is generated over $\mathbb{Z}$ by classes of curves.*

Our proof of Theorem 4.1.6 relies on an inequality of Bogomolov type for Fano manifolds with $b_2 = 1$ that may be of independent interest. Recall that if (X, H) is a polarized projective manifold such that T_X is H-semistable, then the famous Bogomolov inequality gives a lower bound for the second Chern class of X

$$c_2(X) \cdot H^{n-2} \geq \frac{n-1}{2n} c_1(X)^2 \cdot H^{n-2}. \tag{4.1}$$

Following an idea developed by Hwang in [Hwa98], we prove the following weaker version of inequality (4.1) without assuming the (semi-)stability of T_X. We will focus on the case $n \geq 7$ and $r_X \geq 2$ since T_X is semistable if $r_X = 1$ or $n \leq 6$ (cf. [Rei77, Hwa98]).

4.1.8. Theorem (= Theorem 5.2.2). *Let X be a n-dimensional Fano manifolds with $b_2(X) = 1$ such that $n \geq 7$. Let H be the fundamental divisor of X and denote by r_X the index of X.*

(1) If $r_X = 2$, then

$$c_2(X) \cdot H^{n-2} \geq \frac{11n - 16}{6n - 6} H^n.$$

(2) *If $3 \leq r_X \leq n$, then*

$$c_2(X) \cdot H^{n-2} \geq \frac{r_X(r_X - 1)}{2} H^n.$$

It is easy to check that the inequalities given in Theorem 5.2.2 are strictly weaker than the expected inequality (4.1) except $r_X = n$, so the statement is only interesting for $2 \leq r_X \leq (n+1)/2$. Moreover, recall that $r_X > n$ if and only if $X \cong \mathbb{P}^n$ and $r_X = n+1$, and the stability of $T_{\mathbb{P}^n}$ is well-known.

As a second application of the existence of good ladder, we consider the anticanonical geometry of weak Fano varieties. Given a canonical weak $\mathbb{Q}$-Fano projective variety X such that $-K_X \sim_{\mathbb{Q}} rH$ for some nef and big Cartier divisor H and $r > 0$, the rational map $\Phi_{|mH|}$ defined by the linear system $|mH|$ plays a key role in the study of the explicit birational geometry of X. In particular, if X is Gorenstein, then we can set $r = 1$ and $H = -K_X$, and the map $\Phi_{|mH|}$ is just the m-th pluri-anticanonical map Φ_{-m} corresponding to $|-mK_X|$. Since H is big, $\Phi_{|mH|}$ is a birational map onto its image when m is sufficiently large. Therefore it is interesting to find a number m_n, independent of X, which guarantees the birationality of $\Phi_{|mH|}$. The existence of such a constant m_n is due to the boundedness result proved by Birkar in arbitrary dimension (cf. [Bir16a, Theorem 1.2]) (see also [KMMT00, Theorem 1.2]) for threefolds). On the other hand, by effective Basepoint Free Theorem [Kol93, Theorem 1.1], the linear system $|mH|$ is basepoint free for $m = 2(n+2)!(n+1)$ where $n = \dim(X)$. This leads us to ask the following two natural questions.

4.1.9. Question. *Let X be a weak Fano variety with at most canonical Gorenstein singularities.*

(1) *Find the optimal constant $f(n)$ depending only on $\dim(X) = n$ such that the linear system $|-mK_X|$ is basepoint free for all $m \geq f(n)$.*
(2) *Find the optimal constant $b(n)$ depending only on $\dim(X) = n$ such that the rational map Φ_{-m} is a birational map for all $m \geq b(n)$.*

Question (1) is closely related to Fujita's basepoint freeness conjecture, and Question (2) has attracted a lot of interest over the past few decades [And87, Che11, CJ16, Fuk91], etc. On the other hand, we remark that the constants $f(n)$ and $b(n)$ in Question 4.1.9 are invariant if we replace weak Fano varities by Fano varieties (cf. Proposition 5.3.1). In dimension two, we have the following known result.

4.1.10. Theorem [And87, Rei88]. *Let S be a projective surface with at most canonical singularities. If the anticanonical divisor $-K_S$ is nef and big, then*

(1) *the linear system $|-mK_S|$ is basepoint free for all $m \geq 2$;*
(2) *the morphism Φ_{-m} is birational for all $m \geq 3$.*

The example of a degree one del Pezzo surface shows that the bounds in the theorem are optimal, so $f(2) = 2$ and $b(2) = 3$. Borrowing some tools from [OP95] and applying [Rei88, Corollary 2] and a recent result of Jiang [Jia16, Theorem 1.7], we generalize this theorem to weak Fano varieties of dimension at most four.

4.1.11. Theorem (= Theorem 5.3.2). *Let X be a n-dimensional weak Fano variety with at worst Gorenstein canonical singularities such that $-K_X = (n-2)H$ for some nef and big Cartier divisor H. Then*

(1) *the linear system $|mH|$ is basepoint free for $m \geq 2$;*
(2) *the morphism $\Phi_{|mH|}$ is birational for $m \geq 3$.*

The lower bounds given in the theorem are both optimal. To see this, we consider the following two examples: $X = S_1 \times \mathbb{P}^1$ where S_1 is a del Pezzo surface of degree one, and a general smooth hypersurface of degree six in the weighted projective space $\mathbb{P}(1, \cdots, 1, 3)$. In particular, if $n = 3$, we get $f(3) = 2$ and $b(3) = 3$. One can also derive the basepoint freeness of $|-2K_X|$ by the classification of Gorenstein canonical Fano threefolds with non-empty $\mathrm{Bs}\,|-K_X|$ given in [JR06, Theorem 1.1]. Moreover, the Fano threefolds with canonical Gorenstein singularities such that $|-K_X|$ is basepoint free, but not very ample, are called *hyperelliptic*, and they have been classified by Przyjalkowski-Cheltsov-Shramov in [PCS05, Theorem 1.5].

4.1.12. Theorem. *Let X be a weak Fano fourfold with at worst Gorenstein canonical singularities. Then*

(1) the linear system $|-mK_X|$ is basepoint free for $m \geq 7$;
(2) the rational map Φ_{-m} is birational for $m \geq 5$.

A general hypersurfaces of degree ten in the weighted projective space $\mathbb{P}(1,1,1,1,2,5)$ guarantees that the estimate given by Theorem 4.1.12 (ii) is best, i.e., $b(4) = 5$. We remark that the variety X is actually a smooth Fano fourfold with index one, and the base locus of $|-K_X|$ is zero dimensional and of degree ten, given by an equation of the type $x_5^2 + x_4^5 = 0$ in $\mathbb{P}(2,5)$ (see [Heu16, Theorem 6.22]). In higher dimension, if X is a Fano manifold with coindex four, then we can construct a ladder $X = X_0 \supset X_1 \supset \cdots \supset X_{n-3}$ such that $X_{i+1} \in |H|_{X_i}|$ et X_i has at worst canonical singularities. In particular, the same argument in dimension four gives a similar result for Fano manifolds of coindex four.

4.1.13. Theorem (= Theorem 5.3.8). *Let X be a n-dimensional Fano manifold with index $n-3$ and let H be the fundamental divisor. Then*

(1) the linear system $|mH|$ is basepoint free for $m \geq 7$;
(2) the linear system $|mH|$ gives a birational map for $m \geq 5$.

As in dimension four, the estimate in Theorem 4.1.13 (ii) is optimal as shown by a general hypersurface of degree ten in the weighted projective space $\mathbb{P}(1,\cdots,1,2,5)$. The statement Theorem 4.1.13 (i) is weaker than the estimate expected by Fujita's conjecture, which asserts basepoint freeness for $m \geq 4$.

The effective birationality and effective basepoint freeness discussed above measure the global positivity of the fundamental divisors, or equivalently the anticanonical divisors. Motivated in part by the study of linear series in connection with Fujita's conjecture, Demailly introduced the *Seshadri number* of nef line bundles in [Dem92] to quantify how much of the positivity of an ample line bundle can be localized at a given point of a variety.

4.1.14. Definition. *Let X be a projective normal variety and let L be a nef line bundle on X. To every smooth point $x \in X$, we attach the number*

$$\varepsilon(X,L;x) :\, = \inf_{x\in C} \frac{L\cdot C}{\nu(C,x)},$$

which is called the Seshadri constant of L at x. Here the infimum is taken over all irreducible curves C passing through x and $\nu(C,x)$ is the multiplicity of C at x.

The Seshadri constant is a lower-continuous function over X in the topology where closed sets are countable unions of Zariski closed sets. Moreover, there is a number, which we denote by $\varepsilon(X,L;1)$, such that it is the maximal value of Seshadri constant on X. This maximum is attained for a very general point $x \in X$. Unfortunately, it is very difficult to compute or estimate $\varepsilon(X,L;1)$ in general. For the upper bound, an elementary observation shows that $\varepsilon(X,L;1) \leq \sqrt[n]{L^n}$. There have been many works in trying to give lower bound for this invariant. Ein and Lazarsfeld shows that if X is surface and L is ample, then $\varepsilon(X,L;1) \geq 1$ (see [EL93, Theorem]). In higher dimension, Ein, Küchle and Lazarsfeld proved that $\varepsilon(X,L;1) \geq 1/\dim(X)$ for any ample line bundle L over X (see [EKL95, Theorem 1]) and this bound has been improved by Nakamaye in [Nak05]. In general, we have the following conjecture.

4.1.15. Conjecture [Laz04, Conjecture 5.2.4]. *Let L be an ample line bundle over a projective manifold X. Then $\varepsilon(X,L;1) \geq 1$.*

If X is a n-dimensional Fano manifold, Bauer and Szemberg showed that we have $\varepsilon(X,-K_X;1) \leq n+1$ with equality if and only if X is isomorphic to some projective space and this is generalized by Liu and Zhuang to $\mathbb{Q}$-Fano varieties in [LZ17]. On the other hand, as predicted by Conjecture 4.1.15, we shall have $\varepsilon(X,-K_X;1) \geq 1$. Thus it is natural to ask when the equality $\varepsilon(X,-K_X;1) = 1$ holds. In dimension two, the Seshadri constants of anticanonical divisors of del Pezzo surfaces are computed by Broustet in [Bro06]. As a corollary, we have following result.

4.1.16. Theorem [Bro06, Théorème 1.3]. *Let S be a smooth del Pezzo surface. Then $\varepsilon(S, -K_S; 1) = 1$ if and only if S is a del Pezzo surface of degree one, or equivalently $r_S = 1$ and $|-K_S|$ is not basepoint free.*

In higher dimension, if H is the fundamental divisor of X, then we have $\varepsilon(X, H; 1) \geq 1$ in the case $r_X \geq n-2$ (cf. [Bro09]). In the following theorem, we give a slight generalization of this result as an application of the existence of good ladder.

4.1.17. Theorem (= Theorem 6.3.1). *Let X be a n-dimensional Fano manifold such that $r_X \geq n-3$, then $\varepsilon(X, -K_X; 1) \geq r_X$.*

As a consequence, if $r_X \geq n-3$, then the condition $\varepsilon(X, -K_X; 1) = 1$ implies $r_X = 1$. Moreover, in dimension three, we can establish a similar result of Broustet's theorem.

4.1.18. Theorem (= Corollary 6.4.14). *Let X be a smooth Fano threefold very general in its deformation family. Then $\varepsilon(X, -K_X; 1) = 1$ if and only if $r_X = 1$ and $|-K_X|$ has a basepoint. More precisely, X is isomorphic to one of the following.*

(1) $\mathbb{P}^1 \times S_1$, where S_1 is a del Pezzo surface of degree one.
(2) The blow-up of V_1 along an elliptic curve which is an intersection of two divisors from $|-\frac{1}{2}K_{V_1}|$, where V_1 is a del Pezzo threefold of degree one.

Theorem 4.1.18 is actually a direct consequence of the explicit computation of the Seshadri constants of the anticanonical divisors of Fano manifolds with coindex at most three. If X is a very general smooth Fano threefold with Picard number one, the Seshadri constant $\varepsilon(X, -K_X; 1)$ is calculated by Ito via toric degenerations (cf. [Ito14, Theorem 1.8]). In higher dimension, we define ℓ_X to be the *minimal anticanonical degree* of a minimal covering family of rational curves on X, so that $\ell_X \in \{2, \cdots, n+1\}$. In particular, we have $\varepsilon(X, -K_X; 1) \leq \ell_X$ by the definition.

4.1.19. Theorem (= Theorem 6.3.3). *Let X be a n-dimensional Fano manifold with index r_X. Assume moreover that $r_X \geq \max\{2, n-2\}$. Then passing through every point $x \in X$, there is a rational curve C such that $-K_X \cdot C = r_X$. In particular, we have $\varepsilon(X, -K_X; 1) = \ell_X = r_X$.*

Now it remains to consider the Fano threefolds X such that $\rho(X) \geq 2$ and $r_X = 1$. Such manifolds are classified by Mori and Mukai in [MM81] and [MM03]. In the following theorem, we follow the numbering in [MM81] and [MM03] (see also Appendix B).

4.1.20. Theorem (= Theorem 6.5.16). *Let X be a smooth Fano threefold with $\rho(X) \geq 2$.*

(1) $\varepsilon(X, -K_X; 1) = 1$ if and only if X carries a del Pezzo fibration of degree one ($n^o\, 1$ in Table 2 and $n^o\, 8$ in Table 5).
(2) $\varepsilon(X, -K_X; 1) = 4/3$ if and only if X carries a del Pezzo fibration of degree two ($n^o\, 2, 3$ in Table 2, and $n^o\, 7$ in Table 5).
(3) $\varepsilon(X, -K_X; 1) = 3/2$ if and only if X carries a del Pezzo fibration of degree three ($n^o\, 4, 5$ in Table 2, $n^o\, 2$ in Table 3 and $n^o\, 6$ in Table 5).
(4) $\varepsilon(X, -K_X; 1) = 3$ if X is isomorphic to the blow-up of $\mathbb{P}^3$ along a smooth plane curve C of degree at most three ($n^o\, 28, 30, 33$ in Table 2).
(5) $\varepsilon(X, -K_X; 1) = 2$ otherwise.

The key ingredient of the proof of Theorem 4.1.20 is the existence of splitting and free splitting of anticanonical divisors (cf. [MM81, Corollary 11]). On the other hand, inspired by our results above, we propose the following stronger conjecture. For more evidence in the complete intersections case, we refer to [IM14] and Proposition 6.3.5.

4.1.21. Conjecture. *Let X be a smooth Fano threefold with index $r_X = 1$. Then $\varepsilon(X, -K_X; 1) = 1$ if and only if $|-K_X|$ is not basepoint free.*

Now we consider the good divisor problem in a larger category. This is inspired both by the studying of non-projective Moishezon manifolds and the rigidity of Fano manifolds (cf. §7.1). Kähler manifolds and Moishezon manifolds are the natural generalizations of projective manifolds. However, in general, a Kähler manifold may not have any submanifold, so we will consider only the Moishezon manifolds. As the positivity of anticanonical divisors can be destroyed by blow-up, we will restrict ourselves to Moishezon manifolds with Picard number one. More precisely, we focus on the following analytic analogue of good divisor problem.

4.1.22. Question. *Let X be a Moishezon manifold such that $\mathrm{Pic}(X) = \mathbb{Z}L$ for some big line bundle H. Assume that $h^0(X, L) > 0$ and $-K_X = rL$ for some $r > 0$. Does there exist a smooth element $Y \in |L|$?*

In the case $\dim(X) = 3$, there is an affirmative answer in the the following cases : $r_X \geq 3$ ([Nak87, Nak88, Kol91b]), $r_X = 2$ and $h^0(X, H) \geq 6$ [Kol91b]. In fact, if $r_X \geq 3$, Kollár proved that X is even projective. On the other hand, if $r_X = 2$, we have many non-projective examples (see § 7.2.3). Our first result is as follows.

4.1.23. Theorem (= Theorem 7.3.11). *Let X be a smooth Moishezon threefold such that $\mathrm{Pic}(X) = \mathbb{Z}L$ for some big line bundle L and $-K_X = 2L$. Assume moreover that $h^0(X, L) \geq 3$. Let D_1, D_2 be two general members of $|H|$. Then the complete intersection $C\colon = D_1 \cap D_2$ contains at least one mobile component A. Moreover, if A intersects the union of other components of $D_1 \cap D_2$ in at least two points, then there exists a smooth element D in $|L|$.*

Combining this result with [Kol91b, Theorem 5.3.12], we get the following result.

4.1.24. Corollary (= Corollary 7.3.13). *Let X be a smooth Moishezon threefold such that $\mathrm{Pic}(X) = \mathbb{Z}L$ for some big line bundle H and $-K_X = 2L$. Assume moreover that $h^0(X, L) \geq 5$. Then there exists a smooth element D in $|L|$.*

Kollár proved in [Kol91b, Theorem 5.3.12] that $|L|$ is actually basepoint free if $h^0(X, L) \geq 6$ and our result follows immediately from Bertini's theorem in this case. However, if $h^0(X, L) = 5$, there exists examples showing that $\mathrm{Bs}\,|L|$ is not empty (see § 7.2.3). Inspired by the known examples, we consider the classification of certain non-projective Moishezon manifolds. Using the recent achievement in the minimal model program, it is possible to show that every Moishezon variety which is not projective contains a rational curve.

4.1.25. Theorem [BCHM10, Corollary 1.4.6]. *Assume X is a Moishezon manifold which is not projective. Then X contains a rational curve.*

This theorem was proved by Peternell in dimension three [Pet86c] and it was completely settled in [BCHM10]. Roughly speaking, the non-projectivity of a Moishezon manifold may be attributed to the existence of bad (rational) curves (see also [Pet86c]). On the other hand, since every Moishezon manifold becomes projective after a finite number of blow-ups with smooth centres [GPR94] and a compact complex manifold X is projective if and only if the blow-up of X at a point x is projective, it follows that a non-projective Moishezon manifold X is of dimension at least three. In the rest of this part, we consider the non-projective Moishezon manifolds with Picard number one which become projective after blowing-up along a smooth curve. This kind of Moishezon manifolds may be viewed as the simplest non-projective Moishezon manifolds. To be more precise, let X be a Moishezon manifold of dimension n with $\mathrm{Pic}(X) = \mathbb{Z}L$ for some big line bundle L, and let C be a smooth curve on X. Assume that the blow-up $\pi\colon \widehat{X} \to X$ along C is projective. Let us denote the exceptional divisor of π by E. Since $\widehat{X}$ is obtained by blowing-up, the anticanonical divisor $K_{\widehat{X}}$ is not nef and we have an extremal contraction $f\colon \widehat{X} \to Y$ by Théorème 0.5.3. Using results about the structure of extremal contractions of projective manifolds, we get the following theorem.

4.1.26. Theorem (= Theorem 7.5.8). *Notation and assumptions as above.*

(1) If L is nef or K_X is big, then $\dim(X) = 3$ *and the induced map* $\phi \colon X \dashrightarrow Y$ *is a birational morphism such that* ϕ *contracts* C *to the only ordinary node point* P *of* Y*.*
(2) If $K_X = 0$ *and* L *is not nef, then* $\dim(X) = 3$ *and* X *is obtained by a flop from a projective manifold of dimension* 3 *with trivial canonical bundle and Picard number one.*
(3) If $-K_X$ *is big but not nef, and if* f *is not birational, then* f *induces a conic bundle over a Fano manifold of Picard number one.*
(4) If $-K_X$ *is big but not nef, and if* f *is birational, then* f *is the blow-up of a projective Fano manifold of dimension* n *along a submanifold with codimension two.*

Finally we turn our attention to the case where $-K_X$ is big but not nef and f is birational. Working under an additional assumption that $\pi_* f^*$ defines an isomorphism between the Picard groups of X and Y, we prove :

4.1.27. Theorem (= Corollary 7.5.13). *Assume that* $-K_X$ *is big but not nef and* f *is birational. If* E *is not the exceptional divisor of* f *and* $\pi_* f^*$ *defines an isomorphism between the Picard groups of* X *and* Y*, then* $n = 3$ *and* X *is isomorphic to* $B_{(3,6-m)}(\mathbb{P}^3)$ $(m \leq 2)$ *or* $B_{(2,6-m)}(Q^3)$ $(m \leq 3)$ [1].

4.2 Organization

This part is organized as follows. Chapter 5 is devoted to study the good divisor problem and the anticanonical geometry of Fano varieties of large index. Chapter 6 is devoted to study the local positivity of the anticanonical divisors of Fano manifolds with large index and we calculate explicitly the Seshadri constant of the anticanonical divisors in this case. Chapter 7 is devoted to study the good divisor problem over Moishezon manifolds with Picard number one and we discuss its links with rigidity problem. In Appendix A, we recall the classification of Fano manifolds with Picard number at least two and large index given by Wiśniewski in [Wiś91b, Wiś94]. In particular, we compute the dimension $h^0(X, H)$ for the fundamental divisor H of X. In Appendix B, we recall the classification of smooth Fano threefolds with Picard number at least three given in [MM81, MM03]. In particular, we give the free splitting type of anticanonical divisors in each case. More precisely, the organization of each chapter is as follows.

Chapter 5 : In Section 5.1, we recall the definition of log pairs, slope stability of vector bundles and several known results needed in the later. In Section 5.2, we study the second Chern classes of Fano manifolds with Picard number one. In particular, we will prove Theorem 4.1.8. In Section 5.3, we investigate the effective birationality and effective basepoint freeness in various cases as an application of the existence of goood ladder. In particular, Theorem 4.1.11, Theorem 4.1.12 and Theorem 4.1.13 are proved. In Section 5.4, we show that the effective nonvanishing conjecture 4.1.5 reduces the problem in higher dimension to lower dimension.

Chapter 6 : In Section 6.1, we recall some basic materials about pencil of surfaces and the classification of extremal contractions in dimension three. In Section 6.2, we investigate the rational curves of low degree on Fano manifolds, in particular, we establish the existence of covering family of lines on del Pezzo manifolds and Mukai manifolds. In Section 6.3, we will prove Theorem 4.1.17 and Theorem 4.1.19. In Section 6.4, we study the relation between the Seshadri constant of the anticanonical divisors of Fano threefolds and del Pezzo fibrations of small degree. In particular, we prove Theorem 4.1.18. In Section 6.5, we work out all the Fano threefolds admitting a del Pezzo fibration of small degree. In particular, Theorem 4.1.20 will be proved. In Appendix I, we give a complete proof of a result due to Mori-Mukai which is used in Section 6.5.

Chapter 7 : In Section 7.1, we review the known results on the rigidity problems of Fano manifolds under various additional conditions. In Section 7.2, we present some known results on Moishezon manifolds, in particular, the vanishing theorems on Moishezon manifolds. We also collect some examples of non-projective Moishezon manifolds with Picard number one. In Section 7.3, we consider the good divisor

1. see Section (7.2.3.3) for definition.

problem on Moishezon manifolds. In particular, we prove Theorem 4.1.23 and Corollary 4.1.24. In Section 7.4, we discuss the global deformation of Fano manifolds with Picard number one. In Section 7.5, we try to classify certain non-projective Moishezon manifolds and Theorem 4.1.26 and Theorem 4.1.27 will be proved.

4.3 Convention and notations

Throughout, we work over the field $\mathbb{C}$ of complex numbers unless otherwise stated. A complex model space $(A, \mathcal{O}_A)$ is an analytic set A in a domain $U \subset \mathbb{C}^n$ and there is a locally finite analytic sheaf of ideals in the sheaf $\mathcal{O}_U$ such that $\mathcal{I} = \mathcal{O}_U$ over $U \setminus A$ and $\mathcal{O}_A = (\mathcal{O}_U/\mathcal{I})|_A$. A complex space is a $\mathbb{C}$-ringed space $(X, \mathcal{O}_X)$ such that X is a Hausdorff space and every point of X has an open neighborhood U with $(U, \mathcal{O}_U)$ isomorphic to some complex model space $(A, \mathcal{O}_A)$. Moreover, all the complex spaces are assumed to be compact. We denote by X_{sing} the set of singular points of X. For any coherent analytic sheaf $\mathcal{F}$ over X, we set

$$\chi(X, \mathcal{F}) = \sum_{i=0}^{\dim(X)} (-1)^i h^i(X, \mathcal{F}).$$

If Y_1 and Y_1 are two closed subspaces of X, the intersection $Y_1 \cap Y_2$ is the subspace defined the ideal $\mathcal{I}_{Y_1} + \mathcal{I}_{Y_1}$, where $\mathcal{I}_{Y_1}$ and $\mathcal{I}_{Y_2}$ are the defining ideal sheaves of Y_1 and Y_2, respectively. For any ring A, we denote by A_{red} the quotient of A by its ideal of nilpotent elements. For any complex space $(X, \mathcal{O}_X)$, the associated reduced complex space of X is the complex space $(X, \mathcal{O}_{X,\text{red}})$ such that $\mathcal{O}_{X,\text{red}}$ is the sheaf associated to the presheaf $U \mapsto \mathcal{O}_X(U)_{\text{red}}$.

Let $(Z, \mathcal{O}_Z)$ be a complex space of pure dimension n such that Z_{red} has only finitely many irreducible components. Then n-cycle associated with the complex space Z is denoted by $\sum_{i\in I} n_i Z_{i,\text{red}}$, where n_i is the multiplicity of $Z_{i,\text{red}}$ in Z.

Chapitre 5

Second Chern class, effective nonvanishing and anticanonical geometry

This chapter is devoted to study the positivity of the second Chern classes of Fano manifolds with Picard number one. As an application, we prove an effective nonvanishing theorem for Fano manifolds with coindex four, which implies the existence of good ladder. Moreover, we consider also the effective birationality and effective basepoint freeness of fundamental divisors of Fano varieties in some special cases. Most results in this chapter are included in the paper [Liu18].

5.1 Preliminaries and notations

Throughout this chapter, all the projective varieties are assumed to be Cohen-Macaulay. Let X be a n-dimensional projective normal variety and let X_0 be the regular part of X with inclusion map $i\colon X_0 \to X$. Let $\omega_{X_0} = \Omega^n_{X_0}$ be the sheaf of regular n-forms over X_0. The *canonical divisor* K_X is a Weil divisor on X such that

$$\mathcal{O}_X(K_X)|_{X_0} \cong \omega_{X_0}.$$

A normal projective variety X is said to be $\mathbb{Q}$-*Gorenstein*, if some multiple mK_X is a Cartier divisor. If $m=1$, then the projective variety X is called *Gorenstein*. Let $\mu\colon X' \dashrightarrow X$ be a birational map between normal projective varieties. If $\Delta \subset X'$ is a $\mathbb{Q}$-Weil divisor, we denote by $\mu_*(\Delta)$ its strict transform. A *log-pair* is a tuple (X,Δ), where X is a normal projective variety and $\Delta = \sum d_i \Delta_i$ is a $\mathbb{Q}$-divisor on X with $0 \leq d_i \leq 1$ for all i such that $-(K_X+\Delta)$ is a $\mathbb{Q}$-Cartier divisor. If $\Delta = 0$, we will abbreviate the log-pair $(X,0)$ as X. For the terminologies of singularities of pairs, we refer to § 0.4.

A *weak* $\mathbb{Q}$-*Fano variety* X is a n-dimensional $\mathbb{Q}$-Gorenstein projective variety with nef and big anti-canonical divisor $-K_X$. A weak $\mathbb{Q}$-Fano variety X is called $\mathbb{Q}$-*Fano*, if the anti-canonical divisor $-K_X$ is ample. The *index* of a Fano variety X is defined as

$$r_X = \sup\{t \in \mathbb{Q} \mid -K_X \sim_{\mathbb{Q}} tH, H \text{ is ample and Cartier}\}.$$

If X has at worst log terminal singularities, then the Picard group $\mathrm{Pic}(X)$ is finitely generated and torsion free, so the Cartier divisor H such that $-K_X \sim_{\mathbb{Q}} r_X H$ is determined up to linear equivalence and we call it the *fundamental divisor* of X.

Let $\mathcal{E}$ be a torsion free coherent sheaf over a projective manifold X. The *discriminant* of $\mathcal{E}$ by definition is the characteristic class

$$\Delta(\mathcal{E})\colon = 2\mathrm{rk}(\mathcal{E})c_2(\mathcal{E}) - (\mathrm{rk}(\mathcal{E})-1)c_1^2(\mathcal{E}).$$

For the definition of stability of torsion free coherent sheaves, we refer to § 0.2. For a torsion free coherent sheaf $\mathcal{E}$ over a polarized projective manifold (X,H), there exists a unique filtration, the so called *Harder-Narasimhan filtration*, by coherent subsheaves

$$0 = \mathcal{E}_0 \subsetneq \mathcal{E}_1 \subsetneq \cdots \subsetneq \mathcal{E}_{s-1} \subsetneq \mathcal{E}_s = \mathcal{E}$$

such that all the factors $G_i = \mathcal{E}_i/\mathcal{E}_{i-1}$ for $i = 1, \cdots, s$ are H-semistable sheaves. The sheaf $\mathcal{E}_1$ is called the *maximal H-destabilising subsheaf* of $\mathcal{E}$. We furthermore introduce the following numbers attached to $\mathcal{E}$

$$\mu_H^{\max}(\mathcal{E}) = \mu_H(\mathcal{E}_1) \quad \text{and} \quad \mu_H^{\min}(\mathcal{E}) = \mu_H(\mathcal{E}/\mathcal{E}_{s-1}).$$

The following theorem reduces the study of varieties with only canonical singularities to the study of varieties with only $\mathbb{Q}$-factorial terminal singularities.

5.1.1. Theorem [BCHM10, Corollary 1.4.4]. *Let X be a normal projective variety with only canonical singularities. Then there is a birational morphism $\mu\colon Y \to X$, where Y has only $\mathbb{Q}$-factorial terminal singularities such that $K_Y = \mu^* K_X$. Such a variety Y is called a terminal modification of X.*

Using this existence theorem, it is easy to yield the following result from the terminal setting.

5.1.2. Theorem [Rei88, Corollary 2][Jia16, Theorem 1.7]. *Let X be a n-dimensional normal projective variety with $K_X \sim 0$. Assume that X has at worst canonical singularities and L is a nef and big Cartier divisor on X. Then*

(1) the linear system $|mL|$ gives a birational map for $m \geq 3$ if $n = 2$;
(2) the linear system $|mL|$ gives a birational map for $m \geq 5$ if $n = 3$.

We recall an easy lemma which is nevertheless the key ingredient of our inductive approach to generalize Theorem 4.1.10 to higher dimension. In [Ogu91], it was proved for projective manifolds, but the proof still works for normal projective varieties.

5.1.3. Lemma [Ogu91, Lemma 1.3]. *Let X be a normal projective variety. Consider an effective Cartier divisor E and an irreducible reduced Cartier divisor F such that $\dim|F| \geq 1$. Suppose that the restriction $|E+F|_D$ gives a birational map for a general element D in $|F|$. Then $|E+F|$ gives a birational map on X.*

5.2 Second Chern classes of Fano manifolds

As a starting point, the following result relates the unstability of T_X to certain special foliation on X, which gives also some restrictions of the projective geometry of certain special subvarieties in $\mathbb{P}(T_X)$.

5.2.1. Proposition [Hwa98, Proposition 1 and 2]. *Let X be a n-dimensional Fano manifold such that $b_2(X) = 1$. Let H be the ample generator of $\mathrm{Pic}(X)$, and $\mathcal{V}$ a minimal covering family of rational curves over X. If T_X is not H-semistable, then the maximal H-destabilizing subsheaf $\mathcal{F}$ of T_X defines a foliation on X, and general curves in $\mathcal{V}$ are not tangent to $\mathcal{F}$.*

As an application of Proposition 5.2.1, we prove the following theorem by using the strenghtening of Bogomolov's inequality due to Langer [Lan04, Theorem 5.1].

5.2.2. Theorem. *Let X be a n-dimensional Fano manifolds with $b_2(X) = 1$ such that $n \geq 7$. Let H be the fundamental divisor of X and let r_X be the index of X.*

(1) If $r_X = 2$, then

$$c_2(X) \cdot H^{n-2} \geq \frac{11n-16}{6n-6} H^n.$$

(2) If $3 \leq r_X \leq n$, then

$$c_2(X) \cdot H^{n-2} \geq \frac{r_X(r_X-1)}{2} H^n.$$

Proof. Without loss of generality, we may assume that T_X is not semistable, in particular, we will assume that $r_X \leq n-2$ (cf. [Hwa01, PW95]). Let $\mathcal{F}$ be the maximal H-destabilizing subsheaf of T_X. Then $\mathcal{F}$ defines a Fano foliation on X. We denote by $r_{\mathcal{F}}$ (>0) the index of $\mathcal{F}$, i.e., $-K_{\mathcal{F}} \sim r_{\mathcal{F}} H$. Fix a minimal covering family $\mathcal{V}$ of rational curves on X and write

$$T_X|_C = \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus p} \oplus \mathcal{O}_{\mathbb{P}^1}^{\oplus(n-p-1)}$$

for a general point $[C] \in \mathcal{V}$. Since $T_X/\mathcal{F}$ is torsion-free, we may assume that the sheaves $\mathcal{F}$ is a subbundle of T_X along C [Kol96, II, Proposition 3.7]. Then the restriction of $\mathcal{F}$ over C is of the following form

$$\mathcal{F}|_C = \mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(a_r), \ a_1 \geq \cdots \geq a_r.$$

Since $\mathcal{F} \subset T_X$, it follows $a_1 \leq 2$, $a_i \leq 1$ for $2 \leq i \leq p+1$, and $a_j \leq 0$ for $p+2 \leq j \leq r$. However, note that C is not tangent to $\mathcal{F}$ (cf. Proposition 5.2.1), we have actually $a_1 \leq 1$. As a consequence, we obtain $r_{\mathcal{F}} < r_X$ and there exists some $1 \leq d \leq r$ such that

$$a_1 = \cdots = a_d = 1 > a_{d+1} \geq \cdots \geq a_r.$$

On the other hand, if $\mathrm{rk}(\mathcal{F}) = r = d$, then $\mathcal{F}|_C$ is ample. As $\rho(X) = 1$, this implies $X \cong \mathbb{P}^n$ [ADK08, Proposition 2.7] (see also Corollary 2.1.10). This contradicts our assumption that the tangent bundle T_X is not semistable. So we have $d < r$. Moreover, by definition we have

$$c_1(\mathcal{F}) \cdot C = r_{\mathcal{F}} H \cdot C = \sum_{i=1}^{r} a_i \leq d.$$

Since $\mathcal{F}|_C$ is a subbundle of $T_X|_C$, it follows that the positive part

$$(\mathcal{F}|_C)^+ := \mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(a_d) \cong \mathcal{O}_{\mathbb{P}^1}(1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(1)$$

of $\mathcal{F}|_C$ is a subbundle of $T_X|_C$. In particular, the ample vector bundle $(\mathcal{F}|_C)^+$ is also a subbundle of the positive part

$$(T_X|_C)^+ := \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus p}$$

of $T_X|_C$. This implies that $d \leq p+1$. However, if $d = p+1$, then $(\mathcal{F}|_C)^+$ and $(T_X|_C)^+$ have the same rank, and $(\mathcal{F}|_C)^+$ is a subbundle of $(T_X|_C)^+$ if and only if $(\mathcal{F}|_C)^+ = (T_X|_C)^+$. This is impossible. Therefore, we have $d \leq p$. In summary, we have

$$1 \leq r_{\mathcal{F}} \leq r_X - 1, \ r_{\mathcal{F}} H \cdot C \leq d \leq p = r_X H \cdot C - 2, \ 1 \leq d \leq r-1. \tag{5.1}$$

Furthermore, since X is a Fano manifold, its tangent bundle T_X is generically ample [Pet12, Theorem 1.3], and it follows $\mu_H^{\min}(T_X) > 0$. We denote by

$$0 = \mathcal{E}_0 \subsetneq \mathcal{E}_1 \subsetneq \cdots \subsetneq \mathcal{E}_{s-1} \subsetneq \mathcal{E}_s = T_X$$

the Harder-Narasimhan filtration of T_X. By our assumption, we have $s \geq 2$, $\mathcal{E}_1 = \mathcal{F}$ and

$$\mu_H^{\min}(T_X) = \mu_H(T_X/\mathcal{E}_{s-1}).$$

As $\rho(X) = 1$ and $\mu_H^{\min}(T_X) > 0$, we have $c_1(T_X/\mathcal{E}_{s-1}) = iH$ for some $i \in \mathbb{Z}_{>0}$. Moreover, note that $\mathrm{rk}(T_X/\mathcal{E}_{s-1}) \leq n-1$, it follows

$$\mu_H^{\min}(T_X) = \mu_H(T_X/\mathcal{E}_{s-1}) = \frac{i}{\mathrm{rk}(T_X/\mathcal{E}_{s-1})} H^n \geq \frac{1}{n-1} H^n.$$

Proof of (1). In view of (5.1), we have $r_{\mathcal{F}} = 1$. Moreover, we have

$$0 < H \cdot C \leq d \leq p = 2H \cdot C - 2.$$

This implies $H \cdot C \geq 2$. Then we have

$$\mu_H^{\max}(T_X) = \mu_H(\mathcal{F}) = \frac{1}{r} H^n \leq \frac{1}{d+1} H^n \leq \frac{1}{H \cdot C + 1} H^n \leq \frac{1}{3} H^n.$$

Now the strenghtening of Bogomolov inequality (see [Lan04, Theorem 5.1])

$$H^n \cdot \left(\Delta(T_X) \cdot H^{n-2}\right) + n^2 \left(\mu_H^{\max}(T_X) - \mu_H(T_X)\right)\left(\mu_H(T_X) - \mu_H^{\min}(T_X)\right) \geq 0,$$

implies

$$H^n \cdot \left(\Delta(T_X) \cdot H^{n-2}\right) + n^2 \left(\frac{1}{3} - \frac{2}{n}\right) H^n \cdot \left(\frac{2}{n} - \frac{1}{n-1}\right) H^n \geq 0.$$

By the definition of $\Delta(T_X)$, after simplifying the expression, we conclude

$$c_2(X) \cdot H^{n-2} \geq \frac{11n - 16}{6n - 6} H^n.$$

Proof of (2). According to (5.1), we have $r_{\mathcal{F}} \leq r_X - 1$. Similar to the proof of (i), we establish an upper bound for $\mu_H^{\max}(T_X)$ in terms of the index of X

$$\mu_H^{\max}(T_X) \leq \frac{r_X - 2}{r_X - 1} H^n.$$

If $1 \leq r_{\mathcal{F}} \leq r_X - 2$, by (5.1), one can derive

$$\mu_H^{\max}(T_X) = \mu_H(\mathcal{F}) = \frac{r_{\mathcal{F}}}{r} H^n \leq \frac{r_{\mathcal{F}}}{d+1} H^n \leq \frac{r_{\mathcal{F}}}{r_{\mathcal{F}} H \cdot C + 1} H^n.$$

Note that $H \cdot C \geq 1$ and $r_{\mathcal{F}} \leq r_X - 2$, we conclude

$$\mu_H^{\max}(T_X) \leq \frac{r_{\mathcal{F}}}{r_{\mathcal{F}} + 1} H^n \leq \frac{r_X - 2}{r_X - 1} H^n.$$

If $r_{\mathcal{F}} = r_X - 1$, the inequality given in (5.1)

$$0 < r_{\mathcal{F}} H \cdot C = (r_X - 1) H \cdot C \leq p = r_X H \cdot C - 2,$$

yields $H \cdot C \geq 2$. Nevertheless, as $r_X \geq 3$, it follows

$$\mu_H(\mathcal{F}) \leq \frac{r_{\mathcal{F}}}{d+1} H^n \leq \frac{r_X - 1}{(r_X - 1) H \cdot C + 1} H^n \leq \frac{r_X - 2}{r_X - 1} H^n.$$

Now, by the strenghtening of Bogomolov inequality again, it follows

$$H^n \cdot \left(\Delta(T_X) \cdot H^{n-2}\right) + n^2 \left(\frac{r_X - 2}{r_X - 1} - \frac{r_X}{n}\right) H^n \cdot \left(\frac{r_X}{n} - \frac{1}{n-1}\right) H^n \geq 0.$$

After simplifying the expression, one can derive the following inequality

$$c_2(X) \cdot H^{n-2} \geq \frac{r_X(r_X - 1)}{2} H^n + \frac{2n(r_X - 1) - r_X^2}{2(n-1)(r_X - 1)} H^n.$$

Then we can conclude by observing that the number $2n(r_X - 1) - r_X^2$ is positive for $3 \leq r_X \leq n - 2$ and $n \geq 7$. □

5.2.3. Remark. If $n \geq 7$, one can easily check that the upper bound of $\mu_H^{\max}(T_X)$ given in the proof is strictly smaller than $\mu_H(T_X)$ if $r_X \geq n-1$. This actually recovers the semi-stability of T_X in the case $r_X \geq n-1$ and $n \geq 7$.

5.3 Anticanonical geometry of Fano varieties

This section is devoted to study Question 4.1.9 in several special cases. The crucial ingredient is the existence of good ladder on Fano varieties with mild singularities. The following reduction result is obtained as an application of the Basepoint Free Theorem.

5.3.1. Proposition. *In Question 4.1.9, one may assume that X is a Fano variety with at worst Gorenstein canonical singularities.*

Proof. Since $-K_X$ is a nef and big Cartier divisor, by the Basepoint Free Theorem, there exists a projective birational morphism $\phi\colon X \to X'$ to a normal projective variety X' with connected fibers such that $-kK_X \sim \phi^* A$ for some ample divisor A on X' and some positive integer k. It follows that $-kK_{X'} \sim A$ as $\phi_* K_X \sim K_{X'}$. In particular, we have $\phi^* K_{X'} \sim K_X$. Moreover, by Basepoint free Theorem, we may assume that $|-kK_X|$ and $|-(k+1)K_X|$ are both basepoint free. Then it is easy to see that the factorizations of Stein of $\Phi_{|-kK_X|}$ and $\Phi_{|-(k+1)K_X|}$ are the same. In particular, it follows that $-K_{X'}$ is Cartier. As a consequence the variety X' has at worst canonical Gorenstein singularities. On the other hand, we have $h^0(X, -mK_X) = h^0(X', -mK_{X'})$ for any $m \in \mathbb{Z}$ as $\phi_* \mathcal{O}_X = \mathcal{O}_{X'}$. In particular, the pluri-anti-canonical maps $\Phi_{|-mK_X|}$ of X factor as ϕ followed by the pluri-anti-canonical maps $\Phi_{|-mK_{X'}|}$ of X'. As a consequence, the map $\Phi_{|-mK_X|}$ is birational if and only if $\Phi_{|-mK_{X'}|}$ is birational, and the complete linear system $|-mK_X|$ is basepoint free if and only if the complete linear system $|-mK_{X'}|$ is basepoint free. □

5.3.1 Fano threefolds and Fano fourfolds

Theorem 5.3.2 is an improvement of a result of Fukuda [Fuk91, Main Theorem] and a result of Chen and Jiang [CJ16, Corollary 5.13]. We shall prove it as a consequence of Lemma 5.1.3 and [Amb99, Main Theorem].

5.3.2. Theorem. *Let X be a n-dimensional weak Fano variety with at worst Gorenstein canonical singularities such that $-K_X = (n-2)H$ for some nef and big Cartier divisor H. Then*

(1) the linear system $|mH|$ is basepoint free for $m \geq 2$;
(2) the morphism $\Phi_{|mH|}$ is birational for $m \geq 3$.

Proof. Thanks to Proposition 5.3.1, we may assume that X is a Fano variety with at worst Gorenstein canonical singularities. By [Amb99, Main theorem] and adjunction formula, there exists a ladder $X = X_0 \supset X_1 \supset \cdots \supset X_{n-2}$ such that $X_{i+1} \in |H|_{X_i}|$ and X_i has at worst Gorenstein canonical singularities. In particular, $S := X_{n-2}$ is a surface with trivial canonical class.

Step 1. Basepoint freeness of $|mH|$ for $m \geq 2$. Since $H|_S$ is nef and big, $K_S \sim 0$ and S has at worst canonical singularities, the complete linear system $|mH|_S|$ is basepoint free for $m \geq 2$ by [Kaw00, Theorem 3.1]. By Kawamata-Viehweg vanishing theorem and Serre duality, the natural restriction map

$$H^0(X, mH) \to H^0(S, mH|_S)$$

is surjective for $m \in \mathbb{Z}$. In particular, we have $\operatorname{Bs}|mH| = \operatorname{Bs}|mH|_S|$ for $m \in \mathbb{Z}$. As a consequence, the linear system $|mH|$ is basepoint free for $m \geq 2$.

Step 2. The morphism given by $|mH|$ is birational for $m \geq 3$. By Kawamata-Viehweg vanishing theorem, tt is easy to see that we have $h^0(X, H) = h^0(S, H|_S) + (n-2)$. Moreover, by [Kaw00, Theorem 3.1], we have $h^0(S, H|_S) \geq 1$. Therefore, we obtain dim $|H| \geq n-2$. On the other hand, the restriction map

$$H^0(X, mH) \to H^0(X_1, mH|_{X_1})$$

is surjective for all $m \geq 0$, so the rational map defined by $|mH|_{X_1}|$ coincides with the restriction of the rational map given by $|mH|$ over X_1 for $m \geq 0$. Owe to Theorem 5.1.2 (i), we can suppose that the result holds for X_i ($i \geq 1$). Then Lemma 5.1.3 implies that the linear system $|mH|$ gives a birational map for $m \geq 3$. □

5.3.3. **Remark**. As mentioned in the introduction, In dimension three, one can also derive the same result from the classification given in [JR06, PCS05]. The advantage of our argument is that it may be applied in higher dimension to get some upper bound of $f(n)$, the disadvantage is that we do not get any information about the explicit geometry of X or any information about the base locus Bs $|-K_X|$.

5.3.4. **Theorem**. *Let X be a weak Fano fourfold with at worst Gorenstein canonical singularities. Then*

(1) the linear system $|-mK_X|$ is basepoint free for $m \geq 7$;
(2) the rational map Φ_{-m} is birational for $m \geq 5$.

Proof. By Proposition 5.3.1, we may assume that X is a Fano fourfold with at worst Gorenstein canonical singularities. By [Flo13, Proposition 3.2], we obtain that $h^0(X, -K_X) \geq 2$. Let $Y \in |-K_X|$ be a general member. Then Y has only Gorenstein canonical singularities by [Kaw00, Theorem 5.2]. Owe to [OP95, Theorem 2], the linear system $|-mK_X|_Y|$ is basepoint free for $m \geq 7$. It follows that the linear system $|-mK_X|$ is basepoint free for $m \geq 7$ as Bs $|-mK_X| =$ Bs $|-mK_X|_Y|$ by Kawamata-Viehweg vanishing theorem. Furthermore, applying Theorem 5.1.2 (2) and the same argument in the proof of Theorem 5.3.2, one can conclude that the complete linear system $|-mK_X|$ gives a birational map for $m \geq 5$. □

5.3.5. **Remark**. Let X be a n-dimensional normal projective variety with at worst canonical singularities such that $K_X \sim 0$. Let L be an arbitrary nef and big line bundle over X. Then the complete linear system $|mL|$ is basepoint free for $m \geq f(n+1)$ if $n = 2$ (cf. Theorem 5.3.2 and [Kaw00, Theorem 3.1]), and the morphism $\Phi_{|mL|}$ corresponding to $|mL|$ is birational for $m \geq b(n+1)$ if $n = 2$ or 3 (cf. Theorem 5.3.2, Theorem 5.3.4 and Theorem 5.1.2). So it is natural and interesting to ask if these results still hold in higher dimension.

5.3.2 Fano manifolds of coindex four

In this subsection, we investigate Fano manifolds with coindex four. First, we derive an effective non-vanishing theorem as an application of Theorem 5.2.2.

5.3.6. **Theorem**. *Let X be a Fano manifold of dimension $n \geq 4$ and index $n-3$. Let H be the fundamental divisor. Then $h^0(X, H) \geq n-2$.*

Proof. By [Flo13, Theorem 1.2], we may assume that X is a Fano manifold of dimension 6 or 7. If $\rho(X) \geq 2$, according to the explicit classification results given in [Wiś91b] and [Wiś94], one can easily conclude it (see also Appendix A for the details).

Now we consider the case $\rho(X) = 1$. Note that in this case T_X is semistable if $n = 6$ [Hwa98, Theorem 3], and our result follows from [Flo13, Proposition 3.3]. It remains to consider the case $n = 7$ and $\rho(X) = 1$. By Theorem 5.2.2 (ii), we get

$$c_2(X) \cdot H^5 \geq 6H^7.$$

In view of the proof of [Flo13, Proposition 3.3], we have

$$h^0(X, H) = -\frac{1}{3}H^7 + \frac{1}{12}c_2(X) \cdot H^5 + 4 \geq \frac{1}{6}H^7 + 4 > 4.$$

This completes the proof. ☐

The Hodge conjecture states that the space $Hdg^{2i}(X)$ of rational Hodge classes on a projective manifold X is generated over $\mathbb{Q}$ by classes of algebraic cycles of codimension i on X. This conjecture is known to hold for $i = n-1$. If we replace the rational coefficients by integral coefficients, by Kollár's counterexample [Kol92, Lemma] (cf. [HV11, Theorem 1.1]), the group $Hdg^{2n-2}(X, \mathbb{Z})$ is not generated by algebraic curves. In general, the integral Hodge conjecture holds for $i = n-1$ if and only if the following finite group

$$Z^{2n-2}(X) \colon = Hdg^{2n-2}(X, \mathbb{Z})/\langle [Z], \mathrm{codim} Z = n-1 \rangle$$

is trivial. Moreover, $Z^{2n-2}(X)$ is actually a birational invariant of X. Kollár's counterexamples are hypersurfaces of general type and it is conjectured that $Z^{2n-2}(X) = 0$ if X is a rationally connected manifold. In particular, we expect $Z^{2n-2}(X) = 0$ for all Fano manifolds X. By [HV11, Theorem 1.4] and Theorem 5.3.6, we get the following corollary.

5.3.7. **Corollary.** *Let X be a n-dimensional Fano manifold with index $n-3$. Then the group $H^{2n-2}(X, \mathbb{Z})$ is generated over $\mathbb{Z}$ by classes of curves, or equivalently $Z^{2n-2}(X) = 0$.*

As the second application, combining Theorem 5.3.6 with [Flo13, Theorem 1.1] gives the existence of ladder with canonical singularities over Fano manifolds of coindex four. Moreover, if X is a smooth Fano fourfold. It was proved in [Heu15] that a general element D of $|-K_X|$ has at worst isolated terminal singularities. But the proof therein depends on the smoothness of X, and so far we do not know if it still holds for weak Fano fourfolds with Gorenstein terminal singularities.

5.3.8. **Theorem.** *Let X be a n-dimensional Fano manifold with index $n-3$ and let H be the fundamental divisor. Then*

(1) the linear system $|mH|$ is basepoint free for $m \geq 7$;
(2) the linear system $|mH|$ gives a birational map for $m \geq 5$.

Proof. By Theorem 5.3.6 and [Flo13, Theorem 1.1], there exists a good ladder. More precisely, there is a descending sequence of subvarieties

$$X \supsetneq Y_1 \supsetneq Y_2 \supsetneq \cdots \supsetneq Y_{n-4} \supsetneq Y_{n-3}$$

such that the variety $Y_{i+1} \in |H|_{Y_i}|$ has at most Gorenstein canonical singularities and $K_{Y_{n-3}} \sim 0$.

From [OP95, Theorem 2], the linear system $\left|mH|_{Y_{n-3}}\right|$ is basepoint free when $m \geq 7$. By Kawamata-Viehweg vanishing theorem, the natural restriction

$$H^0(X, mH) \to H^0(Y_{n-3}, mH|_{Y_{n-3}})$$

is surjective for $m \geq 0$, so the linear system $|mH|$ is basepoint free for $m \geq 7$.

Note that the rational map defined by $|mH|_{Y_{n-3}}|$ is birational from $m \geq 5$ (cf. Theorem 5.1.2). By the same argument that we used in the proof of Theorem 5.3.2, we conclude that the rational map given by $|mH|_{Y_{n-4}}|$ is birational for $m \geq 5$. Therefore, after an inductive argument, we see that the rational map given by $|mH|$ is birational for $m \geq 5$. ☐

5.3.9. **Remarks.**

(1) At the present times, observe that the classification of Fano n-folds of index $n-3$ is very far from being known even in dimension four, so our result above might be a starting point of the classification : one may try to describe the Fano n-folds of index $n-3$ such that Φ_{-m} is not birational for all $m \leq 4$.
(2) Let X be a n-dimensional weak Fano variety with at worst Gorenstein canonical singularities. Let H be its fundamental divisor. If its index r_X is equal to $n-1$ or $n-2$, then the same argument in the proof of Theorem 5.3.8 together with Theorem 4.1.10, Theorem 5.3.2 and Theorem 5.1.2 shows that

the complete linear system $|mH|$ is basepoint free for $m \geq 2$ and the morphism $\Phi_{|mH|}$ is birational for $m \geq 3$. The existence of good ladder on such varieties was proved in [Amb99, Main Theorem] (see also [Mel99, Kaw00]).

(3) Theorem 5.3.6 and Theorem 5.3.8 are true even for weak Fano fivefolds with at worst canonical Gorenstein singularties. However, in higher dimension, the argument of Theorem 5.2.2 relies on the smoothness, so it may be interesting to find an alternative proof of Theorem 5.3.6 which does not depend on the smoothness ; then, we expect that Theorem 5.3.8 still holds for weak Fano varieties with at worst canonical Gorenstein singularities.

5.4 Weak Fano varieties in higher dimension

In this section, we investigate weak Fano varieties in higher dimension. First we show that Kawamata's effective nonvanishing conjecture implies a positive answer for good divisor problem. We propose the following effective non-vanishing conjecture which is a special case of Conjecture 4.1.5.

5.4.1. Conjecture. *Let (X, Δ) be a klt pair. Assume that H is a nef and big Cartier divisor on X such that $-(K_X + \Delta) \sim_{\mathbb{Q}} rH$ for some positive $r \in \mathbb{Q}$. Then $H^0(X, H) \neq 0$.*

An affirmative answer to Conjecture 5.4.1 is given in the case $r > \dim X - 3$. In particular, it holds in dimension at most three (cf. [Amb99]). The following two useful lemmas are also proved by Ambro in [Amb99].

5.4.2. Lemma. *Let (X, Δ) be a klt pair. Let $|H|$ be a linear system on X and let $D \in |H|$ be a general member. Let c be the log canonical threshold of D with respect to (X, Δ). If the pair $(X, \Delta + D)$ is not plt, then there exists a lc center of $(X, \Delta + cD)$ contained in* Bs $|H|$.

Proof. If $h^0(X, H) \geq 2$, this was proved in [Amb99, Lemma 5.1]. If $h^0(X, H) = 1$, then every lc center of $(X, \Delta + cD)$ is contained in D since the log-pair (X, Δ) is klt. □

5.4.3. Lemma. *Let X be a normal $\mathbb{Q}$-Gorenstein projective variety of dimension ≥ 2 and D an ample effective Cartier divisor. Let c be the lc threshold of D. If the pair (X, D) is not plt and there exists a minimal center W of (X, cD) of codimension one, then $c \leq 1/2$.*

Proof. If D is not reduced, then $c \leq 1/2$. Now we assume that D is reduced. Since W is of codimension one, it is a component of D and $c = 1$. Note that D is connected [Har77, III, Corollary 7.9], then we have $W = D$ since W is minimal and any intersection of lc centers is also a union of lc centers. However, as the pair (X, D) is not plt, there is another lc center W' of codimension at least two. Since $W = D$ is ample, we have $W \cap W' \neq \emptyset$ and $W \cap W' \subsetneq W$, this contradicts the minimality of W. □

The following theorem was proved for Fano fourfolds in [Kaw00, Theorem 5.2]. A similar result still holds for fundamental divisor of Fano varieties with coindex four [Flo13, Theorem 1.1]. Our proof below is inspired by both the work of Floris [Flo13] and the work of Heuberger on Fano manifolds [Heu16, Chapter 5].

5.4.4. Theorem. *Let X be a weak Fano variety of dimension n with at worst Gorenstein canonical singularities. Assume $n \geq 2$ and $H^0(X, -K_X) \neq 0$. If Conjecture 4.1.5 holds in dimension at most $n-2$ and Conjecture 5.4.1 holds in dimension $n-1$, then the log-pair (X, D) is plt for a general element $D \in |-K_X|$. In particular, $K_D \sim 0$ and D has only Gorenstein canonical singularities.*

Proof. To prove our theorem, we assume to the contrary that the pair (X, D) is not plt. Let c be the log canonical threshold of D. Then $0 < c \leq 1$. By Lemma 5.4.2, it is enough to prove that the pair (X, cD) has no centers of log canonicity contained in Bs $|D|$.

Since D is nef and big, own to Basepoint Free Theorem [KM98, Theorem 3.3], there exists a proper surjective birational morphism $\phi\colon X \to X'$ with connected fibers to a normal projective variety such that $D \sim \phi^* D'$ for some ample Cartier divisor D' on X'. Furthermore, we also have $-K_{X'} \sim D'$ since $\phi_* K_X \sim K_{X'}$. It follows that we have also $\mathrm{Bs}\,|D| = \phi^{-1}(\mathrm{Bs}\,|D'|)$, in particular, X' is a Fano variety with at worst canonical singularities and c is also the log canonical threshold of D'. Let W be a minimal lc center of (X, cD). Then $\phi(W)$ is a minimal lc center of (X', cD'). Replacing X and D by X' and D' respectively, we may assume that X is Fano and D is ample. Now applying the argument of [Flo13, Proposition 4.1], we obtain that there exists an arbitrary small positive rational number η and an effective $\mathbb{Q}$-divisor Δ_W depending on η over W such that the pair (W, Δ_W) is a klt pair and the following Kawamata's subadjunction formula holds

$$K_W + \Delta_W \sim_{\mathbb{Q}} -(1 - c - \eta) D|_W. \tag{5.2}$$

Moreover, the restriction map $H^0(X, D) \to H^0(W, D|_W)$ is surjective (cf. [Fuj11, Theorem 2.2]).

If W is a subvariety of dimension at most $n-2$, Conjecture 4.1.5 implies $H^0(W, D|_W) \neq 0$ if we choose η small enough by our assumption. If W is a subvariety of dimension $n-1$, then Lemma 5.4.3 implies $c \leq 1/2$. Since η can be arbitrary small, Kawamata's subadjunction formula (5.2) yields $1 - c - \eta > 0$. Thus Conjecture 5.4.1 in dimension $n-1$ implies $H^0(W, D|_W) \neq 0$. As a consequence, W is not contained in $\mathrm{Bs}\,|D|$ in both cases. Since W is arbitrary, it follows that there are no lc centers of (X, cD) contained in $\mathrm{Bs}\,|D|$. Hence the pair (X, D) is plt.

Furthermore, by inversion of adjunction [KM98, Theorem 5.50], D has only klt singularities. Note that the canonical divisor $K_D = (K_X + D)|_D \sim 0$ is Cartier, we conclude that D has only canonical singularities. ☐

Now we are in the position to prove our main theorem in this section. Let $\ell(n)$ be the smallest positive number such that $|mL|$ is basepoint free for any n-dimensional projective variety X with at worst canonical singularities and $K_X \sim 0$ and any nef and big line bundle L over X. The existence of $\ell(n)$ is a consequence of effective Basepoint Freeness Theorem [Kol93].

5.4.5. **Theorem**. *Assume that Conjecture 4.1.5 holds in dimension $\leq n-2$ and Conjecture 5.4.1 holds in dimension $n-1$ and n. Then $f(n) \leq \ell(n-1)$.*

Proof. Let X be a weak Fano variety with at worst Gorenstein canonical singularities. By assumption, we have $h^0(X, -K_X) \geq 1$. Thus the pair (X, Y) is plt for a general $Y \in |-K_X|$ by Theorem 5.4.4, i.e., Y has only Gorenstein canonical singularities and $K_Y \sim 0$. By Kawamata-Viehweg vanishing theorem and Serre duality, the restriction map

$$H^0(X, -mK_X) \longrightarrow H^0(Y, -mK_X|_Y)$$

is surjective for all $m \geq 1$. It follows $f(n) \leq \ell(n-1)$. ☐

5.4.6. **Remark**. It is interesting to ask if similar results hold for birationality. However, the existence of universal bound for effective birationality of Calabi-Yau varieties is still not clear.

Chapitre 6

Seshadri constants of the anticanonical divisors of Fano manifolds of large index

This chapter is devoted to study the Seshadri constants of the anticanonical divisors of Fano manifolds with large index. In particular, we try to relate the calculation to the existence of certain rational curves of small degree and the anticanonical geometry of Fano manifolds.

6.1 Notation and basic material

6.1.1 Pencil of surfaces

A projective morphism $f\colon X \to C$ from a smooth projective threefold X onto a smooth curve C with connected fibers is called a *del Pezzo fibration of degree* d if its general fiber S is a del Pezzo surface of degree d. Let X be a smooth projective threefold, and let D be a Cartier divisor on X such that $N\colon = h^0(X, D) \geq 2$. We denote by Φ_D the rational map defined by $|D|$, say

$$\Phi_D\colon X \dashrightarrow \mathbb{P}^{N-1}.$$

By Hironaka's big theorem, we can take successive blow-ups $\pi\colon Y \to X$ such that :

(1) Y is a projective manifold ;
(2) the movable part $|M|$ of $|\pi^* D|$ is basepoint free and the rational map $\alpha\colon = \Phi_D \circ \pi$ is a morphism.

Let $Y \xrightarrow{g} \Gamma \xrightarrow{f} Z$ be the Stein factorization of α with $Z = \alpha(Y) \subset \mathbb{P}^{N-1}$. We have the following commutative diagram.

$$\begin{tikzcd} Y \arrow[r, "g"] \arrow[d, "\pi"'] \arrow[dr, "\alpha"] & \Gamma \arrow[d, "f"] \\ X \arrow[r, dashed, "\Phi_D"] & Z \end{tikzcd}$$

Case (f_{np}). If $\dim(\Gamma) \geq 2$, a general member of $|M|$ is a irreducible smooth surface by Bertini's theorem. We say that $|D|$ *is not composed with a pencil of surfaces.*

Case (f_{p}). If $\dim(\Gamma) = 1$, then a general fiber S of g is an irreducible smooth projective surface by Bertini's theorem. We may write

$$M \sim \sum_{i=1}^{a} S_i \equiv aS,$$

where S_i are smooth fibers of g for all i. We say that $|D|$ is *composed with a pencil of surfaces.* It is clear that $a \geq N-1$. Furthermore, $a = N-1$ if and only if $\Gamma \cong \mathbb{P}^1$, and then we say that $|D|$ is *composed with a rational pencil of surfaces.* In particular, if $q(X) = 0$, then $\Gamma \cong \mathbb{P}^1$ since $g(\Gamma) \leq q(X)$.

6.1.2 Extremal contractions in dimension three

The classification of extremal rays on smooth projective threefold was obtained by Mori in [Mor82]. Let X be a smooth projective threefold and let R be an extremal ray of $\overline{\mathrm{NE}}(X)$. Recall that the length of R is defined as

$$l(R) := \min\{-K_X \cdot C | [C] \in R \text{ is a rational curve}\}.$$

By Mori's Cone Theorem (see Théorème 0.5.3), there is an extremal contraction $\varphi\colon X \to Y$ corresponding to R. In the following, we list the possibilities for the extremal contractions (see [Mor82] and [Kol91a])

Case 1. φ is birational. In this case, φ is a divisorial contraction. We denote by E the exceptional divisor of φ.

Type of R	φ and Y	$l(R)$
E_1	$\varphi(E)$ is a smooth curve and Y is a smooth threefold	1
E_2	$\varphi(E)$ is a point and Y is a smooth threefold, $E \cong \mathbb{P}^2$ and $\mathcal{O}_E(E) \cong \mathcal{O}_{\mathbb{P}^2}(-1)$	2
E_3	$\varphi(E)$ is an ordinary double point, $E = \mathbb{P}^1 \times \mathbb{P}^1$ and $\mathcal{O}_E(E) \cong \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(-1,-1)$	1
E_4	$\varphi(E)$ is a double (cDV)-point, E is a quadric cone in $\mathbb{P}^3$, and $\mathcal{O}_E(E) \cong \mathcal{O}_E \otimes \mathcal{O}_{\mathbb{P}^3}(-1)$	1
E_5	$\varphi(E)$ is a quadruple non-Gorenstein point on Y, $E \cong \mathbb{P}^2$ and $\mathcal{O}_E(E) \cong \mathcal{O}_{\mathbb{P}^2}(-2)$	1

Case 2. φ is not birational. In this case, $\dim(Y) \leq 2$ and Y is nonsingular.

Case 2.a) Y is a smooth surface.

Type of R	φ and Y	$l(R)$
C_1	φ has singular fiber	1
C_2	φ is a smooth morphism	2

Case 2.b) Y is a smooth curve.

Type of R	φ and Y	$l(R)$
D_1	the general fiber of φ is a del Pezzo surface of degree d, $1 \leq d \leq 6$	1
D_2	the general fiber of φ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, and singular fibers to a quadric cone $Q \subset \mathbb{P}^3$	2
D_3	φ is a $\mathbb{P}^2$-bundle	3

Case 2.c) Y is a point. Then X is a Fano manifold with Picard number one and $l(R) = r_X$, where r_X is the index of X (see [Sho79] and Theorem 6.3.3).

6.2 Lines on Fano manifolds with large index

6.2.1 Lines on polarized projective manifolds

Let $X \subset \mathbb{P}^N$ be a nondegenerate irreducible projective manifold of dimension $n \geq 1$. We denote by $\mathbb{P}(T^{\vee}_{X,x}) \subset \mathbb{P}^N$ the projective tangent space of X at x. Let $\mathcal{L}_{x,X}$ denote the Hilbert scheme of lines contained in X and passing through the point x (see [Rus16, Chapter 2]). For a line $\ell \subset X$ passing through x, we let $[\ell] \in \mathcal{L}_{x,X}$ be the corresponding point. Let $x \in X$ be a general point. If $\mathcal{L}_{x,X} \neq \emptyset$, then for every $[\ell] \in \mathcal{L}_{x,X}$ we have

$$\dim_{[\ell]}(\mathcal{L}_{x,X}) = -K_X \cdot \ell - 2.$$

To study the problem about low degree rational curves in arbitrary polarized projective manifolds, we introduce the following definition.

6.2.1. Definition. *Let (X, H) be a polarized projective manifold. A line in X is rational curve $C \subset X$ such that $H \cdot C = 1$. X is covered by lines if through any point x of X there is a line contained in X.*

In general X cannot be embedded into projective spaces in such a way that a line C on X becomes a projective line. If (X, H) is covered by lines, then by definition it is easy to see $\varepsilon(X, H; x) \leq 1$ for any point $x \in X$. If H is in addition globally generated, then $\varepsilon(X, H; x) \geq 1$ holds for any point $x \in X$. In particular, this implies $\varepsilon(X, H; x) = 1$ for every point $x \in X$.

6.2.2. Lemma. *Let (X, H) be a polarized uniruled projective manifold. Assume moreover that through a very general point there is a rational curve $C_x \subset X$ of degree d passing through x. Then there exists an irreducible closed subvariety W of* Chow(X) *such that*

(1) the universal cycle over W dominates X, and
(2) the subset of points in W parametrizing the rational curves C_x (viewed as 1-cycles on X) is open in W.

Proof. Recall that Chow(X) has countably many irreducible components. On the other hand, since we are working over $\mathbb{C}$, we have uncountably many lines on X. Then the existence of W is clear. □

6.2.3. Remark. Let (X, H) be a polarized projective manifold. If through a very general point $x \in X$ there is a line $x \in \ell \subset X$, then X is covered by lines. In fact, let us denote by W the subvariety provided in Lemma 6.2.2. We remark that every cycle $[C]$ in W is irreducible and reduced as $H \cdot C = 1$. Let $\mathcal{V}$ be an irreducible component of RatCurves$^n(X)$ containing W. Then $\mathcal{V}$ is an unsplit minimal covering family of rational curves. Let U be the universal family over $\mathcal{V}$. Then the evaluation map $e\colon U \to X$ is surjective ; that is, X is covered by lines.

The following result is an easy corollary of Mori's "bend and break" lemma.

6.2.4. Lemma. *Let X be a Fano manifold, and let H be the fundamental divisor. If $2r_X > n + 1$, then (X, H) is covered by lines.*

Proof. By [Kol96, V, Theorem 1.6], through every point $x \in X$, there is a rational curve $C_x \subset X$ passing through x such that $-K_X \cdot C_x \leq n + 1$. By our assumption, we get

$$H \cdot C_x = -\frac{1}{r_X} K_X \cdot C_x \leq \frac{n+1}{r_X} < 2.$$

This implies that C_x is a line and X is covered by lines. □

In general, it is not clear if lines should exist on Fano manifolds (see [Sho79] for the results of Fano threefolds). However, the existence of lines can be established for many special examples.

6.2.5. Example.

(1) [Deb01, Proposition 2.13] Let $X \subset \mathbb{P}^N$ be a smooth complete intersection of $\mathbb{P}^N$ defined by equations of degree $(d_1, \cdots, d_r)$. Assume $|\mathbf{d}| = \sum_{i=1}^r d_i \leq N - 1$. Then through any point x of X, there is a line $\ell \subset X$.
(2) [Kol96, V, 4.11] Let $X = X_{d_1,\cdots,d_k} \subset \mathbb{P}(a_0, \cdots, a_n)$ be a weighted complete intersection of degree $(d_1, \cdots, d_k)$. If $|\mathbf{d}| = \sum_{i=1}^k d_i \leq \sum_{i=0}^n a_i - 2$, then through every point $x \in X$, there is a rational curve $C \subset X$ such that $\deg(\mathcal{O}_X(1)|_C) = 1$. If X is smooth, then X is covered by lines.

(3) [Kol96, V, Theorem 1.15] Let X be a rational homogeneous space of Picard number one. Then X is covered by lines.

6.2.6. Lemma. *Let $Y \subset \mathbb{P}^N$ be a subvariety of $\mathbb{P}^N$ such that $\mathcal{O}_Y(-K_Y) \cong \mathcal{O}_Y(r_Y)$ for some positive integer r_Y. Let $X \subset Y$ be a complete intersection of degree $(d_1, \cdots, d_r)$. Assume moreover that we have $\dim \mathcal{L}_{x,Y} \geq r_Y - 2$ for a general point $x \in X$. If $|\mathbf{d}| = \sum_{i=1}^r d_i \leq r_Y - 2$, then $(X, \mathcal{O}_X(1))$ is covered by lines.*

Proof. Assume that X is defined as $Y \cap \{f_1 = \cdots = f_r = 0\}$ in $\mathbb{P}^N$. Let V be the variety defined as $\{f_1 = \cdots = f_r = 0\}$. In view of the proof of [Deb01, Proposition 2.13], for a general point $x \in X$, the Hilbert scheme $\mathcal{L}_{x,V}$ of lines contained in V and passing through x is a subset of $\mathbb{P}(T^\vee_{\mathbb{P}^N,x}) \cong \mathbb{P}^{N-1}$ defined by $|\mathbf{d}|$ equations depending on the equations f_j's. On the other hand, since $\mathcal{L}_{x,Y}$ is a subvariety of $\mathbb{P}(T^\vee_{\mathbb{P}^N,x})$ of dimension at least $r_Y - 2$, it follows that $\mathcal{L}_{x,Y} \cap \mathcal{L}_{x,V} \neq \emptyset$. Equivalently, there exists a line contained in X and passing through x. As a consequence, X is covered by lines (cf. Remark 6.2.3). □

As an immediate application, one can derive the following result.

6.2.7. Proposition. *Let G/P be a rational homogeneous space with Picard number one. Let H be the ample generator of $\mathrm{Pic}(G/P)$. Let X be a smooth complete intersection of G/P such that $-K_X = rH|_X$ for some $r \geq 2$. Then $(X, H|_X)$ is covered by lines.*

Proof. Since $Y = G/P$ is covered by lines (cf. Example 6.2.5), then for any point $x \in X$, we have $\dim \mathcal{L}_{x,Y} = r_Y - 2$. Then we conclude by Lemma 6.2.6. □

6.2.2 Lines on del Pezzo manifolds and Mukai manifolds

In this subsection, we establish the existence of covering families of lines on Fano manifolds with coindex at most three : projective spaces, quadric hypersurfaces, del Pezzo manifolds and Mukai manifolds. The existences of covering families of lines on projective spaces and quadric hypersurfaces are clear, so we focus on del Pezzo manifolds and Mukai manifolds. Recall that del Pezzo manifolds were classified by Fujita in [Fuj82a] and [Fuj82b] and we refer the reader to [IP99] for the description.

6.2.8. Proposition. *Let X be a del Pezzo manifold of dimension at least 3. Let H be the fundamental divisor. Then (X, H) is covered by lines.*

Proof. Case 1. $\rho(X) = 1$. If $d = H^n \leq 4$, then X is a smooth complete intersection in projective spaces or weighted projective spaces, so X is covered by lines (cf. Example 6.2.5). If $d = H^n = 5$, then X is a linear section of the rational homogeneous space $Gr(2,5) \subset \mathbb{P}^9$. Then Proposition 6.2.7 shows that X is also covered by lines.

Case 2. $\rho(X) \geq 2$. Then X is isomorphic to one of the following manifolds,

$$\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1,\ \mathbb{P}(T_{\mathbb{P}^2}),\ \mathbb{P}^2 \times \mathbb{P}^2,\ \mathbb{P}(\mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}).$$

In particular, X carries a $\mathbb{P}^r$-bundle structure such that $H|_F$ is isomorphic to $\mathcal{O}_{\mathbb{P}^r}(1)$ over any fiber F $(r = 1, 2)$. Hence, (X, H) is covered by lines. □

Now we turn our attention to Mukai manifolds. The classification of Mukai manifolds was firstly announced in [Muk89] under the existence of good ladder which was confirmed later by Mella in [Mel99]. We refer reader to [IP99] and the references therein for the details. If X is a n-dimensional Mukai manifolds with $\rho(X) = 1$, then the *genus* of X is the integer g such that

$$d = H^n = 2g - 2,$$

where H is the fundamental divisor of X. In general, a smooth Mukai threefold is not necessarily covered by lines, for instance, a smooth quartic threefold is not covered by lines [Col79]. In fact, in general, we have the following result.

6.2.9. Proposition. *Let X be a Fano manifold with index $r_X = 1$. Then $(X, -K_X)$ is not covered by lines.*

Proof. This follows directly from the fact that the minimal anticanonical degree of a minimal covering family of rational curves on X is at least 2. □

6.2.10. Proposition. *Let X be a Mukai manifold of dimension n at least 4. Let H be the fundamental divisor. Then (X, H) is covered by lines.*

Proof. *Case 1.* $\rho(X) = 1$. In this case we have $g \leq 10$ (cf. [IP99, Theorem 5.2.3]). Moreover, if $g \leq 5$, then X is a complete intersection in projective spaces or weighted projective spaces and X is covered by lines (cf. Example 6.2.5). If $g \geq 6$, then the linear system $|H|$ defines an embedding and X is a linear section of a variety

$$\Sigma_{2g-2}^{n(g)} \subset \mathbb{P}^{g+n(g)-2}$$

of dimension $n(g)$ and degree $2g-2$ (cf. [IP99, Corollary 2.1.17]). If $6 \leq g \leq 9$, then $\Sigma_{2g-2}^{n(g)}$ is a Mukai manifold of dimension $n(g) \geq 6$. In particular, by Lemma 6.2.4, $\Sigma_{2g-2}^{n(g)}$ is covered by lines. If $g = 10$, then $\Sigma_{18}^{n(10)}$ is a 5-dimensional homogeneous manifold. According to Example 6.2.5, $\Sigma_{18}^{n(10)}$ is also covered by lines. Therefore, for a general point $x \in \Sigma_{2g-2}^{n(g)-2}$, we have

$$\dim\left(\mathcal{L}_{x,\Sigma_{2g-2}^{n(g)}}\right) = -K_{\Sigma_{2g-2}^{n(g)}} \cdot C - 2 = (n(g)-2)H \cdot C - 2 = n(g) - 4,$$

where C is a line passing through x. Thanks to Lemma 6.2.6, the polarized manifold (X, H) is also covered by lines.

Case 2. $\rho(X) \geq 2$. By Lemma 6.2.4, we may assume that $\dim(X) = 4$ or 5. If $\dim(X) = 5$, by [Wiś91a], X is isomorphic to one of the following manifolds,

$$\mathbb{P}^2 \times Q^3,\ \mathbb{P}(T_{\mathbb{P}^3}),\ \mathbb{P}(\mathcal{O}_{\mathbb{P}^3}(2) \oplus \mathcal{O}_{\mathbb{P}^3}(1) \oplus \mathcal{O}_{\mathbb{P}^3}(1)),$$

where Q^3 is a smooth 3-dimensional hyperquadric. In particular, X carries a $\mathbb{P}^2$-bundle structure such that $H|_F$ is isomorphic to $\mathcal{O}_{\mathbb{P}^2}(1)$ for every fiber F. Hence, (X, H) is covered by lines.

It remains to consider the case $\dim(X) = 4$. If $\rho(X) \geq 3$, by [Wiś94], X is isomorphic to one of the following manifolds,

$$\mathbb{P}^1 \times \mathbb{P}(T_{\mathbb{P}^2}),\ \mathbb{P}^1 \times \mathbb{P}(\mathcal{O}_{\mathbb{P}^2}(2) \oplus \mathcal{O}_{\mathbb{P}^2}(1)),\ \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1,$$

so X carries a $\mathbb{P}^1$-bundle structure such that $H|_F$ is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(1)$ for every fiber F. Thus, X is covered by lines. If $\rho(X) = 2$, by [Wiś94], there exists a fibration $p\colon X \to Y$ such that Y is a Fano manifold of dimension r $(2 \leq r \leq 3)$ and the general fiber F of p is isomorphic to $\mathbb{P}^1$ or Q^2. Moreover, the restriction $H|_F$ is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(1)$ or $\mathcal{O}_{Q^2}(1)$. As a consequence, there exists a line passing through a general point $x \in X$ and X is covered by lines. □

6.3 Seshadri constants in higher dimension

This section is devoted to study the Seshadri constants of the anticanonical divisors of Fano manifolds of coindex at most 4. Let L be a nef line bundle on a projective manifold X. Recall that $\varepsilon(X, L; 1)$ is defined to be $\varepsilon(X, L; x)$ for a very general point $x \in X$. For an irreducible and reduced curve $C \subset X$ passing through x, we recall that $\nu(C, x)$ is the multiplicity of C at x. Our first result is a slight generalization of [Bro06, Théorème 1.5].

6.3.1. Theorem. *Let X be a n-dimensional Fano manifold with index $r_X \geq n-3$. Let H be the fundamental divisor. Then $\varepsilon(X, H; 1) \geq 1$.*

Proof. If $r_X \geq n-2$, this was proved in [Bro09, Théorème 1.5]. Thus, it remains to consider the case $r_X = n-3$. Thanks to Theorem 5.3.6 and [Flo13, Theorem 1.1], there is a descending sequence of subvarieties

$$X = X_0 \supsetneq X_1 \supsetneq X_2 \supsetneq \cdots \supsetneq X_{n-4} \supsetneq X_{n-3}$$

such that $-K_{X_i} \sim (n-i-3)H$ and $X_{i+1} \in |H|_{X_i}|$ is a projective variety with at worst canonical singularties. Let us denote $H|_{X_{n-3}}$ by L. By Kawamata-Viehweg vanishing theorem, we have

$$h^0(X_{n-3}, L) = h^0(X, H) - (n-3) \geq (n-2) - (n-3) = 1.$$

Equivalently, there is an effective Cartier divisor $D \in |L|$ such that $L - D \sim 0$. According to [Bro09, Lemme 4.11], it follows $\varepsilon(X_{n-3}, L; 1) \geq 1$. Let x be a very general point on X_{n-3}. If $C \subset X_{n-4}$ is a curve passing through x that is contained in X_{n-3}, then we obtain

$$(H|_{X_{n-4}}) \cdot C = (H|_{X_{n-3}}) \cdot C = L \cdot C \geq \nu(C, x).$$

Moreover, if $C \subset X_{n-4}$ is a curve passing through x that is not contained in X_{n-3}, then we get

$$(H|_{X_{n-4}}) \cdot C = X_{n-3} \cdot C \geq \nu(C, x),$$

since X_{n-3} passes through x, but not containing C. Hence, we have $\varepsilon(X_{n-4}, H|_{X_{n-4}}, x) \geq 1$. By the maximality of $\varepsilon(X_{n-4}, H|_{X_{n-4}}; 1)$, we conclude that $\varepsilon(X_{n-4}, H|_{X_{n-4}}; 1) \geq 1$. Similarly, after an inductive argument, we obtain that $\varepsilon(X, H; 1) \geq 1$. □

6.3.2. Remark. If X is a smooth Fano fourfold, this theorem was proved in [Bro09, Théorème 1.6]. Our proof above is essentially the same as that in [Bro09].

As an application of the existence of lines on del Pezzo manifolds and Mukai manifolds, one can easily derive the following theorem.

6.3.3. Theorem. *Let X be a n-dimensional Fano manifold of index $r_X \geq \max\{2, n-2\}$. Then passing through every point $x \in X$, there is a rational curve C such that $-K_X \cdot C = r_X$. In particular, we have $\varepsilon(X, -K_X; 1) = \ell_X = r_X$.*

Proof. Let H be the fundamental divisor of X. If $r_X \geq n$, then X is isomorphic to $\mathbb{P}^n$ or Q^n. In particular, (X, H) is covered by lines. By Proposition 6.2.8 and Proposition 6.2.10, we conclude that (X, H) is covered by lines if $r_X \geq n-2$. This implies $\varepsilon(X, -K_X; 1) \leq \ell_X = r_X$. Then we conclude by Theorem 6.3.1 (see also [Bro09, Théorème 1.5]). □

The same argument can be applied to higher dimensional Fano manifolds with coindex four.

6.3.4. Proposition. *Let X be a n-dimensional Fano manifold such that $n \geq 8$. If $r_X = n-3$, then we have $\varepsilon(X, -K_X; 1) = \ell_X = n-3$.*

Proof. This follows directly from Theorem 6.3.1 and Lemma 6.2.4. □

In general, let X be a projective variety embedded in $\mathbb{P}^N$, and let $\mathcal{O}_X(1) = \mathcal{O}_{\mathbb{P}^N}(1)|_X$. Then it is easy to see that $\varepsilon(X, \mathcal{O}_X(1); 1) \geq 1$. Furthermore it is known that $\varepsilon(X, \mathcal{O}_X(1); 1) = 1$ holds if and only if $(X, \mathcal{O}_X(1))$ is covered by lines (cf. [Cha10, Theorem 1.2]). In [IM14], Ito and Miura give a formula for the Seshadri constant $\varepsilon(X, \mathcal{O}_X(1); 1)$ of a complete intersection X in a rational homogeneous space Y of Picard number one. In particular, using the classification of Fano threefolds with Picard number one together with [IM14, Theorem 3], we get the following results.

6.3.5. Proposition. *Let X be a smooth Fano threefold with Picard number one such that $r_X = 1$. Assume moreover that $4 \leq g(X) \leq 10$ and $g(X) \neq 6$.*

(1) $\varepsilon(X, -K_X; 1) = 3/2$ if $g(X) = 4$; that is, X is a complete intersection of a quadric and a cubic in $\mathbb{P}^5$.

(2) $\varepsilon(X, -K_X; 1) = 2$ *otherwise.*

Proof. By the classification of Mukai manifolds, under our assumption, $|-K_X|$ defines a morphism $\Phi_{|-K_X|}\colon X \to \mathbb{P}^{g+1}$ which is an embedding and its image is a complete intersection in some rational homogeneous space Y of Picard number one. Moreover, we know that $(X, -K_X)$ is not covered by lines. Then the result follows directly from [IM14, Theorem 3]. $\square$

6.3.6. Remark. If X is a smooth Fano threefold with Picard number one and index $r_X = 1$, by [IM14, Theorem 2], one can derive that we have $\varepsilon(X, -K_X; 1) > 1$ if Φ_{-1} is an embedding. In particular, if $g(X) \geq 4$, then we have $\varepsilon(X, -K_X; 1) > 1$.

6.4 Seshadri constants of the anticanonical divisors of Fano threefolds

This section is devoted to compute the Seshadri constant of anticanonical divisors of Fano threefolds with Picard number at least two. These manifolds were classified by Mori and Mukai in [MM81, MM83, MM86, MM03]. There are exactly 88 types of such manifolds.

6.4.1 Splitting and free splitting of anticanonical divisors

The following concept plays a key role in the classification of Fano threefolds, and it is also the starting point of our argument.

6.4.1. Definition. *Let X be a projective manifold. A Weil divisor D on X has a splitting if there are two non-zero effective divisors D_1 and D_2 such that $D_1 + D_2 \in |D|$. The splitting is called free if the linear systems $|D_1|$ and $|D_2|$ are basepoint free.*

6.4.2. Remark. We warn the reader that we used also D_1 and D_2 for types of extremal contractions in dimension three (cf. Section 6.1.2). However, it will be easy for the reader to understand from the context whether we are considering a splitting or the type of an extremal contraction.

The following criterion was frequently used in [MM86] to check the free splitting of anticanonical divisors.

6.4.3. Definition-Proposition [MM86, Proposition 2.10]. *Let X be a projective manifold. Assume that C is a smooth proper closed subscheme of X. Let $\mathcal{I}_C$ be the sheaf of ideals of C in X, and let $\mathcal{L}$ be an invertible sheaf over X with the attached complete linear system $|\mathcal{L}|$. Let $f\colon Y \to X$ be the blow-up of X along C. We denote by E the exceptional divisor of f. We say that C is an intersection of members of $|\mathcal{L}|$ when the equivalent conditions below are satisfied.*

(1) $H^0(X, \mathcal{L} \otimes \mathcal{I}_C) \otimes \mathcal{O}_X \twoheadrightarrow \mathcal{L} \otimes \mathcal{I}_C$*,*
(2) $f^*\mathcal{L} \otimes \mathcal{O}_Y(-E)$ *is generated by global sections.*

The following theorem is the key ingredient of our proof of Theorem 4.1.20. It was claimed in [MM81] and the proof was provided in [MM86] except for one case.

6.4.4. Theorem [MM86, Theorem 3]. *Let X be a smooth Fano threefold with $\rho(X) \geq 2$, then the anticanonical divisor $-K_X$ has a splitting. Furthermore, $-K_X$ has a free splitting if and only $|-K_X|$ is basepoint free.*

Proof. This theorem was proved in [MM86, §7] except for n^o 13 in Table 4 given in [MM03]. In the latter case, X is the blow-up of $Y = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ with center a curve C of tridegree $(1, 1, 3)$. Let $f\colon Y \to \mathbb{P}^1 \times \mathbb{P}^1$ be the projection to the first two factors. Then the curve $\widetilde{C} = f(C)$ is a divisor of bidegree $(1, 1)$ on $\mathbb{P}^1 \times \mathbb{P}^1$ and C is a curve of bidegree $(1, 3)$ on the surface $Q = \widetilde{C} \times \mathbb{P}^1$. From the exact sequence

$$0 \to \mathcal{O}_Y(1, 0, 1) \to \mathcal{O}_Y(2, 1, 1) \to \mathcal{O}_Q(3, 1) \to 0,$$

one obtains the surjectivity of $H^0(Y, \mathcal{O}_Y(2,1,1)) \to H^0(Q, \mathcal{O}_Q(3,1))$ from $H^1(Y, \mathcal{O}_Y(1,0,1)) = 0$. Therefore, C is an complete intersection of two divisors $Q \in |\mathcal{O}_Y(1,1,0)|$ and $D \in |\mathcal{O}_Y(2,1,1)|$. Thus, C is an intersection of members of $|\mathcal{O}_Y(2,1,1)|$. Let $g\colon Y \to \mathbb{P}^1 \times \mathbb{P}^1 = S$ be the projection to the last two factors. Then $g|_C\colon C \to S$ is an embedding as $\deg \mathcal{O}_Y(0,1,0)|_C = 1$. We set $N = \mathcal{O}_S(1,1)$ on S. Then C is an intersection of members of $|-K_Y - g^*N|$. By [MM86, Proposition 2.13], X is a Fano manifold and $-K_X$ has a free splitting. □

In the case where $|-K_X|$ is not basepoint free, we can prove the following proposition by reducing it to K3 surface.

6.4.5. Proposition. *Let X be a n-dimensional Mukai manifold with H the fundamental divisor. If $|H|$ is not basepoint free, then $\varepsilon(X, H; 1) = 1$.*

Proof. If $n \geq 4$, this follows from Theorem 4.1.19. Now we consider the case $n = 3$; that is, X is a smooth Fano threefold with index one. According to [Bro09, Théorème 1.5], it is enough to show $\varepsilon(X, H; 1) \leq 1$. By [Sho79], a general element $S \in |H|$ is a smooth K3 surface. By Kodaira's vanishing theorem, the natural restriction

$$H^0(X, H) \longrightarrow H^0(S, H|_S)$$

is surjective. In particular, as $|H|$ is not basepoint free, $|H|_S|$ is also not basepoint free. According to [SD74], the divisor $H|_S$ is of the form

$$H|_S \sim aE + \Gamma,$$

where $E \subset S$ is an elliptic curve, $\Gamma \subset S$ is a smooth rational curve with $E \cdot \Gamma = 1$ and $a \geq 3$. Moreover, the pencil $|E|$ gives an elliptic fibration $S \to \mathbb{P}^1$. As $L \cdot E = 1$, it follows that we have $\varepsilon(S, H|_S; 1) \leq 1$. By the definition, for any $x \in S$, we have

$$\varepsilon(X, H; x) \leq \varepsilon(S, H|_S; x) \leq \varepsilon(S, H|_S; 1) \leq 1.$$

Recall that the set of points

$$Z\colon = \{\, x \in X \mid \varepsilon(X, H; x) < \varepsilon(X, H; 1) \,\}$$

is contained in a countable union of proper Zariski subsets of X and $\dim |H| \geq 1$. It follows that there exists a smooth element $S \in |H|$ such that $S \not\subset Z$. Hence, we have $\varepsilon(X, H; 1) \leq 1$. □

6.4.6. Remark. If X is a del Pezzo manifold, then the same argument together with [Bro06, Théorème 1.3] shows that we have also $\varepsilon(X, H; 1) = 1$ if $|H|$ is not basepoint free. Thus it is natural and interesting to ask if this still holds for Fano manifolds of higher coindex.

6.4.2 Small Seshadri constant and del Pezzo fibrations

In this section, we study smooth Fano threefolds X such that $\varepsilon(X, -K_X; 1) < 2$. The following result is a consequence of the existence of free splitting of anticanonical divisors (cf. Theorem 6.4.4).

6.4.7. Theorem. *Let X be a smooth Fano threefold with $\rho(X) \geq 2$ such that $|-K_X|$ is basepoint free. Let D_1 and D_2 be a free splitting of $-K_X$. Then $\varepsilon(X, -K_X; 1) < 2$ if and only if $|D_1|$ or $|D_2|$ defines a del Pezzo fibration $X \to \mathbb{P}^1$ of degree $d \leq 3$. Moreover, we have $\varepsilon(X, -K_X; 1) = \varepsilon(S_d, -K_{S_d}; 1)$ in this case, where S_d is a del Pezzo surface of degree d.*

Proof. If X admits a del Pezzo fibration of degree $d \leq 3$, by [Bro06], then we have $\varepsilon(X, -K_X; 1) \leq \varepsilon(S_d, -K_{S_d}; 1) < 2$, where S_d is a general fiber. Now, we suppose that $\varepsilon(X, -K_X; 1) < 2$. Denote by g_1 and g_2 the contractions defined by $|D_1|$ and $|D_2|$, respectively. Then we have $\dim(g_1(X)) \geq 1$ (resp. $\dim(g_2(X)) \geq 1$) since D_1 (resp. D_2) is non zero and free. Let $x \in X$ be a very general point and let $|D_1|_x \subset |D_1|$ (resp. $|D_2|_x \subset |D_2|$) be the subset of elements of $|D_1|$ (resp. $|D_2|$) passing through x. Let C be an irreducible and reduced curve passing through x.

Case 1. C is not contracted by g_1 and g_2. In this case, there exist $D_1' \in |D_1|_x$ and $D_2' \in |D_2|_x$ such that $C \not\subset D_1'$ and $C \not\subset D_2'$. In particular, we get

$$-K_X \cdot C = (D_1' + D_2') \cdot C \geq 2\nu(C, x).$$

Case 2. C is contracted by g_1 or g_2. Without loss of generality, we can assume that C is contracted by g_1. By the generality of x, this implies that g_1 is not birational and $\dim(g_1(X)) \leq 2$.

Case 2.a) $\dim(g_1(X)) = 2$. It follows that C is a smooth curve by generic smoothness. In particular, we have $\nu(C, x) = 1$. We claim that $-K_X \cdot C \geq 2$. Otherwise, we assume that $-K_X \cdot C = 1$. Since $|-K_X|$ is basepoint free and $-K_X$ is ample, the morphism

$$\phi\colon = \Phi_{|-K_X|}\colon X \to \mathbb{P}^{N'}$$

is finite. In particular, we have $-K_X \cdot C = \deg(\phi|_C) H \cdot \phi_* C$, where H is a hyperplane of $\mathbb{P}^{N'}$. This forces that the restriction $\phi|_C$ is birational and $H \cdot \phi_* C = 1$. It follows that $\phi_* C$ is a projective line and C is a line of the polarized pair $(X, -K_X)$. In particular, $(X, -K_X)$ is covered by lines. Due to Lemma 6.2.2, we get $\ell_X = 1$, which is absurd. Hence, we have $-K_X \cdot C \geq 2\nu(C, x)$.

Case 2.b) $\dim(g_1(X)) = 1$. Since X is Fano, it follows that $g_1(X) = \mathbb{P}^1$ and the general fiber of g_1 is a smooth surface by generic smoothness. Let S be the fiber of g passing through x. Since x is very general, we may assume that S is smooth. By adjunction formula, we have

$$-K_S = -(K_X + S)|_S = -K_X|_S.$$

Thus, S is a smooth del Pezzo surface of degree d. Then, by [Bro06, Théorème 1.3], we have

$$-K_X \cdot C = -K_S \cdot C \geq 2\nu(C, x)$$

unless $d \leq 3$. In summary, we have proved that if C is an irreducible curve passing through a very general point $x \in X$, then we have

$$-K_X \cdot C \geq 2\nu(C, x)$$

unless g_1 or g_2 defines a del Pezzo fibration of degree d at most 3 and C is contained in the fibers. Then the assumption $\varepsilon(X, -K_X; 1) < 2$ implies g_1 or g_2 defines a del Pezzo fibration of degree $d \leq 3$. Moreover, in this case, we have $\varepsilon(X, -K_X; 1) \geq \varepsilon(S_d; -K_{S_d}; 1)$ by our argument above, and this completes the proof. □

6.4.8. Corollary. *Let X be a smooth Fano threefold such that $\rho(X) \geq 2$. Then $\varepsilon(X, -K_X; 1) = 1$ if and only if $|-K_X|$ is not basepoint free.*

Proof. If $|-K_X|$ is not basepoint free, this is proved in Proposition 6.4.5. Now we consider the converse by assuming $\varepsilon(X, -K_X; 1) = 1$. If $|-K_X|$ is basepoint free, by Theorem 6.4.7, there exists a del Pezzo fibration $f\colon X \to \mathbb{P}^1$ of degree one. Let S be a general fiber. Then the natural map

$$H^0(X, -K_X) \to H^0(S, -K_S)$$

shows that $|-K_X|$ is not basepoint free, since S is a del Pezzo surface of degree one, $|-K_S|$ is not basepoint free. We get a contradiction. □

6.4.9. Remark. By our argument above, in Theorem 6.4.7, we have actually $d = 2$ or 3.

The following example shows that a smooth Fano threefold may be released as two del Pezzo fibrations of different degree.

6.4.10. Example. Let $X = S_d \times \mathbb{P}^1$, where S_d is a smooth del Pezzo surface of degree d such that S_d is isomorphic to neither $\mathbb{P}^2$ nor a smooth quadri surface Q^2. Then there is natural fibration $S_d \to \mathbb{P}^1$ with general fiber $\mathbb{P}^1$. Denote the induced fibration $X \to \mathbb{P}^1$ by p_1 and the second projection $S_d \times \mathbb{P}^1 \to \mathbb{P}^1$ by p_2. Then the general fiber of p_1 is a smooth quadric surface, while the fiber of p_2 is a del Pezzo surface of degree d.

In the following proposition, we show that this cannot happen for del Pezzo fibrations of small degrees.

6.4.11. Proposition. *Let X be a smooth Fano threefold with $\rho(X) \geq 2$ such that $\varepsilon(X, -K_X; 1) < 2$. Then there exists a splitting D_1 and D_2 of $-K_X$ such that $|D_1|$ is basepoint free and it induces a del Pezzo fibration $f\colon X \to \mathbb{P}^1$ of degree d such that $\varepsilon(X, -K_X; 1) = \varepsilon(S_d, -K_{S_d}; 1)$, where S_d is a general fiber of f. Moreover, if $g\colon X \to \mathbb{P}^1$ is any del Pezzo fibration of degree $d' \leq 3$, then $g = f$ and $d = d'$.*

Proof. If $|-K_X|$ is basepoint free, then the existence of f is proved in Theorem 6.4.7. We start with the case where $|-K_X|$ is not basepoint free.

Case 1. $|-K_X|$ is not basepoint free. In this case, X is isomorphic to one of the following.

(1) $X \cong \mathbb{P}^1 \times S_1$.
(2) X is the blow-up of V_1 with center an elliptic curve which is a complete intersection of two members of $|H|$, where V_1 is a smooth del Pezzo threefold of degree 1 and H is the fundamental divisor of V_1.

We claim that there exists a splitting D_1 and D_2 of $-K_X$ such that D_2 is nef and big and $|D_1|$ is free which induces a del Pezzo fibration of degree 1.

In Case (1), we choose $D_1 \in |p_1^*\mathcal{O}_{\mathbb{P}^1}(1)|$ and $D_2 \in |p_1^*\mathcal{O}_{\mathbb{P}^1}(1) \otimes p_2^*\mathcal{O}_{S_1}(-K_{S_1})|$, where p_i is the projection of the i-th factor. Then it is easy to see that $|D_1|$ is basepoint free and D_2 is nef. On the other hand, one observe $D_2^3 = 1$, so D_2 is actually big.

In Case (2), denote by $\pi\colon X \to V_1$ the blow-up and E the exceptional divisor of π. Choose $D_1 \in |\pi^*H - E|$ and $D_2 \in |\pi^*H|$. Then it is easy to see that D_2 is nef and big and $-K_X \sim D_1 + D_2$. Moreover, by Definition-Proposition 6.4.3, the linear system $|\pi^*H - E|$ is basepoint free.

Now we assume that $g \neq f$. Note that $D_2|_{S_{d'}}$ is again nef and big since $S_{d'}$ is general. Thanks to [Bro09, Proposition 4.12], we have $\varepsilon(S_{d'}, D_2|_{S_{d'}}; 1) \geq 1$. As $g \neq f$, thus $S_{d'}$ is not contracted by f and we have $f(S_{d'}) = \mathbb{P}^1$. Let $x \in S_{d'}$ be a very general point, and let $C \subset S_{d'}$ be an irreducible reduced curve passing through x. Let S_x be the fiber of f passing through x. If $f(C)$ is not a point, then C is not contained in S_x and $S_x \sim D_1$. Note that we have $-K_X|_{S_{d'}} = -K_{S_{d'}}$, it follows

$$-K_{S_{d'}} \cdot C = D_1|_{S_{d'}} \cdot C + D_2|_{S_{d'}} \cdot C \geq S_x|_{S_{d'}} \cdot C + \nu(C, x) \geq \nu(C, x) + \nu(C, x) = 2\nu(C, x).$$

If $f(C)$ is a point, as x is very general, we can assume that C is a smooth curve by generic smoothness. In particular, we have $\nu(C, x) = 1$. Moreover, since $-K_{S_{d'}}$ is ample, we have

$$\deg(K_C) = (K_{S_{d'}} + C) \cdot C = K_{S_{d'}} \cdot C < 0.$$

Note that $\deg(K_C) = 2g(C) - 2 \geq -2$ is even, we get $K_{S_{d'}} \cdot C = -2$. In particular, we have

$$-K_{S_{d'}} \cdot C = 2 = 2\nu(C, x).$$

Hence, we conclude that $\varepsilon(S_{d'}, -K_{S_{d'}}; 1) \geq 2$. This contradicts our assumption that $d' \leq 3$ (cf. [Bro06, Théorème 1.3]). As a consequence, the general fibers of g are contracted by f. In particular, this implies that we have $f^*\mathcal{O}_{\mathbb{P}^1}(1) \cong g^*\mathcal{O}_{\mathbb{P}^1}(1)$. Hence, we get $g = f$.

Case 2. $|-K_X|$ is basepoint free. In this case, the existence of D_1 and D_2 is proved in Theorem 6.4.7 (after exchanging D_1 and D_2 if necessary). Moreover, D_1 and D_2 is actually a free splitting of $-K_X$. If $g \neq f$, then $f(S_{d'}) = \mathbb{P}^1$ since $S_{d'}$ is not contracted by f. We claim that $D_1|_{S_{d'}}$ and $D_2|_{S_{d'}}$ is a free splitting of $-K_{S_{d'}}$. In fact, as $K_X|_{S_{d'}} \sim K_{S_{d'}}$, it is enough to show that $D_1|_{S_{d'}}$ and $D_2|_{S_{d'}}$ are not zero. The divisor $D_1|_{S_{d'}}$ is nonzero from the fact that f is defined by $|D_1|$ and $f(S_{d'})$ is not a point. To see that $D_2|_{S_{d'}}$ is nonzero, note that $-K_{S_{d'}}$ is ample while $D_1|_{S_{d'}} \cdot F = 0$ for any curve F contained in the fiber of $f|_{S_{d'}}\colon S_{d'} \to \mathbb{P}^1$.

Now we show that we have actually $\varepsilon(S_{d'}, -K_{S_{d'}}; 1) \geq 2$. The proof is the same as that of Theorem 6.4.7. Let $x \in S_{d'}$ be a very general point, and let $C \subset S_{d'}$ be an irreducible reduced curve passing through x. Denote by C_1 (resp. C_2) the divisor $D_1|_{S_{d'}}$ (resp. $D_2|_{S_{d'}}$) and denote by $|C_1|_x$ (resp. $|C_2|_x$) the subset of elements of $|C_1|$ (resp. $|C_2|$) passing through x. If there exist two curves $C_1' \in |C_1|_x$ and $C_2' \in |C_2|_x$ such that $C \not\subset C_1'$ and $C \not\subset C_2'$, then we have

$$-K_{S_{d'}} \cdot C = (C_1' + C_2') \cdot C \geq 2\nu(C, x).$$

Otherwise, we may assume that C is contained in C_1' for every curve $C_1' \in |C_1|_x$. Let $\mu\colon S_{d'} \to \mathbb{P}^1$ be the fibration defined by $|C_1|$. Then C is a general fiber of μ. In particular, C is a smooth rational curve. This implies $-K_{S_{d'}} \cdot C = 2$ and $\nu(C, x) = 1$. It follows that $\varepsilon(S_{d'}, -K_{S_{d'}}; 1) \geq 2$. This again contradicts our assumption that $d' \leq 3$. Hence the general fibers of g are contracted by f. As the previous case, we conclude that $g = f$. □

6.4.3 Large Seshadri constant and minimal anticanonical degree

Let X be a n-dimensional Fano manifold. Recall that ℓ_X is the *minimal anticanonical degree* of a minimal covering family of rational curves. Then we have $\varepsilon(X, -K_X; 1) \leq \ell_X$ by definition. On the other hand, there are many efforts to characterize projective spaces and quadric hypersurfaces by the values of ℓ_X (cf. [CMSB02, Keb02, Miy04, CD15, DH17] etc.). This indicates that we can also try to characterize projective spaces and quadric hypersurfaces by the values of $\varepsilon(X, -K_X; 1)$. In particular, by [LZ17, Theorem 2] (cf. [CMSB02, BS09]), $\varepsilon(X, -K_X; 1) > n$ if and only if X is isomorphic to $\mathbb{P}^n$.

6.4.12. Proposition. *Let X be a n-dimensional Fano manifold such that $\rho(X) \geq 2$ and $n \geq 3$. Then $\varepsilon(X, -K_X; 1) > n - 1$ if and only if X is isomorphic to the blow-up of $\mathbb{P}^n$ along a smooth subvariety Z of dimension $n - 2$ and degree $d \in \{1, \cdots, n\}$, contained in a hyperplane. In particular, we have $\varepsilon(X, -K_X; 1) = n$ in this case.*

Proof. If $\epsilon(X, -K_X; 1) > n - 1$, then for a very general point $x \in X$ and any curve C passing through x, we have $-K_X \cdot C \geq n$. This implies $\ell_X \geq n$. By [CMSB02, Corollary 0.3] and [CD15, Theorem 1.4] (see also [DH17, Theorem 1.3]), it follows that X is isomorphic to the blow-up of $\mathbb{P}^n$ along a smooth subvariety Z of dimension $n - 2$ and degree $d \in \{1, \cdots, n\}$, containing in a hyperplane. On the other hand, as shown in [LZ17, Theorem 3], if X is a such Fano manifold, then we have indeed $\varepsilon(X, -K_X; 1) = n$. □

The proposition above is actually a special case of [LZ17, Theorem 3]. In fact, let X be a projective manifold and let L be a nef divisor on X. Fix a point $x \in X$, and let $\mu\colon \widehat{X} = Bl_x(X) \to X$ be the blow-up of X at x with exceptional divisor $E \subset \widehat{X}$. Then by [Laz04, Proposition 5.1.5] the Seshadri constant of L at x is equal to

$$\max\{\, \varepsilon \geq 0 \mid \mu^* L - \varepsilon E \text{ is nef} \,\}.$$

In particular, if L is ample, Definition 4.1.14 coincides with that given in [LZ17]. Let us conclude this section with a structure theorem of Fano threefolds via Seshadri constants.

6.4.13. Theorem. *Let X be a smooth Fano threefold with $\rho(X) \geq 2$.*

(1) $\varepsilon(X, -K_X; 1) = 1$ if and only if there is a fibration in del Pezzo surfaces $X \to \mathbb{P}^1$ of degree one.
(2) $\varepsilon(X, -K_X; 1) = 4/3$ if and only if there is a fibration in del Pezzo surfaces $X \to \mathbb{P}^1$ of degree two.
(3) $\varepsilon(X, -K_X; 1) = 3/2$ if and only if there is a fibration in del Pezzo surfaces $X \to \mathbb{P}^1$ of degree three.
(4) $\varepsilon(X, -K_X; 1) = 3$ if and only if X is isomorphic to the blow-up of $\mathbb{P}^3$ along a smooth curve C of degree d at most three which is contained in a hyperplane.
(5) $\varepsilon(X, -K_X; 1) = 2$ otherwise.

Proof. It follows directly from Theorem 6.4.7, Proposition 6.4.11 and Proposition 6.4.12. □

The following result in dimension two is a direct consequence of [Bro06, Théorème 1.3].

6.4.14. Corollary. *Let X be a smooth Fano threefold which is very general in its deformation family. Then $\varepsilon(X, -K_X; 1) = 1$ if and only if $r_X = 1$ and $|-K_X|$ is not basepoint free.*

Proof. This follows from Theorem 6.4.13 and [Ito14, Theorem 1.8] (cf. Corollary 6.4.8). □

6.4.15. Remark. It is interesting to ask if this still holds for any smooth Fano threefolds or in higher dimension. For more evidence, we refer to [IM14] and Proposition 6.3.5.

6.5 Fano threefolds admitting del Pezzo fibrations

This section is devoted to investigate Fano threefolds which can be regarded as a del Pezzo fibration of degree d at most 3.

6.5.1. Assumption. As we have seen, the linear system $|-K_X|$ is not basepoint free if and only if there is a fibration $X \to \mathbb{P}^1$ in del Pezzo surfaces of degree one. Throughout this section, we will assume that $|-K_X|$ is basepoint free.

6.5.1 Fibration in surfaces and Fano threefolds of type I

The following result is a simple criterion to determine whether a basepoint free linear system is composed with a pencil of surfaces.

6.5.2. Lemma. *Let X be a smooth projective threefold. Let D be a divisor such that $|D|$ is basepoint free. Then the following four statements are equivalent.*

(1) $|D|$ is composed with a pencil of surfaces.
(2) $D^2 \cdot S = 0$ for any irreducible surface $S \subset X$.
(3) $D^2 \cdot A = 0$ for any nef and big divisor A on X.
(4) $D^2 \cdot A = 0$ for some nef and big divisor A on X.

Proof. Let $\Phi_D \colon X \to Z = \Phi_D(X) \subset \mathbb{P}^N$ be the morphism defined by $|D|$. Let $X \xrightarrow{g} \Gamma \xrightarrow{f} Z$ be the Stein factorization of Φ_D.

$(1) \Rightarrow (2)$. Suppose that Γ is a curve. In particular, $g(S)$ is a point or a curve. Since $|D|$ is basepoint free, the restriction $D|_S$ is nef, but not big. As a consequence, we obtain

$$D^2 \cdot S = (D|_S)^2 = 0.$$

$(2) \Rightarrow (3)$. Since A is nef and big, then there exists a positive integer m such that $|mA|$ is not empty. Let $S \in |mA|$ be a member. By assumption, then we have

$$D^2 \cdot A = \frac{1}{m} D^2 \cdot S = 0.$$

$(3) \Rightarrow (4)$. Obvious.

$(4) \Rightarrow (1)$. Since A is nef and big, there exists an ample $\mathbb{Q}$-divisor H and an effective $\mathbb{Q}$-divisor E such that $A \sim_{\mathbb{Q}} H + E$. Then $D^2 \cdot A = 0$ implies $D^2 \cdot H = 0$ as D is nef. By Bertini's theorem, there exists a smooth surface S such that $S \sim mH$ for some positive integer m. Then we get $D^2 \cdot S = 0$. As a consequence, $g(S)$ is a point or a curve. Nevertheless, if $\dim(\Gamma) \geq 2$, then S can not be an ample divisor, a contradiction. Hence, we have $\dim(\Gamma) = \dim(Z) = 1$. □

6.5.3. Lemma. *Let $f \colon X \to Y$ be the blow-up of a smooth projective threefold Y along a smooth (may be disconnected) curve C. Denote by E the exceptional divisor of f. If L is a line bundle on Y such that $|f^*L - E|$ is basepoint free and $|f^*L - E|$ is composed with a pencil of surfaces, then we have*

$$L^2 \cdot N = N \cdot C,$$

for every nef divisor N over Y. Moreover, C is a complete intersection of two members of $|L|$ in this case.

Proof. By Definition-Proposition 6.4.3, C is an intersection of members of $|L|$. Let A be an ample divisor over Y. Then the pull-back f^*A is nef and big. Since $|f^*L-E|$ is basepoint free and $\dim \Phi_{|f^*L-E|}=1$, it follows that $(f^*L-E)^2\cdot f^*A=0$ by Lemma 6.5.2. Then we get

$$L^2\cdot A=(f^*L)^2\cdot f^*A=-E^2\cdot f^*A=A\cdot C.$$

Since every nef divisor is a limit of ample divisors, we conclude that the same equality holds also for nef divisors. On the other hand, if $C\subset D_1\cap D_2$ and A is an ample divisor, then

$$L^2\cdot A=D_1\cdot D_2\cdot A=A\cdot C.$$

Since A is ample, this implies that $C=D_1\cap D_2$ as 1-cycles. Thus, we obtain $C=D_1\cap D_2$ since C is smooth. □

For convenience, we introduce the following notation for Fano threefolds, and we refer to [MM86, §2] for more details.

6.5.4. Notation. *Let $f\colon X\to Y$ be the blow-up of a smooth projective threefold Y along a smooth (but possibly disconnected) curve C such that X is a smooth Fano threefold.*

(1) X is called of Type I_1 if C is an intersection of members of a complete linear system $|L|$ such that $-K_Y-L$ is ample and $|-K_Y-L|$ is basepoint free.
(2) X is called of Type I_2 if Y has a structure of a $\mathbb{P}^1$-bundle $g\colon Y\to S$ over a smooth surface S, such that $g|_C\colon C\to S$ is an embedding, and there is an very ample divisor N on S such that C is an intersection of members of $|L|$, where $L=-K_Y-g^*N$.
(3) X is called of Type I_3 if Y has a structure of a $\mathbb{P}^2$-bundle $g\colon Y\to\mathbb{P}^1$, and that the curve $C\subset Y$ is an intersection of members of the complete linear system $|L|$ with $L=-K_Y-g^*\mathcal{O}_{\mathbb{P}^1}(1)$, and $g|_C\colon C\to\mathbb{P}^1$ is sujective and of degree at most 5, and there is an irreducible divisor Q of Y containing C such that the fiber Q_t over every point $t\in\mathbb{P}^1$ is a smooth conic of $Y_t=\mathbb{P}^2$.

X is called of Type I if X satisfies one of the conditions above.

The following property about Fano threefolds of Type I will play an important role in the classification of smooth Fano threefolds admitting a del Pezzo fibration of small degree.

6.5.5. Proposition. *Let X be a smooth Fano threefold of Type I. Denote by E the exceptional divisor of the blow-up $f\colon X\to Y$. Then $-K_X$ has a free splitting D_1 and D_2 such that the following statements hold.*

(1) $D_1\sim f^*L-E$.
(2) D_2 is composed with a rational pencil of surfaces if and only if X is of Type I_3.

Proof. The proposition follows directly from [MM86, Proposition 2.12, 2.13 and 2.14]. □

6.5.2 Fano threefolds with Picard number two

The classification of Fano threefolds with Picard number two was given in Table 2 in [MM81]. Let D_1 and D_2 be a free splitting of $-K_X$. Then the morphisms defined by $|D_1|$ and $|D_2|$ coincide with the two extremal contractions of X. Recall that a smooth Fano threefold is *imprimitive* if it is isomorphic to the blow-up of a smooth Fano threefold along an irreducible smooth curve. A smooth Fano threefold is *primitive* if it is not imprimitive.

6.5.6. Proposition. *Let X be a smooth Fano threefold with $\rho(X)=2$. Assume that $|-K_X|$ is basepoint free. If there is a fibration in del Pezzo surfaces $X\to\mathbb{P}^1$ of degree d at most 3, then X is isomorphic to one of the following.*

(1) *X is a double covering of $\mathbb{P}^1 \times \mathbb{P}^2$ whose branch locus is a divisor of bidegree $(2, 4)$.*
(2) *X is the blow-up of Y along a complete intersection of two members of $\left|-\frac{r-1}{r}K_Y\right|$, where Y is isomorphic to V_2, V_3 or $\mathbb{P}^3$ and r is the index of Y.*

6.5.7. **Remark**. This theorem is a direct corollary of the classification given in [MM81]. A sketchy proof was given in [MM83, §5]. For the reader's convenience, we give a complete proof in our situation.

Proof. Let R_1 and R_2 be the extremal rays of X. Then we can assume that R_1 is of type D_1 (cf. Proposition 6.4.11). We denote by $\varphi_i \colon X \to Y_i$ the extremal contraction associated to R_i.

Case 1. X is primitive. By [MM86, Theorem 3.7], the extremal contraction R_2 is of type C_1 or C_2. By [MM83, §5], $\varphi = (\varphi_1, \varphi_2) \colon X \to \mathbb{P}^1 \times \mathbb{P}^2$ is a double covering since the general fiber of φ_1 is a del Pezzo surface of degree at most 3 (See also Remark I.2). Denote by H_1 (resp. H_2) the line bundle $\mathcal{O}(1, 0)$ (resp. $\mathcal{O}(0, 1)$) over $\mathbb{P}^1 \times \mathbb{P}^2$. Let B be the branch locus. We can write $B \sim b_1H_1 + b_2H_2$. Thus, we get

$$-K_X = \varphi^*\left(-K_{\mathbb{P}^1\times\mathbb{P}^2} - \frac{1}{2}B\right) = \varphi^*\left(\left(2 - \frac{b_1}{2}\right)H_1 + \left(3 - \frac{b_2}{2}\right)H_2\right).$$

Moreover, as $-K_X = \varphi^*(H_1 + H_2)$ (see [MM83, Theorem 5.1] and Theorem I.1), it follows that $b_1 = 2$ and $b_2 = 4$. Moreover, let S be a general fiber of φ_1. Then the degree of S is equal to

$$K_S^2 = (K_X + \varphi^*H_1)^2 \cdot \varphi^*H_1 = (\varphi^*H_2)^2\varphi^*H_1 = 2.$$

Case 2. X is imprimitive. By the definition, R_2 is of type E_1. Let E be the exceptional divisor of φ_2. Then Y_2 is a smooth Fano threefold of index $r \geq 2$ (see [IP99, Proposition 7.1.5]). We denote $\varphi_1^*\mathcal{O}_{\mathbb{P}^1}(1)$ by H_1, and denote $\varphi_2^*(-\frac{1}{r}K_{Y_2})$ by H_2. Then we have $-K_X \sim H_1 + H_2$ [MM83, Theorem 5.1] (cf. Theorem I.1). On the other hand, as $K_X \sim \varphi_2^*K_{Y_2} + E$, it follows that $H_1 \sim (r-1)H_2 - E$. Moreover, since H_1 is basepoint free, $C = \varphi_2(E)$ is an intersection of members of $|-\frac{r-1}{r}K_{Y_2}|$ (see Definition-Proposition 6.4.3). Let S be a general fiber of φ_1. Then the degree d of S is equal to

$$d = (K_X + H_1)^2 \cdot H_1 = H_2^2 \cdot H_1 = (r-1)H_2^3.$$

Since $d = 2$ or 3, it follows that the possibility of the pair (r, H_2^3) are as follows,

$$(2, 2),\ (3, 1),\ (2, 3),\ (4, 1).$$

If $r = 3$, then Y_2 is isomorphic to a quadric and we have $H_2^3 = 2$. This is a contradiction. Thus, $(3, 1)$ is impossible. Since $|H_1|$ is basepoint free and is composed with a rational pencil of surfaces, according to Lemma 6.5.3, C is indeed a complete intersection of two members of $|-\frac{r-1}{r}K_{Y_2}|$. □

6.5.8. **Remark**. In (1), φ_1 is a del Pezzo fibration of degree 2, and in (2) φ_1 is a del Pezzo fibration of degree $(r-1)H_2^3$.

6.5.3 Fano threefolds of Picard number at least three

In this subsection, we will consider Fano threefolds of Picard number at least three. The classification of such manifolds was given in Tables 3, 4 and 5 in [MM81]. First we consider these Fano threefolds which are not of type I. There are exactly 17 families of such manifolds (see Appendix B.1), and we will divide them into five different groups.

(6.5.3.1) Double cover of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$.

In this case, X is a double covering of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ whose branch locus is a divisor of tridegree $(2, 2, 2)$ (n^o 1 in Table 3 in [MM81]).

6.5.9. Proposition. *Let $f\colon X \to \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ be a double covering of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ whose branch locus is a divisor of tridegree $(2,2,2)$. Then $\varepsilon(X, -K_X; 1) = 2$.*

Proof. Let $p_i\colon Y = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ be the i-th projection. Denote by H_i the divisor $p_i^*\mathcal{O}_{\mathbb{P}^1}(1)$. Then $-K_Y = 2H_1 + 2H_2 + 2H_3$. By ramification formula, we get

$$K_X = f^*K_Y + \frac{1}{2}f^*(2H_1 + 2H_2 + 2H_3) = -f^*(H_1 + H_2 + H_3).$$

Let $x \in X$ be a very general point and let C be a curve passing through x. If two of the images $p_i \circ f(C)$ are curves, for example $i = 1$ and 2, then we can chose $D_1 \in |f^*H_1|$ and $D_2 \in |f^*H_2|$ such that D_1 and D_2 pass through x and C is not contained in D_1 or D_2. This implies

$$-K_X \cdot C \geq D_1 \cdot C + D_2 \cdot C \geq 2\nu(C, x).$$

Otherwise, C is a complete intersection of two divisors $D_1 \in |f^*H_1|$ and $D_2 \in |f^*H_2|$. In particular, C is smooth at x and we have $-K_X \cdot C \geq 2$. Therefore, we have $\varepsilon(X, -K_X; x) \geq 2$. Thanks to Proposition 6.4.12, we have $\varepsilon(X, -K_X; 1) \leq 2$. Then by lower semicontinuity, we get $\varepsilon(X, -K_X; 1) = 2$. □

(6.5.3.2) Divisors

In this case, X is given as a smooth member of a divisor class of a smooth projective fourfolds (n^o 2, 3, 8, 17 in Table 3 and n^o 1 in Table 4). First we consider n^o 2 in Table 3.

6.5.10. Proposition. *Let $\mathcal{E}$ be the vector bundle $\mathcal{O} \oplus \mathcal{O}(-1,-1)^{\oplus 2}$ over $\mathbb{P}^1 \times \mathbb{P}^1$. Let $f\colon Y \to \mathbb{P}^1 \times \mathbb{P}^1$ be the $\mathbb{P}^2$-bundle $\mathbb{P}(\mathcal{E})$. Set $L = \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$. Let X be a member of $|L^{\otimes 2} \otimes f^*\mathcal{O}(2,3)|$ such that $X \cap S$ is irreducible, where S is a member of $|L|$. Then $-K_X$ has a free splitting D_1 and D_2 such that $|D_2|$ is composed with a rational pencil of del Pezzo surfaces of degree 3.*

Proof. We choose $H_1 \in |f^*\mathcal{O}(1,0)|$, $H_2 \in |f^*\mathcal{O}(0,1)|$ and $\xi \in |L|$. Then, by adjunction formula, we obtain

$$-K_X = (-K_Y - X)|_X = (\xi + 2H_1 + H_2)|_X.$$

Set $D_1 = (\xi + H_1 + H_2)|$ and set $D_2 = H_1|_X$. Then D_1 and D_2 give a free splitting of $-K_X$ and $|D_2|$ is composed with a rational pencil of surfaces. In particular, we have

$$\begin{aligned}
(K_X + D_2)^2 \cdot D_2 &= (K_Y + X + H_1)^2 \cdot X \cdot H_1 \\
&= (-\xi - 2H_1 - H_2 + H_1)^2 \cdot (2\xi + 2H_1 + 3H_2) \cdot H_1 \\
&= (-\xi - H_1 - H_2)^2(2\xi + 2H_1 + 3H_2)H_1 \\
&= (2\xi^3 + 7\xi^2 H_2) \cdot H_1
\end{aligned}$$

Let $\mathcal{E}' \to \mathbb{P}^1$ be the vector bundle $\mathcal{O} \oplus \mathcal{O}(-1) \oplus \mathcal{O}(-1)$. Choose a divisor $\xi' \in |\mathcal{O}_{\mathbb{P}(\mathcal{E}')}(1)|$. Then we get $\xi^3 \cdot H_1 = \xi'^3$. We denote by π the natural projection $\mathbb{P}(\mathcal{E}') \to \mathbb{P}^1$. Recall that we have the following relation

$$\pi^*c_0(\mathcal{E}')\xi'^3 - \pi^*c_1(\mathcal{E}')\xi'^2 + \pi^*c_2(\mathcal{E}')\xi' - \pi^*c_3(\mathcal{E}') = 0.$$

It follows

$$\xi'^3 = \pi^*c_1(\mathcal{E}')\xi'^2 = -2F \cdot \xi'^2 = -2,$$

where F is a fiber of π. As $\xi^2 H_2 H_1 = 1$, we get $(K_X + D_2)^2 \cdot D_2 = 3$. On the other hand, let $X \xrightarrow{g} \mathbb{P}^1 \xrightarrow{h} \mathbb{P}^1$ be the factorization of Stein of Φ_{D_2}. Then we have $D_2 \sim aS$ for some $a > 0$, where S is general fiber of g. Since $|-K_X|$ is basepoint free, it follows that $K_S^2 > 1$. As $(K_X + aS)^2 \cdot (aS) = 3$, we obtain $a = 1$. Hence, h is an isomorphism and g is indeed a del Pezzo fibration of degree 3. □

6.5.11. Proposition. *For n^o 3, 8, 17 in Table 3 and n^o 1 in Table 4, $-K_X$ has a free splitting D_1 and D_2 such that $|D_1|$ and $|D_2|$ are both not composed with a rational pencil of surfaces.*

Proof. For n^o 3, 17 in Table 3 and n^o 1 in Table 4, X is a divisor of $Q^2 \times \mathbb{P}^2$ or $Q^2 \times Q^2$, where Q^2 is smooth quadric surface. Then it is easy to see that $-K_X$ has a free spliting D_1 and D_2 such that $|D_1|$ and $|D_2|$ are not composed with a pencil of surfaces. For n^o 8 in Table 3, X is a member of the linear system

$$|p_1^* g^* \mathcal{O}_{\mathbb{P}^2}(1) \otimes p_2^* \mathcal{O}_{\mathbb{P}^2}(2)|$$

on $\mathbb{F}_1 \times \mathbb{P}^2$, where p_i $(i = 1, 2)$ is the projection to the i-th factor and $g\colon \mathbb{F}_1 \to \mathbb{P}^2$ is the blow-up. Let C be the exceptional curve of g. Then we have

$$-K_X = (2H_1 - E + H_2)\,|_X$$

where $H_1 \in |p_1^* g^* \mathcal{O}_{\mathbb{P}^2}(1)|$, $H_2 \in |p_2^* \mathcal{O}_{\mathbb{P}^2}(1)|$ and $E = p_1^* C$. Set $D_1 = (2H_1 - E)|_X$ and set $D_2 = H_2|_X$. Then D_1 and D_2 form a free splitting of $-K_X$. It is easy to see that $|D_2|$ is not composed with a rational pencil of surfaces. On the other hand, we have

$$D_1^2 \cdot (H_1 + H_2)|_X = (2H_1 - E)^2 (H_1 + H_2)(H_1 + 2H_2) = 8H_1^2 H_2^2 + 2E^2 H_2^2 = 6 > 0.$$

Since $H_1 + H_2$ is nef and big, by Lemma 6.5.2, $|D_1|$ is also not composed with a rational pencil of surfaces. □

(6.5.3.3) Product of $\mathbb{P}^1$ and a del Pezzo surface

In this case, X is isomorphic to the product $\mathbb{P}^1$ and a del Pezzo surface S of degree d (n^o 27, 28 in Table 3, n^o 10 in Table 4, n^o $3 - 8$ in Table 5). Let p_i $(i = 1, 2)$ be the projection to the i-th factor. Choose $D_1 \in |p_1^* \mathcal{O}(2)|$ and choose $D_2 \in | - p_2^* K_S|$. Then D_1 and D_2 give a free splitting of $-K_X$ unless S is a del Pezzo surface of degree one. In particular, $|D_1|$ is composed with a rational pencil of del Pezzo surfaces of degree d.

(6.5.3.4) Blow-up of points

In this case, $f\colon X \to Q^3$ is the blow-up of $Q^3 \subset \mathbb{P}^4$ with center two points p and q on it which is not colinear (n^o 19 in Table 3). Denote by L the line bundle $\mathcal{O}_{\mathbb{P}^4}(1)|_{Q^3}$. Then $\{p, q\}$ is an intersection of members of $|L|$ since p and q is not colinear. By [MM86, Propostion 2.12], one has

$$-K_X \sim f^*(-K_Y - 2L) + 2(f^*L - E_1 - E_2),$$

where E_1 and E_2 are the exceptionl divisors over p and q, respectively. Set $D_1 = f^*(-K_Y - 2L)$ and set $D_2 = 2(f^*L - E_1 - E_2)$. Since D_1 is nef and big, $|D_1|$ is not composed with a rational pencil of surfaces. On the other hand, we have

$$D_2^2 \cdot f^*L = 4(f^*L)^3 = 8 > 0.$$

By Lemma 6.5.2, $|D_2|$ is also not composed with a rational pencil of surfaces.

(6.5.3.5) Projective bundles

In this case, the Fano manifold X is the $\mathbb{P}^1$-bundle $\pi\colon \mathbb{P}(\mathcal{E}) \to \mathbb{P}^1 \times \mathbb{P}^1$, where $\mathcal{E} = \mathcal{O} \oplus \mathcal{O}(1, 1)$ (n^o 31 in Table 3). We choose $D_1 \in |\mathcal{O}_{\mathbb{P}(\mathcal{E})}(2)|$ and $D_2 \in |\pi^* \mathcal{O}(1, 1)|$. Then $-K_X \sim D_1 + D_2$. Moreover, $|D_1|$ and $|D_2|$ are basepoint free and $|D_2|$ is not composed with a pencil of surfaces. On the other hand, we have

$$D_1^2(D_1 + 3D_2) = (-K_X)^3 - 3D_1 D_2^2 = (-K_X)^3 - 6\xi D_2^2 = 40 > 0,$$

where $\xi \in |\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)|$. Hence, by Lemma 6.5.2, $|D_1|$ is not composed with a rational pencil of surfaces.

(6.5.3.6) Fano threefolds of type I

For the rest of Tables 3, 4 and 5 in [MM81], X is of Type I. More precisely, X is the blow-up of a smooth projective threefold Y along a smooth (not necessarily irreducible) curve C such that C is an intersection of members of a complete linear system $|L|$ over Y. Moreover, $-K_X$ has a free splitting

$$-K_X \sim D_1 + D_2$$

satisfying $D_1 \sim f^*L - E$, where f denotes the blow-up $X \to Y$ and E is the exceptional divisor. The linear system $|D_2|$ is composed with a rational pencil of surfaces if and only if Y lies in $n^o\,5$ in Table 3 (cf. Proposition 6.5.5, Appendix B.2 and Appendix B.3).

6.5.12. Proposition. *For $n^o\,5$ in Table 3, $|D_2|$ is composed with a rational pencil of del Pezzo surfaces of degree 4.*

Proof. In this case, $Y \cong \mathbb{P}^2 \times \mathbb{P}^1$, C is a curve of bidegree $(2,5)$, and $D_2 \in |H_2|$, where $H_2 = p_2^*\mathcal{O}_{\mathbb{P}^1}(1)$ and $p_2 \colon Y \to \mathbb{P}^1$ is the projection to the second factor. Then the morphism defined by $|D_2|$ coincides with the morphism $X \xrightarrow{\pi} Y \xrightarrow{p_2} \mathbb{P}^1$. We will denote it by f. Then the general fiber S of f is isomorphic to the blow-up of $\mathbb{P}^2$ at 5 points, it follows that the degree of S is 4. This completes the proof. □

6.5.13. Remark. An alternative way to see this is to calculate the following quantity,

$$(K_X + \pi^*H_2)^2 \cdot \pi^*H_2 = (\pi^*(-3H_1 - 2H_2) + E + \pi^*H_2)^2 \cdot \pi^*H_2,$$

where $H_1 = p_1^*\mathcal{O}_{\mathbb{P}^2}(1)$ with $p_1 \colon Y \to \mathbb{P}^2$ the first projection and E is the exceptional divisor of π. Note that $E^2 \cdot \pi^*H_2 = -H_2 \cdot C = -5$, it follows that we have

$$(K_X + \pi^*H_2)^2 \cdot \pi^*H_2 = 9\pi^*(H_1)^2 \cdot \pi^*H_2 + E^2 \cdot \pi^*H_2 = 9 - 5 = 4.$$

Now we assume that $|D_2|$ is not composed with a rational pencil of surfaces. Then $|D_1|$ is composed with a rational pencil of surfaces if and only if C is a complete intersection of two members of $|L|$ (cf. Lemma 6.5.3). This happens for $n^o\,4, 7, 11, 24, 26$ in Table 3, and $n^o\,4, 9$ in Table 4, and $n^o\,1$ in Table 5 (cf. [MM86, §7], Appendix B.2 and Appendix B.3).

6.5.14. Proposition. *For $n^o\,4, 7, 11, 24, 26$ in Table 3, $n^o\,4, 9$ in Table 4 and $n^o\,1$ in Table 5. the linear system $|D_1| = |f^*L - E|$ is composed with a rational pencil of del Pezzo surfaces of degree at least 4.*

Proof. Let $X \xrightarrow{g} \mathbb{P}^1 \xrightarrow{h} \mathbb{P}^1$ be the factorization of Stein of Φ_{D_1}. We claim that h is an isomorphism. In fact, let E be the exceptional divisor of f and let F be a fiber of the morphism $f|_E \colon E \to C$. Then we have $g(F) = \mathbb{P}^1$, since $f^*L - E$ is strictly positive over F. As $(f^*L - E) \cdot F = 1$, we get $(f^*L - E)|_F = \mathcal{O}_{\mathbb{P}^1}(1)$. It follows that $g|_F \colon F \to \mathbb{P}^1$ is birational and $D_1 = g^*\mathcal{O}_{\mathbb{P}^1}(1)$. As an consequence, we obtain that h is birational and $D_2 = h^*g^*\mathcal{O}_{\mathbb{P}^1}(1)$. Let S be a general fiber of g. Then we have $S \sim D_1 \sim f^*L - E$ and

$$K_S^2 = (K_X + f^*L - E)^2 \cdot (f^*L - E) = (f^*K_Y + f^*L)^2 \cdot f^*L = (K_Y + L)^2 \cdot L.$$

Then an easy calculation shows that the degree of S is at least 4 (see Appendix B.2). □

Conclusion. Summarizing, we have proved in this section the following theorem.

6.5.15. Theorem. *Let X be a smooth Fano threefold with $\rho(X) \geq 2$. Assume moreover that $|-K_X|$ is basepoint free and there is a fibration $X \to \mathbb{P}^1$ of del Pezzo surfaces of degree d at most 3. Then $d \geq 2$ and we have*

(1) $d = 2$ if and only if X lies in $n^o\,2, 3$ in Table 2 or $n^o\,7$ in Table 5 ;
(2) $d = 3$ if and only if X lies in $n^o\,4, 5$ in Table 2, $n^o\,2$ in Table 3 or $n^o\,6$ in Table 5.

Now our main theorem follows immediately.

6.5.16. Theorem. *Let X be a smooth Fano threefold with $\rho(X) \geq 2$.*

(1) $\varepsilon(X, -K_X; 1)$=1 if X carries a del Pezzo fibration of degree one. To be explicit, X is isomorphic to one of the following.
   (1.1) The blow-up of V_1 with center an elliptic curve which is a complete intersection of two members of $\left|-\frac{1}{2}K_{V_1}\right|$ (n° 1 in Table 2).
   (1.2) The product $\mathbb{P}^1 \times S_1$ (n° 8 in Table 5).

(2) $\varepsilon(X, -K_X; 1) = 4/3$ if X carries a del Pezzo fibration of degree two. To be precise, X is isomorphic to one of the following.
   (2.1) A double cover of $\mathbb{P}^1 \times \mathbb{P}^2$ whose branch locus is a divisor of bidegree $(2, 4)$ (n° 2 in Table 2).
   (2.2) The blow-up of V_2 with center an elliptic curve which is a complete intersection of two members of $\left|-\frac{1}{2}K_{V_2}\right|$ (n° 3 in Table 2).
   (2.3) The product $\mathbb{P}^1 \times S_2$ (n° 7 in Table 5).

(3) $\varepsilon(X, -K_X; 1) = 3/2$ if X carries a del Pezzo fibration of degree three. More precisely, X is isomorphic to one of the following.
   (3.1) The blow-up of $\mathbb{P}^3$ with center an intersection of two cubics (n° 4 in Table 2).
   (3.2) The blow-up of $V_3 \subset \mathbb{P}^4$ with center a plane cubic on it (n° 5 in Table 2).
   (3.3) A member of $|\mathcal{O}_{\mathbb{P}(\mathcal{E})}(2) \otimes \pi^*\mathcal{O}(2, 3)|$ on the $\mathbb{P}^2$-bundle $\pi\colon \mathcal{E} \to \mathbb{P}^1 \times \mathbb{P}^1$ such that $X \cap Y$ is irreducible, where $\mathcal{E} = \mathcal{O} \oplus \mathcal{O}(-1, -1)^{\oplus 2}$ and $Y \in |\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)|$ (n° 2 in Table 3).
   (3.4) The product $\mathbb{P}^1 \times S_3$ (n° 6 in Table 5).

(4) $\varepsilon(X, -K_X; 1) = 3$ if X is isomorphic to the blow-up of $\mathbb{P}^3$ along a smooth plane curve C of degree at most three (n° 28, 30, 33 in Table 2).

(5) $\varepsilon(X, -K_X; 1) = 2$ if X lies in the other class in Tables 2, 3, 4 and 5.

Proof. It follows from Proposition 6.4.12, Theorem 6.4.13, Corollary 6.4.8 and Theorem 6.5.15. □

I Appendix : Anticanonical divisor of Fano threefolds with $\rho = 2$

In the proof of Proposition 6.5.6 , we used the following theorem. In [MM83], Mori and Mukai explained the general principle to prove this theorem. For lack of details in the literature, we give a complete proof of the cases used in Proposition 6.5.6 for reader's convenience. We denote by $(\star - \star\star)$ the case in which the ray R_1 is of $\star$-type and R_2 is of $\star\star$-type.

I.1. Theorem [MM83, Theorem 5.1]. *Let X be a smooth Fano threefold with $\rho(X) = 2$. Let R_i be the extremal rays of X with associated contractions $\varphi_i\colon X \to Y_i$. Let H_i be the generator of the group $\mathrm{Pic}(Y_i)$. Then we have $\mathrm{Pic}(X) = \mathbb{Z}\varphi_1^*H_1 + \mathbb{Z}\varphi_2^*H_2$ and*

$$-K_X \sim l(R_1)\varphi_1^*H_1 + l(R_2)\varphi_2^*H_2.$$

Proof. We focus on the cases $(D_1 - C_1)$, $(D_1 - C_2)$ and (D_1, E_1). In view of the proof of [MM83, Theorem 5.1], there is an integer a such that

$$\mathrm{Pic}(X)/\left(\mathbb{Z}\varphi_1^*H_1 Z + \mathbb{Z}\varphi_2^*H_2\right) \cong \mathbb{Z}/a\mathbb{Z}$$

and

$$a(-K_X) \sim l(R_1)\varphi_1^*H_1 + l(R_2)\varphi_2^*H_2.$$

Moreover, a satisfies the following equality,

$$24a = l(R_2)(\varphi_1^*H_1 \cdot c_2(X)) + l(R_1)(\varphi_2^*H_2 \cdot c_2(X)).$$

Thus, it suffices to show $a = 1$.

Case 1. $(D_1 - C_1)$. Let S_d be the general fiber of f_1, and let Δ be the discriminant locus of f_2. By [MM83, Lemma 5.4], we have

$$c_2(X) \cdot \varphi_1^* H_1 = 12 - (K_{S_d})^2 \quad \text{and} \quad c_2(X) \cdot \varphi_2^* H_2 = \deg \Delta + 6.$$

As $Y_2 \cong \mathbb{P}^2$, by the formula $\Delta \sim -4K_{\mathbb{P}^2} - (\varphi_2)_*(-K_X)^2$, we have $\deg(\Delta) < 12$. Moreover, we have also $l(R_1) = l(R_2) = 1$. It follows that

$$24a < 12 + 6 + 12 = 30.$$

It follows that $a = 1$.

Case 2. $(D_1 - C_2)$. By [MM83, Lemma 5.4], we have $c_2(X) \cdot \varphi_2^* H_2 = 6$. On the other hand, we have $l(R_1) = 1$ and $l(R_2) = 2$. Then, we get

$$24a < 2 \times 12 + 6 = 30.$$

It follows that $a = 1$.

Case 3. $(D_1 - E_1)$. By [MM83, Lemma 5.9], we have $c_2(X) \cdot \varphi_2^* H_2 \leq 31$. Moreover, since we have $l(R_1) = l(R_2)$, we get

$$24a < 12 + 31 = 43.$$

It follows that $a = 1$. □

I.2. **Remark.** In the case $(D_1 - C_2)$, the map $\varphi = (\varphi_1, \varphi_2) \colon X \to \mathbb{P}^1 \times \mathbb{P}^2$ is a finite morphism of degree one, so φ is actually an isomorphism ; that is, $X \cong \mathbb{P}^1 \times \mathbb{P}^2$. Thus, in Proposition 6.5.6, this can not happen as we assume that the general fiber of φ_1 is a del Pezzo surface of degree at most 3.

Chapitre 7

Moishezon manifolds with Picard number one

In this chapter, we study the Moishezon manifolds with Picard number one. Recall that a compact complex manifold X is called *Moishezon* if the transcendence degree of its field of meromorphic functions is equal to its dimension, or equivalently X is bimeromorphically equivalent to some projective manifold. They are natural objects appearing in algebraic geometry even if we consider only the projective manifolds since it is conjectured that the limit of projective manifolds under smooth holomorphic deformation is always a Moishezon manifold (cf. [Pop13, Bar16]). For general definitions of singular complex spaces and the basic properties of analytic coherent sheaves, we refer to the books [Uen75], [GR84] and [GPR94].

7.1 Rigidity problems of Fano manifolds

For a nice historical overview on the rigidity problem of complex structures on irreducible Hermitian symmetric spaces of compact type, we refer the reader to Siu's survey [Siu91].

7.1.1. Question. *Is it possible to characterize the symmetric spaces, or more generally Fano manifolds with Picard number one, by topological or curvature or other conditions ?*

The approach by topological conditions asks under what additional condition a complex manifold diffeomorphic to a given model manifold is biholomorphic to it. The natural additional condition one can consider is the Kähler condition, the Moishezon condition, and the deformation condition.

7.1.1 Kähler Condition

For the case of complex projective spaces, Hirzebruch and Kodaira proved the following result in [HK57].

7.1.2. Theorem. *Let X be a n-dimensional compact Kähler manifold which is C^∞ differentially homeomorphic to the complex projective space $\mathbb{P}^n$. Then X is biholomorphic to $\mathbb{P}^n$ if n is odd. If n is even, the same result holds with the additional condition that the anticanonical line bundle is not negative.*

Yau applied the theory of Kähler-Einstein metrics in [Yau77] to remove the additional assumption of the nonnegativity of the anticanonical line bundle in the case of even-dimensional complex projective spaces. This result was generalized to complex quadric hypersurfaces by Brieskorn in [Bri64, Satz 1].

7.1.3. Theorem. *Let X be a compact Kähler manifold of dimension $n \geq 3$ which is C^∞ differentially homeomorphic to a smooth quadric hypersurface Q^n. Then X is biholomorphic to Q^n if n is odd. If n is even, the same result holds with the additional condition that the anticanonical line bundle is not negative.*

Here one must make the exception $n \neq 2$ since the Hirzebruch surfaces $\Sigma_{2n} = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-2n))$ are diffeomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ for $n > 0$. As so far, whether the additional assumption in the even-dimensional case can be removed is still unknown. We end this subsection with a conjecture for general irreducible Hermitian symmetric manifolds of compact type.

7.1.4. **Conjecture [Siu91, Conjecture 3.3].** *Let X be a compact Kähler manifold that is C^∞ differentially homeomorphic to an irreducible Hermitian symmetric space M of compact type. Then X is biholomorphic to M.*

7.1.2 Moishezon Condition

The first attempt into this direction is to try to generalize the result of Hirzebruch-Kodaira and Brieskorn by replacing the Kähler condition with Moishezon condition. More precisely, we consider the following question.

7.1.5. **Question.** *Let X be a n-dimensional Moishezon manifold which is C^∞ differentially homeomorphic to $\mathbb{P}^n$ (resp. Q^n). Is X biholomorphic to $\mathbb{P}^n$ (resp. Q^n)?*

For $n = 1$ and 2, X is actually a projective manifold. For $n = 3$, Nakamura proved the following result.

7.1.6. **Theorem [Nak87, Nak88].** *Let X be a smooth Moishezon threefold which is C^∞ homeomorphic to the complex projective space $\mathbb{P}^3$ (resp. Q^3). If the Kodaira dimension of X is less than three, then X is biholomorphic to $\mathbb{P}^3$ (resp. Q^3).*

By studying the fundamental linear system over Moishezon threefolds with Picard number one, Kollár proved the following theorem.

7.1.7. **Theorem [Kol91b, Theorem 5.3.4].** *Let X be a smooth Moishezon threefold with* $\mathrm{Pic}(X) = \mathbb{Z}$*. Let L be the big generator of* $\mathrm{Pic}(X)$*. Let r be the integer such that $K_X = -rL$. Then X is biholomorphic to $\mathbb{P}^3$ (resp. Q^3) if $r = 4$ (resp. r=3).*

As a corollary, Kollár gave an affirmative answer to Question 7.1.5 in the case $n = 3$ without extra assumptions (see [Kol91b, Corollary 5.3.5]). We remark that another different proof was given in a series works of Peternell (see [Pet85, Pet86a, Pet86b]). For $n \geq 4$, the solution to Question 7.1.5 is still largely open. Some partial results were obtained by Peternell and Nakamura in [Pet86b, Nak92, Nak94].

7.1.3 Deformation Condition

An easier problem is the question of global deformation of $\mathbb{P}^n$ posed by Kodaira and Spencer [KS58]. The problem is as follows : suppose that $\pi\colon X \to \Delta$ is a smooth holomorphic family of compact complex manifolds parametrized by the unit disk in $\mathbb{C}$ such that for every nonzero t the fiber $X_t := \pi^{-1}(t)$ is biholomorphic to $\mathbb{P}^n$, does it follow that the fiber X_0 at $t = 0$ is also biholomorphic to $\mathbb{P}^n$? More generally, comparing with Conjecture 7.1.4, one may expect that the following stronger rigidity holds.

7.1.8. **Conjecture [Hwa06, Conjecture 3.2].** *Let $\pi\colon X \to \Delta$ be a smooth holomorphic family of compact complex manifolds parametrized by the unit disk Δ in $\mathbb{C}$. If for every nonzero t, the fiber $X_t := \pi^{-1}(t)$ is biholomorphic to a rational homogeneous space G/P with $b_2 = 1$, then the fiber X_0 at $t = 0$ is also biholomorphic to G/P.*

In our situation, the fiber X_0 is Moishezon by an argument of the semicontinuity of $H^0(X_t, -mK_{X_t})$ in t (see Proposition 7.4.2). For the case of $\mathbb{P}^n$ and Q^n, Conjecture 7.1.8 were proved by Siu and Hwang in [Siu89] and [Hwa95], respectively. Moreover, we remark that one can also derive the result for $n = 3$ and 4 from the works of Kollár, Peternell and Nakamura mentioned in the last subsection. Note that by the results of Hirzebruch-Kodaira and Brieskorn, the main difficulty is the lack of Kähler condition on the centeral fiber X_0, or equivalently the projectivity of X_0. Mok and Hwang proved Conjecture 7.1.8 under Kähler deformation in a series of works, we refer the reader to [HM98, HM02, HM05] and the references therein.

7.1.9. Theorem [HM05, Main Theorem]. *Let $\pi\colon X \to \Delta$ be a smooth projective family of compact complex manifolds parametrized on by the unit disk Δ in $\mathbb{C}$. If for every nonzero t, the fiber $X_t :=\pi^{-1}(t)$ is biholomorphic to a rational homogeneous space G/P with Picard number one other than the 7-dimensional Fano homogeneous contact manifold $\mathbb{F}^5$. Then the central fiber X_0 is also biholomorphic to G/P.*

In the exceptional case, it was found by Pasquier-Perrin in [PP10, Proposition 2.3] that $\mathbb{F}^5$ admits a deformation to a non-homogeneous G_2-horospherical variety X^5. Moreover, we remark that the limit of projective manifolds is not necessarily projective as showen by the famous example of Hironaka. However, for Fano manifolds with Picard number one, we have the following conjecture which will imply Conjecture 7.1.8 by combining with the theorem of Mok-Hwang (with the exceptional case $\mathbb{F}^5$).

7.1.10. Question. *Let $\pi\colon X \to \Delta$ be a smooth family of compact complex manifolds parametrized by the unit disk Δ in $\mathbb{C}$. Suppose that for every nonzero t, the fiber $X_t :=\pi^{-1}(t)$ is biholomorphic to a Fano manifold S with Picard number one. Is the central fiber X_0 also projective ?*

By the works of Siu and Hwang, there is an affirmative answer if S is biholomorphic to a projective space or a smooth quadric hypersurface. On the other hand, in view of the index of Fano manifolds, projective spaces and smooth quadric hypersurfaces are the Fano manifolds with coindex at most one, so it is natural to consider the Fano manifolds with coindex two, namely del Pezzo manifolds. In particular, in dimension three, by the works of Kollár, Nakamura and Dorsch, we have a positive answer for almost all del Pezzo threefolds.

7.1.11. Theorem [Kol91b, Nak96, Dor14]. *Let $\pi\colon X \to \Delta$ be a smooth holomorphic family of compact complex manifolds parametrized by the unit disk Δ in $\mathbb{C}$. If for every nonzero t, the fiber $X_t :=\pi^{-1}(t)$ is biholomorphic to a smooth del Pezzo threefold Y with Picard number one such that the degree d of Y is at least 2, then the central fiber X_0 is also a del Pezzo threefold of degree d.*

Recall that for a del Pezzo manifold X, there exists always a smooth member in its fundamental system [Fuj77a]. This leads us to ask the following weaker good divisor question.

7.1.12. Question. *Let $\pi\colon X \to \Delta$ be a smooth family of compact complex manifolds parametrized by the unit disk Δ in $\mathbb{C}$. Suppose that for every nonzero t, the fiber $X_t :=\pi^{-1}(t)$ is biholomorphic to a del Pezzo manifold Y with Picard number one. Let L_0 be the big generator of* $\mathrm{Pic}(X_0)$. *Does there exist a smooth member in the linear system $|L_0|$?*

7.2 Basic materials on Moishezon manifolds

Let X be an irreducible reduced compact complex space. To study X from a bimeromorphic point of view means to study its field of meromorphic functions $K(X)$. In fact, if X' is bimeromorphic to X, then the function fields $K(X)$ and $K(X')$ are isomorphic and vice versa. The size of $K(X)$ is measured by its transcendence degree over $\mathbb{C}$.

7.2.1 Algebraic dimension and basic properties

7.2.1. Definition. *Let X be an irreducible reduced compact complex space. The transcendence degree $a(X)$ of $K(X)$ over $\mathbb{C}$ is called the algebraic dimension of X.*

It is well-known that we have always $a(X) \leq \dim(X)$.

7.2.2. Definition. *Let X be an irreducible reduced compact complex space. Then X is called a Moishezon space if $a(X) = \dim(X)$.*

By using the algebraic reduction and a strong form of elimination of indeterminacies, one can derive the following important characterizations of Moishezon spaces.

7.2.3. Proposition [GPR94, VII, Corollary 6.10]. *Let X be an irreducible reduced Moishezon space. Then there exists a modification $\pi\colon \widehat{X} \to X$ such that $\widehat{X}$ is projective. If X is smooth, we can achieve this also by a finite sequence of blow-ups with smooth centers.*

7.2.4. Definition. *A line bundle L on a compact Moishezon manifold X is said to be*

(1) *nef if for every $C \subset X$ complex compact curve in X, we have $L \cdot C \geq 0$.*
(2) *big if its Kodaira-Iitaka dimension, denoted by $\kappa(L)$, is equal to the dimension of X.*

7.2.5. Remark. In complex geometry, our definition of nefness is usually called algebraically nef or nef in the curve sense. However, on Moishezon manifolds, Păun proved that the nefness in the curve sense is equivalent to the nefness in a metric sense ([Pău98, Corollaire]).

Recall that if X is a compact complex manifold and $x \in X$, then X is projective if and only if the blow-up of X at x is projective. Thus, every smooth Moishezon surface is projective. Moreover, it is not difficult to see that an irreducible reduced complex compact space is Moishezon if and only if it carries a big line bundle L. The following projectivity criterion due to Moishezon is fundamental.

7.2.6. Theorem [Moi66]. *Let X be a Moishezon manifold. Then X is projective if and only if X is Kähler.*

7.2.2 Vanishing theorems on Moishezon manifolds

Instead of Kodaira's vanishing theorem on projective manifolds, we have the following vanishing result due to Kollár for "almost positive" line bundle on Moishezon manifolds. Recall that a line bundle L is called *ample in codimension one* if there exists a codimension two subset $Z \subset X$ and a positive integer m such that the composite map

$$X \setminus Z \hookrightarrow X \overset{\Phi}{\dashrightarrow} \mathbb{P}^N$$

is an embedding, where Φ is the meromorphic map defined by the global sections of $L^{\otimes m}$.

7.2.7. Theorem [Kol91b, Lemma 5.3.8]. *Let X be a normal Moishezon space, and let L be a line bundle over X such that L is ample in codimension one. Set $n = \dim(X)$. Then*

$$H^{n-1}(X, \omega_X \otimes L) = 0.$$

It is well-known that Fano manifolds are simply connected. In the following result, we see that this still holds for Moishezon manifolds with $b_2(X) = 1$ and big anticanonical divisor. Before giving the statement, we recall that the *maximal rationally connected* (MRC) quotient of a compact complex manifold X lying in the Fujiki class is an almost holomorphic $\varphi\colon X \dashrightarrow T$ such that the general fiber of φ is rationally chain-connected and T is not uniruled (see [Cam04, §3.4] and see also [Cam92] and [KMM92] in the projective case).

7.2.8. Proposition. *Let X be a Moishezon manifold with $b_2(X) = 1$ such that ω_X^{-1} is a big line bundle. Then X is simply connected. In particular, we have $H^1(X, \mathcal{O}_X) = 0$.*

Proof. Let $\pi\colon \widehat{X} \to X$ be a modification of X such that $\widehat{X}$ is projective. Then $K_{\widehat{X}} = \pi^* K_X + E$ where E is an effective π-exceptional divisor. In particular, $\widehat{X}$ is uniruled since $K_{\widehat{X}}$ is not pseudoeffective (cf. [BDPP13]). Thus, X is also uniruled. Let $\varphi\colon X \dashrightarrow T$ be the maximally rationally connected quotient (see [Cam04, Theorem 3.23]). As $\rho(X) = 1$, then φ is trivial and X is rationally chain-connected. Therefore, $\widehat{X}$ is rationally connected. As a consequence, $\widehat{X}$ is simply connected (cf. [Deb01, Corollary

4.29]). Since the fundamental group is invariant under the bimeromorphic map (see [Tak03]), it follows that X is also simply connected. Then we can conclude that $H^1(X, \mathcal{O}_X)$ by Hodge decomposition. $\square$

As an immediate application of the two vanishing results above, we get the following corollary.

7.2.9. Corollary. *Let X be a n-dimensional Moishezon manifold with $b_2(X) = 1$ such that $n \geq 3$. Let $\mathcal{O}_X(1)$ be the big generator of* $\mathrm{Pic}(X)$ *and assume that $\omega_X^{-1} = \mathcal{O}_X(r)$ for some $r > 0$. Then we have*

$$h^0(X, \mathcal{O}_X(\ell)) = 0 \text{ if } \ell < 0, \qquad h^n(X, \mathcal{O}_X(\ell)) = 0 \text{ if } \ell > -r;$$
$$h^1(X, \mathcal{O}_X(\ell)) = 0 \text{ if } \ell \leq 0, \qquad h^{n-1}(X, \mathcal{O}_X(\ell)) = 0 \text{ if } \ell \geq -r.$$

Proof. By Serre duality, Theorem 7.2.7 and Proposition 7.2.8, it is enough to show that $\mathcal{O}_X(1)$ is ample in codimension one. Let $\pi\colon \widehat{X} \to X$ be a projective resolution. Let A be a very ample divisor over $\widehat{X}$. Then $\mathcal{O}_X(\pi_* A) \cong \mathcal{O}_X(m)$ for some $m > 0$ and there exists a π-exceptional divisor E such that

$$\mathcal{O}_{\widehat{X}}(A) = \pi^*\mathcal{O}_X(\pi_* A) \otimes \mathcal{O}_{\widehat{X}}(E) = \pi^*\mathcal{O}_X(m) \otimes \mathcal{O}_{\widehat{X}}(E).$$

Since $\mathcal{O}_{\widehat{X}}(A) \otimes \pi^*\mathcal{O}_X(-m)$ is π-nef, by negativity lemma, $-E$ is effective. Denote by Z the subvariety $\pi(Ex(\pi))$. Then $\mathrm{codim}_X(Z) \geq 2$ and the meromorphic map defined by $|\mathcal{O}_X(m)|$ is biholomorphic over $X \setminus Z$ since A is very ample. $\square$

7.2.10. Remark. The negativity lemma is usually stated for proper birational morphisms between projective varieties (cf. [KM98, Lemma 3.39]). Nevertheless, if $f\colon Y \to X$ be a proper holomorphic map of analytic surfaces (not necessarily compact) with exceptional curves E_i. Then the intersection matrix $(E_i \cdot E_j)$ is negative (cf. [Gra62, p.367]). In higher dimension, if $f\colon Y \to X$ be a bimeromorphic projective map between smooth complex manifolds (not necessarily compact), since the negativity lemma is local in X, the proof of [KM98, Lemma 3.39] can be adapted to our situation.

In dimension two, one can easily derive the following result by the vanishing theorem above and the Riemann-Roch formula.

7.2.11. Corollary [Kol91b, Corollary 5.3.10]. *Let X be a Moishezon threefold such that $b_2(X) = 1$. Let L be the big generator of* $\mathrm{Pic}(X)$*. Assume that $-K_X = rL$ for some $r > 0$. Then* $\mathrm{Pic}(X)$ *has no torsion, and*

(1) $r \leq 4$;
(2) *if* $r = 4$, *then* $L^3 = 1$ *and* $\chi(X, kL) = \frac{1}{6}(k+1)(k+2)(k+3) + 1$;
(3) *if* $r = 3$, *then* $L^3 = 2$ *and* $\chi(X, kL) = \frac{1}{6}(k+1)(k+2)(2k+3)$;
(4) *if* $r = 2$, *then* $\chi(X, kL) = \frac{1}{6}L^3 k(k+1)(k+2) + k + 1$.

7.2.12. Notation. *If X is a smooth Moishezon threefold with $b_2(X) = 1$ and L is the big generator of* $\mathrm{Pic}(X)$*, we call r the index of X if $-K_X = rL$.*

Using the results above, Kollár proved the following classification theorem by careful analysis of the rational map defined by $|L|$.

7.2.13. Theorem [Kol91b, Theorem 5.3.4 and Theorem 5.3.12]. *Let X be a smooth Moishezon threefold with $b_2(X) = 1$. Let r be the index of X and let L be the big generator of* $\mathrm{Pic}(X)$*. Then we have the following results.*

(1) $r = 4$ *if and only if* $X \cong \mathbb{P}^3$ *and* $r = 3$ *if and only if* $X \cong Q^3$.
(2) *If* $r = 2$ *and* $N = h^0(X, L) \geq 4$, *then the meromorphic map Φ defined by $|L|$ is bimeromorphic. Moreover, if Φ is not a morphism, then $N \leq 5$ such that one of the following holds.*
  (2.1) $N = 4$, *and* $\Phi\colon X \dashrightarrow \mathbb{P}^3$ *is bimeromorphic.*
  (2.2) $N = 5$, *and* $\Phi\colon X \dashrightarrow Q^3$ *is bimeromorphic.*

7.2.3 Examples of non-projective Moishezon manifolds

In this subsection, we collect various different constructions of non-projective Moishezon threefolds of Picard number one. Let us recall the Nakano-Fujiki criterion for blow-down by a monoidal transformation in analytic category.

7.2.14. Theorem [FN71]. *Let X be a complex manifold, and let A be a closed subspace of the form $\mathbb{P}(\mathcal{E})$, where $\mathcal{E}$ is a locally free sheaf on a complex space B. Let $p\colon A \to B$ be the projection. Let $\mathcal{I}_A$ be the ideal sheaf of A in X. Assume furthermore :*

(1) $\operatorname{codim}_X A = 1$,

(2) $\mathcal{I}_A/\mathcal{I}_A^2 \cong \mathcal{O}_{\mathbb{P}(\mathcal{E}')}(1)$, *where $\mathcal{E}' \cong \mathcal{E} \otimes \mathcal{L}$ for some line bundle $\mathcal{L}$ over B.*

Then there exists a monoidal transformation $\varphi\colon X \to Y$ with center $B \subset Y$ such that $\varphi|_A = p$.

(7.2.3.1) Small resolution of nodal threefolds

Let $Y_n \subset \mathbb{P}^4$ be a nodal hypersurface of degree n in $\mathbb{P}^4$. Thus Y_n is smooth away from a finite number of nodes which are given in local analytic coordinates by an equation

$$Q\colon = \{x_1x_2 = x_3x_4\} \subset \mathbb{C}^4.$$

Let $P \in Y_n$ be a node. We have two different ways to get a resolution of P.

(1) **The big resolution.** We can blow up P in the usual way : $\pi\colon \widehat{Y_n} \to Y_n$ is the blow-up of Y_n in P. Then $\pi^{-1}(P) \cong \mathbb{P}^1 \times \mathbb{P}^1$. This kind of resolution is called *big* since P is replaced by a divisor.

(2) **The small resolution.** We can use the local meromorphic function

$$\frac{x_1}{x_3} = \frac{x_4}{x_2} \tag{7.1}$$

to resolve the point P. Denote by U a neighborhood of P in Y_n. Outside the locus where $x_1 = x_3 = 0$, we have a map

$$\begin{aligned} \sigma\colon U &\longrightarrow \quad U \times \mathbb{P}^1 \\ x &\longmapsto (x, [x_1\colon x_3])\,. \end{aligned}$$

Then outside the locus $x_2 = x_4 = 0$, the image of σ coincides with the image of the following map

$$\begin{aligned} \sigma'\colon U' &\longrightarrow \quad U' \times \mathbb{P}^1 \\ x &\longmapsto (x, [x_4\colon x_2])\,. \end{aligned}$$

Thus, outside the point P, we get an embedding $U \to U \times \mathbb{P}^1$. Denote by $\widehat{U}$ the closure of the image. Let $V = U \times \mathbb{C}$ be the open subset of $U \times \mathbb{P}^1$ by identifying (x, y) with $(x, [1 : y])$. Then the subvariety $(V \cap \widehat{U}) \subset \mathbb{C}^4 \times \mathbb{C}$ is defined by the following equations

$$\begin{cases} x_4y = x_2; \\ x_1y = x_3; \\ x_1x_2 = x_3x_4. \end{cases} \tag{7.2}$$

The corresponding Jacobian matrix is as follows :

$$\begin{pmatrix} 0 & -1 & 0 & y & x_4 \\ y & 0 & -1 & 0 & x_1 \\ x_2 & x_1 & -x_4 & -x_3 & 0 \end{pmatrix}$$

Then it is easy to see that $V \cap \widehat{U}$ is smooth and the same argument can be applied to show that $\widehat{U}$ is smooth. We denote by $\pi\colon \widehat{U} \to U$ the natural projection, then $L_p\colon = \pi^{-1}(P) \cong \mathbb{P}^1$. By replacing U with $\widehat{U}$, we obtain a resolution $\widetilde{Y}_n$ in P. The other small resolution $\widetilde{Y}'_n$ is obtained by using the other meromorphic function

$$\frac{x_1}{x_4} = \frac{x_3}{x_2}.$$

Denote by L'_p the preimage of P in $\widetilde{Y}'_n$. Then the blow-up of $\widetilde{Y}_n$ along L_p and the blow-up of $\widetilde{Y}'_n$ along L'_p are both isomorphic to the big resolution $\widehat{Y}_n$. In fact, $\widetilde{Y}_n$ and $\widetilde{Y}'_n$ can be obtained by contracting the different directions in $\pi^{-1}(P) \cong \mathbb{P}^1 \times \mathbb{P}^1$.
Note that the above construction is in the analytic category, so in general the small resolutions are not projective, or equivalently not Kähler. As the small resolution only changes a subvariety of codimension two, so we have the following isomorphism between divisor classes groups

$$Cl(Y_n) \cong Cl(\widetilde{Y}_n) \cong \operatorname{Pic}(\widetilde{Y}_n).$$

In particular, if Y_n is $\mathbb{Q}$-factorial, then we get $\operatorname{Pic}(Y_n) \otimes \mathbb{Q} \cong \operatorname{Pic}(\widetilde{Y}_n) \otimes \mathbb{Q}$, and we see that $\widetilde{Y}_n$ is never projective in this case since L_p is numerically zero. For instance, the classification of small resolutions of nodal cubic threefolds were given by Finkelnberg and Werner in the article [FW89]. However, in general a nodal hypersurface is not necessarily $\mathbb{Q}$-factorial. Recall that the local class group at a node in a threefold has no torsion, and hence each Weil divisor that is $\mathbb{Q}$-Cartier must be a Cartier divisor on a nodal hypersurface. In particular, the $\mathbb{Q}$-factoriality is equivalent to the factoriality on a nodal hypersurface. The following example due to Cheltsov shows that there do exist non factorial nodal hypersurfaces.

7.2.15. Example [Che05, Example 6]. Let $Y_n \subset \mathbb{P}^4$ be a hypersurface defined the following equation

$$x_0 g_{n-1}(x_0, x_1, x_2, x_3, x_4) + x_1 f_{n-1}(x_0, x_1, x_2, x_3, x_4) = 0 \subset \mathbb{P}^4,$$

where g_{n-1} and f_{n-1} are general polynomials of degree $n-1$. Then Y_n is nodal and contains the plane $x_0 = x_1 = 0$. The threefold Y_n is not $\mathbb{Q}$-factorial since the hyperplane $x_0 = 0$ splits into two non Cartier divisors while the Picard group of Y_n is generated by a hyperplane (see [AF59, Theorem 2]). Note that the number of nodes on Y_n is $(n-1)^2$.

In [Che05, Theorem 4], Cheltsove proved that Y_n must be $\mathbb{Q}$-factorial if the number of nodes on Y_n is at most $(n-1)^2/4$. The details of such a construction for cubic threefolds were carried out by Nakamura in [Nak96, §1].

(7.2.3.2) Flops of Calabi-Yau threefolds

(1) The following example is due to Oguiso (see [Ogu94]). A Calabi-Yau threefold of type $(2,4)$ is a complete intersection of a quadric and a quartic in $\mathbb{P}^5$. For any positive integer d, Oguiso proved that there exists a Calabi-Yau threefold X_d of type $(2,4)$, which contains a smooth rational curve C_d of degree d whose normal bundle is $N_{C_d|X_d} = \mathcal{O}_{C_d}(-1)^{\oplus 2}$. Now we consider the pair (C_2, X_2) and take an elementary transformation of X_2 along C_2 :

$$C_2 \subset X_2 \xleftarrow{\pi_1} C_2 \times D = E \subset \widehat{X}_2 \xrightarrow{\pi_2} D \subset Y,$$

where π_1 is the blowing-up of X_2 along $C_2 = \mathbb{P}^1$, $E = C \times D = \mathbb{P}^1 \times \mathbb{P}^1$ is the exceptional divisor on $\widehat{X}_2$. Since we have $E|_E = (-1,-1)$, there exist a contraction π_2 along C (cf. Theorem 7.2.14). Thus, Y is a smooth Calabi-Yau threefold with $\operatorname{Pic} Y = \mathbb{Z}L$, where L is the proper transform of H. Since $H \cdot C_2 = 2$, we have $\pi_1^* H = \pi_2^* L - 2E$. We calculate $L \cdot D = \pi_2^* L \cdot D = (\pi_1^* H + 2E) \cdot D = -2$.

(2) The following example is due to Bonavero (see [Bon96, §3.2]). Thanks to [Bon96, Proposition 1], there exists three homogeneous polynomials h_0, h_1, h_2 of degree four such that the hypersurface $Y \subset \mathbb{P}^4$ of degree five defined by the homogeneous polynomial $g = x_0 h_0 + x_1 h_1 + x_2 h_2 = 0$

is smooth and it contains a line $B = \mathbb{P}^1$ such that $N_{B/Y} \cong \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2}$. Let $f\colon \widehat{X} \to Y$ be the blow-up of Y along B and denote by $E \cong \mathbb{P}^1 \times \mathbb{P}^1$ the exceptional divisor of f. Then Bonavero showed that we have $\mathcal{O}_{\widehat{X}}(E)|_E \cong \mathcal{O}_E(-1,-1)$. Therefore, we can contract the direction which is different from π to get a smooth Moishezon threefold X such that $K_X = 0$ and $\operatorname{Pic}(X) \cong \mathbb{Z}$.

(7.2.3.3) Bimeromorphic models of Q^3 and $\mathbb{P}^3$

There are infinitely many examples of non-projective bimeromorphic models of Q^3 and $\mathbb{P}^3$ with $b_2 = 1$. The following first two families of examples were given in [Kol91b, Examples 5.3.14], and Kollár attributes them to Hironaka and Fujiki.

(1) In $\mathbb{P}^3$, take a smooth quadric Q^2 and blow up a smooth curve of type $(3, 6-m)$ on Q^2 for some integer $m \in \mathbb{Z}$. The proper transform of the quadric has normal bundle $(-1, m-4)$, By Theorem 7.2.14, it can be contracted in one direction to get a smooth Moishezon threefold X with $\operatorname{Pic} X = \mathbb{Z}L$ where L is the strict transform of a hyperplane H in $\mathbb{P}^3$.

$$Q^2 \subset \mathbb{P}^3 \xleftarrow{\pi_1} \widehat{Q}^2 \subset \widehat{X} \xrightarrow{\pi_2} C \subset X.$$

We have $L^3 = m$, $\operatorname{Bs}|L| = C \cong \mathbb{P}^1$ and $L \cdot C = m-3$. We denote X by $B_{(3,6-m)}(\mathbb{P}^3)$. Note that $B_{(3,6-m)}(\mathbb{P}^3)$ is not projective for $m \leq 3$ and L is not nef if and only if $m \leq 2$.

(2) Similarly, taking a smooth quadric Q^2 in Q^3 and blowing-up a curve of type $(2, 6-m)$ for an integer $m \in \mathbb{Z}$. The strict transform of Q^2 has normal bundle $(-1, 5-m)$, hence it can be contracted in one direction to get a smooth Moishezon threefold X with $\operatorname{Pic} X = \mathbb{Z}L$ where L is the strict transform of a hyperplane section of Q^3.

$$Q^2 \subset Q^3 \xleftarrow{\pi_1} \widehat{Q}^2 \subset \widehat{X} \xrightarrow{\pi_2} C \subset X.$$

Moreover, we have $L^3 = m$, $\operatorname{Bs}|L| = C \cong \mathbb{P}^1$ and $L \cdot C = m-4$. We denote X by $B_{(2,6-m)}(Q^3)$. Note that $B_{(2,6-m)}(Q^3)$ is not projective for $m \leq 4$ and L is not nef if and only if $m \leq 3$.

(3) Dorsch's example for reducible base locus [Dor14, §4] : choose two distinct hyperplane H_1 and H_2 in $\mathbb{P}^3$ and choose also four distinct points p_1, p_2, p_3 and p_4 lying on the intersection line $C_0 = H_1 \cap H_2$. Find two smooth plane cubics

$$C_1 \subset H_1, \qquad C_2 \subset H_2$$

such that

$$C_1 \cap C_0 = \{p_1, p_2, p_3\}, \qquad C_2 \cap C_0 = \{p_1, p_2, p_4\}.$$

Let $f\colon Y \to \mathbb{P}^3$ be the twisted blow-up of $\mathbb{P}^3$ along C_1 and C_2, i.e., locally near p_1 we first blow-up C_1 and then the strict transform of C_2 and locally near p_2 we proceed in reversed order. Let $\widehat{f}\colon \widehat{Y} \to Y$ be the blow-up of Y along C_0', where C_0' is the strict transform of C_0 under f. Denote by $\widehat{E} = \mathbb{P}^1 \times \mathbb{P}^1$ the exceptional divisor of $\widehat{f}$, and denote by $\widehat{H}_i$ the strict transform of H_i, respectively. Then we have a contraction $\widetilde{f}\colon \widehat{Y} \to \widetilde{Y}$ contracting $\widehat{E}_0$ to a rational curve in the different direction of $\widehat{f}$. Moreover, the images $\widetilde{H}_i$ of $\widehat{H}_i$ are disjoint and both $\widehat{H}_i$ can be contracted to a rational curve. Finally, we get a smooth Moishezon threefold X such that $\operatorname{Pic}(X) = \mathbb{Z}L$. Then the base locus of $|L|$ consists of the images of $\widetilde{H}_i$. In particular, it is reducible. Here we remark that if we take two general elements D and D' in $|L|$. Then the mobile part of $D \cap D'$ intersects the base locus of $|L|$ in two different points. For the details of the construction of the contractions, we refer the reader to [Dor14].

(4) Compactification of $\mathbb{C}^3$ by an irreducible divisor [Nak96, §2.2] : Nakamura constructed some bimeromorphic models of Q^3 which are compactifications of $\mathbb{C}^3$ by an irreducible divisor. In general, the non-projective compactification X of $\mathbb{C}^3$ by an irreducible divisor D with $b_2(X) = 1$ are investigated in a series works of Furushima (see [Fur94, Fur96, Fur98, Fur99, Fur07] etc.).

7.2.4 Topology of Moishezon threefolds with index two

The key ingredient of the papers [Nak96, Dor14] is to show that a non-projective Moishezon threefold is not homeomorphic to a given Fano manifold. In general, Wall classified smooth, simply-connected closed 6-dimensional manifolds M with torsion-free homology and vanishing second Stiefel-Whitney class $w_2(M) = 0$. More precisely, we have the following result.

7.2.16. Theorem [Wal66, Theorem 5]. *Diffeomorphism classes of oriented closed, simply connected 6-manifolds with torsion free homology and vanishing w_2 correspond bijectively to isomorphism classes of systems of invariants :*

(1) two free Abelian groups H^2 and H^3 ;
(2) a system trilinear map $\mu\colon H^2 \times H^2 \times H^2 \to \mathbb{Z}$ such that $\mu(x,x,y) \equiv \mu(x,y,y) \ (mod\ 2)$ for any $x, y \in H^2$;
(3) a homomorphism $p_1\colon H^2 \to \mathbb{Z}$ such that $p_1(x) \equiv 4\mu(x,x,x) \ (mod\ 24)$.

As an application, we derive the following diffeomorphic criteria for Moishezon manifolds with $b_2 = 1$ (cf. [Nak96, Corollary 1.5]).

7.2.17. Proposition. *Let X and $\widehat{X}$ be two smooth Moishezon threefolds with $b_2 = 1$. Denote by $\mathcal{O}_X(1)$ (resp. $\mathcal{O}_{\widehat{X}}(1)$) the big generator of* $\mathrm{Pic}(X)$ *(resp.* $\mathrm{Pic}(\widehat{X})$*). Assume furthermore that X and $\widehat{X}$ are of the same degree and index 2, i.e.,*

$$\omega_X \cong \mathcal{O}_X(-2),\ \omega_{\widehat{X}} \cong \mathcal{O}_{\widehat{X}}(-2),\ \mathcal{O}_X(1)^3 = \mathcal{O}_{\widehat{X}}(1)^3.$$

Then X is diffeomorphic to $\widehat{X}$ if and only if $H^3(X,\mathbb{Z}) \cong H^3(\widehat{X},\mathbb{Z})$.

Proof. By our assumption, we see that $H^2(X,\mathbb{Z})$ and $H^2(\widehat{X},\mathbb{Z})$ are isomorphic and the intersection forms are the same. Moreover, by Theorem 7.2.13, we have

$$\chi(X,\mathcal{O}_X) = \chi(\widehat{X},\mathcal{O}_{\widehat{X}}) = 1.$$

By Riemann-Roch theorem, we have

$$\frac{c_2(X)c_1(X)}{24} = \chi(X,\mathcal{O}_X),\ \text{and}\ \frac{c_2(\widehat{X})c_1(\widehat{X})}{24} = \chi(\widehat{X},\mathcal{O}_{\widehat{X}}).$$

It follows that we have $c_2(X)\mathcal{O}_X(1) = c_2(\widehat{X})\mathcal{O}_{\widehat{X}}(1) = 12$. Recall that the Pontrjagin classes of X and $\widehat{X}$ are defined as

$$p_1(X) = c_1(X)^2 - 2c_2(X),\ \text{and}\ p_1(\widehat{X}) = c_1(\widehat{X})^2 - 2c_2(\widehat{X}).$$

Then we obtain

$$p_1(X)\cdot\mathcal{O}_X(m) \equiv 4m\mathcal{O}_X(1)^3 (mod\ 24)\ \text{and}\ p_1(\widehat{X})\cdot\mathcal{O}_{\widehat{X}}(m) \equiv 4m\mathcal{O}_{\widehat{X}}(1)^3 (mod\ 24).$$

Note that we have $4m \equiv 4m^3$ (mod 24) for any $m \in \mathbb{Z}$. Thus, all the data of X and $\widehat{X}$ given in Theorem 7.2.16 are the same except $H^3(X,\mathbb{Z})$ and $H^3(\widehat{X},\mathbb{Z})$. Then we can conclude by Theorem 7.2.16. □

7.3 Existence of good divisors on Moishezon threefolds

In this section, we study the good divisor problem for smooth Moishezon threefolds. The argument is based on the analysis of the complete intersection of two general members.

7.3.1 Cohen-Macaulay spaces and dualizing sheaves

In this subsection, we recall some basic materials on Cohen-Macaulay spaces and their dualizing sheaves. For details we refer to [GPR94, Chapter II].

7.3.1. Definition.

(1) A complex space X is called Cohen-Macaulay if every local ring $\mathcal{O}_{X,x}$, $x \in X$, is a Cohen-Macaulay ring.

(2) Let X be a complex manifold, and let $Y \subset X$ be a closed complex subspace. Y is said to be a local complete intersection if its ideal sheaf $\mathcal{I}_Y$ is locally generated by $\mathrm{codim}_X(Y)$ elements.

7.3.2. Proposition [GPR94, II,Proposition 5.4]. *Let X be a complex manifold. Every local complete intersection $Y \subset X$ is Cohen-Macaulay.*

Let X be a complex manifold. Let $Y \subset X$ be a closed complex subspace of pure codimension r. Then the dualising sheaf of Y is defined as $\omega_Y = \mathcal{E}xt^r_{\mathcal{O}_X}(\mathcal{O}_Y, \omega_X)$. This sheaf is well-defined and it is independent of the embedding [GPR94, II, Corollary 5.23]. We are particularly interested in the case where Y is obtained as a complete intersection in some complex manifold X.

7.3.3. Proposition. *Let Y be a complete intersection in a complex manifold X such that Y is generically reduced. Then Y is reduced and its conormal sheaf $\mathcal{N}_{Y/X}$ is locally free. Moreover, we have the following adjunction formula*

$$\omega_Y \cong \omega_X|_Y \otimes \det(\mathcal{N}_{Y/X}).$$

Proof. Since Y is a complete intersection, Y is a Cohen-Macaulay space (cf. Proposition 7.3.2). Thus Y is actually reduced as it is generically reduced [GPR94, II, Corollary 5.10]. Then the sheaf $\mathcal{I}_Y/\mathcal{I}_Y^2$ is locally free. By the definition, we see

$$\mathcal{N}_{Y/X} := \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{I}_Y/\mathcal{I}_Y^2, \mathcal{O}_Y)$$

is locally free and the sequence

$$0 \to \mathcal{N}^*_{Y/X} \to \Omega^1_X|_Y \to \Omega^1_Y \to 0$$

is exact. Then, by the local fundamental isomorphism [GPR94, II,Proposition 5.26],

$$\omega_Y \cong \mathcal{H}om_{\mathcal{O}_X}(\wedge^r(\mathcal{I}_Y/\mathcal{I}_Y^2), \omega_X/(\mathcal{I}_Y\omega_X)).$$

where r is the codimension of Y, we get the adjunction formula. □

The name "dualizing sheaf" comes from the Serre duality theorem.

7.3.4. Theorem [GPR94, III, Theorem 4.15a]. *Let X be a compact Cohen-Macaulay space of pure dimension n, $\mathcal{E}$ a coherent sheaf on X and ω_X the dualizing sheaf of X. Then*

$$H^q(X, \mathcal{E})^* \cong Ext^{n-q}_{\mathcal{O}_X}(\mathcal{E}, \omega_X).$$

Moreover, if $\mathcal{E}$ is locally free, then we have

$$Ext^{n-q}_{\mathcal{O}_X}(\mathcal{E}, \omega_X) \cong Ext^{n-q}_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{E}^\vee \otimes \omega_X) \cong H^{n-q}(X, \mathcal{E}^\vee \otimes \omega_X).$$

7.3.2 Complete intersection of divisors

This subsection is devoted to study the structure of C both from the local and global views. Throughout this subsection, we fix the following notation and assumptions.

7.3.5. Assumption. Let X be a smooth Moishezon threefold with $b_2 = 1$ and ω_X^{-1} big. Let L be the big generator of $\mathrm{Pic}(X)$. Assume that $h^0(X, L) \geq 2$ and $\omega_X^{-1} \cong L^{\otimes r}$ for some $r > 0$. Let D_1 and D_2 be two distinct members of the linear system $|L|$. We denote by C the complete intersection of D_1 and D_2.

As an immediate application of Theorem 7.2.9, one can derive the following simple but useful results.

7.3.6. Lemma. *The complex space $(C, \mathcal{O}_C)$ is a Cohen-Macaulay curve with $\omega_C \cong \mathcal{O}_C(2-r)$. Moreover, the following properties hold.*

(1) The map $H^0(X, \mathcal{O}_X(1)) \to H^0(C, \mathcal{O}_C(1))$ is surjective.
(2) If $r \geq 2$, then we have $H^0(C, \mathcal{O}_C) \cong \mathbb{C}$, $H^1(C, \mathcal{O}_C) \cong H^0(X, \mathcal{O}_X(2-r))$ and

$$h^1(C, \mathcal{O}_C(-1)) = h^0(X, \mathcal{O}_X(3-r)) - 2h^0(X, \mathcal{O}_X(2-r)).$$

Proof. Note that C is a local complete intersection, so C is Cohen-Macaulay (cf. Proposition 7.3.2). The dualizing sheaf of C follows from the adjunction formula (cf. Theorem 7.3.4). Consider the natural exact sequences

$$0 \to \mathcal{O}_X(\ell - 1) \to \mathcal{O}_X(\ell) \to \mathcal{O}_D(\ell) \to 0$$

and

$$0 \to \mathcal{O}_D(\ell - 1) \to \mathcal{O}_D(\ell) \to \mathcal{O}_C(\ell) \to 0.$$

Taking $\ell = 0$ and 1, we see that the composite restriction map

$$H^0(X, \mathcal{O}_X(1)) \to H^0(D, \mathcal{O}_D(1)) \to H^0(C, \mathcal{O}_C(1))$$

is surjective and $\mathbb{C} \cong H^0(X, \mathcal{O}_X) \cong H^0(D, \mathcal{O}_D)$ since $H^1(X, \mathcal{O}_X) \cong H^1(D, \mathcal{O}_D) = 0$.
If $r \geq 2$, setting $\ell = -1$, we obtain $H^1(D, \mathcal{O}_D(-1)) = 0$ and it follows that $H^0(D, \mathcal{O}_D) \cong H^0(C, \mathcal{O}_C)$. Moreover, by Serre duality, we have also

$$H^2(D, \mathcal{O}_D(-1)) \cong H^3(X, \mathcal{O}_X(-2)) \cong H^0(X, \mathcal{O}_X(2-r)).$$

Since $H^2(D, \mathcal{O}_D) = H^1(D, \mathcal{O}_D) = 0$, it follows that we have

$$H^1(C, \mathcal{O}_C) \cong H^2(D, \mathcal{O}_D(-1)) \cong H^0(X, \mathcal{O}_X(2-r)).$$

Moreover, as $H^1(D, \mathcal{O}_D(-1)) = 0$, we get

$$h^1(C, \mathcal{O}_C(-1)) = h^2(D, \mathcal{O}_D(-2)) - h^2(D, \mathcal{O}_D(-1)) = h^2(D, \mathcal{O}_D(-2)) - h^0(X, \mathcal{O}_X(2-r)).$$

Then we conclude by the following exact sequence and Serre duality

$$0 \to H^2(D, \mathcal{O}_D(-2)) \to H^3(X, \mathcal{O}_X(-3)) \to H^3(X, \mathcal{O}_X(-2)) \to 0.$$

This completes the proof. □

7.3.7. Lemma. *Let C_{red} be the reduced complex space associated to C and denote by $A_1 + \cdots + A_s$ the decomposition of C_{red} into irreducible components. If $r \geq 3$ or if C_{red} is reducible and $r = 2$, then*

(1) each A_i is a smooth rational curve,
(2) $\sum_{i=1}^{s} h^1(A_i, \mathcal{O}_{A_i}(-1)) \leq h^1(C, \mathcal{O}_C(-1))$.

Proof. As $h^1(A_i, \mathcal{O}_{A_i}) \leq h^1(C, \mathcal{O}_C)$, we obtain $h^1(A_i, \mathcal{O}_{A_i}) = 0$ unless $r = 2$. However, if $r = 2$ and $h^1(C, \mathcal{O}_C) = h^1(A_i, \mathcal{O}_{A_i}) = 1$, we will prove that $C \cong A_i$. In fact, consider the natural exact sequence

$$0 \to \mathcal{I}_i \to \mathcal{O}_C \to \mathcal{O}_{A_i} \to 0.$$

Note that $H^0(C,\mathcal{O}_C) \cong H^0(A_i,\mathcal{O}_{A_i}) \cong \mathbb{C}$, then our assumption implies $H^1(C,\mathcal{I}_i) = 0$. By Serre duality, we obtain

$$0 = H^1(C,\mathcal{I}_i)^* \cong Ext^0_{\mathcal{O}_C}(\mathcal{I}_i,\mathcal{O}_C) \cong \mathrm{Hom}(\mathcal{I}_i,\mathcal{O}_C).$$

Then we see that we have $\mathcal{I}_i = 0$ and $C \cong A_i$. As a consequence, C_{red} is irreducible, which contradicts our assumption. Thus $h^1(A_i,\mathcal{O}_{A_i}) = 0$ and A_i is a smooth rational curve. On the other hand, denote by $\iota_i\colon A_i \to C_{\text{red}}$ the corresponding injection, there is an exact sequence

$$0 \to \mathcal{O}_{C_{\text{red}}}(-1) \to \bigoplus_{i=1}^{s} \iota_{i*}\mathcal{O}_{A_i}(-1) \to \bigoplus_{x\in A_i\cap A_j} \delta_x \to 0.$$

Then we get

$$\sum_{i=1}^{s} h^1(A_i,\mathcal{O}_{A_i}(-1)) \leq h^1(C_{\text{red}},\mathcal{O}_{\text{red}}(-1)) \leq h^1(C,\mathcal{O}_C(-1)).$$

The first equality holds if and only if the natural map

$$\bigoplus_{i=1}^{s} H^0(A_i,\mathcal{O}_{A_i}(-1)) \to \bigoplus_{x\in A_i\cap A_j} H^0(x,\delta_x)$$

is surjective. □

7.3.8. **Remark**. From our proof above, if $r = 2$ and A is complex subspace of C such that $h^1(A,\mathcal{O}_A) = 1$, then $A \cong C$.

The following lemma gives a rough description of the local structure of C at the singular points lying on a generically reduced component.

7.3.9. **Lemma**. *Let A be an irreducible component of C with reduced structure such that A is smooth and C is generically reduced along A. Then we have*

$$2L\cdot A - \deg(K_A) = rL\cdot A + \deg(\delta),$$

where δ is the sheaf defined as $(\mathcal{I}_A/(\mathcal{I}_C + \mathcal{I}_A^2)) \otimes \mathcal{O}_A$.

Proof. Since C is generically reduced along A, we see that $\mathcal{I}_C$ is isomorphic to $\mathcal{I}_A$ along A outside a finite number of points. We consider the natural exact sequence

$$0 \to \mathcal{K} \xrightarrow{\phi} \mathcal{I}_C/\mathcal{I}_C^2 \otimes \mathcal{O}_A \to \mathcal{I}_A/\mathcal{I}_A^2 \to \delta \to 0.$$

First we show that $\mathcal{K} = 0$. In fact, as C is a locally complete intersection, $\mathcal{I}_C/\mathcal{I}_C^2$ is a locally free sheaf. The the pull-back $\mathcal{I}_C/\mathcal{I}_C^2 \otimes \mathcal{O}_A$ is also locally free. In particular, it is torsion free and, as a consequence, its subsheaf $\mathcal{K}$ is also torsion free. However, note that ϕ is an isomorphism generically over A since C is generically reduced over A. This shows that $\mathcal{K} = 0$.

As A is a smooth curve, we have $\mathcal{I}_A/\mathcal{I}_A^2 \cong N^*_{A/X}$ and by adjunction formula we have

$$c_1(\mathcal{I}_A/\mathcal{I}_A^2) = -rL\cdot A - \deg(K_A).$$

On the other hand, we see $c_1(\mathcal{I}_C/\mathcal{I}_C^2) = -2c_1(L|_C)$ from the Koszul complex associated to $\mathcal{I}_C$. Then we can conclude by the compatibility of Chern classes with the pull-back. □

7.3.3 Application to good divisor problem

Now we are in the position to prove the existence of good divisor on smooth Moishezon threefold. We start with an easy lemma.

7.3.10. Lemma. *Let X be a smooth Moishezon threefold such that* $\mathrm{Pic}(X) = \mathbb{Z}L$ *for some big line bundle H and $-K_X = 2L$. Assume moreover that $h^0(X, L) \geq 3$. Then every element $D \in |L|$ is irreducible and* $\dim \phi_{|L|} \geq 2$.

Proof. Since $\mathrm{Pic}(X) \cong \mathbb{Z}L$, it is clear that every element in $|L|$ is irreducible. Moreover, if $\dim |\Phi_{|L|} = 1$, then a general member of $|L|$ is reducible by Bertini's theorem as $h^0(X, L) \geq 3$ (cf. [Uen75, Theorem 4.21]), a contradiction. □

7.3.11. Theorem. *Let X be a smooth Moishezon threefold such that* $\mathrm{Pic}(X) = \mathbb{Z}L$ *for some big line bundle H and $-K_X = 2L$. Assume moreover that $h^0(X, L) \geq 3$. Let D_1, D_2 be two general members of $|L|$, and let C be the complete intersection $D_1 \cap D_2$. Then C contains at least one mobile irreducible component A. Moreover, if A intersects the union of other components of C in at least two points, then a general member D of $|L|$ is smooth.*

Proof. First we show the existence of A. Since the closure of the image $\Phi_{|L|}(X)$ is of dimension at least 2, for any two general members D_1, $D_2 \in |H|$, the complete intersection C contains at least one mobile component A and C is generically reduced along A since D_1 and D_2 are general. Write

$$C = A_1 + n_2 A_2 + \cdots + n_r A_s$$

such that $A_1 = A$ and A_i are irreducible and reduced curves.

We claim that C is a reduced 1-cycle. If $s = 1$, this is clear, so we shall assume that $s \geq 2$. By Lemma 7.3.7, all the A_i's are smooth rational curves. Then Lemma 7.3.9 implies $\deg(\delta) = 2$ where $\delta = \mathcal{I}_{A_1}/(\mathcal{I}_C + \mathcal{I}_{A_1}^2) \otimes \mathcal{O}_{A_1}$. Without loss of generality, we assume that A_1 intersects A_2 at p_1. Thus we get $\deg(\delta_{p_1}) = 1$ by our assumption. Hence, locally around p_1, we have $\mathcal{I}_{C,p_1} = (x, yz)$ and $\mathcal{I}_{A_1,p_1} = (x, y)$ for some local coordinate functions (x, y, z). In particular, C is generically reduced along A_2 and $n_2 = 1$. Repeating this argument, we eventually obtain $A_2, \cdots, A_s$ such that A_i intersects A_{i+1} transversally at a point p_i and A_s intersects A_1 transversally at p_0. In particular, $n_i = 1$ for $i = 2, \cdots, s$ and C is a reduced 1-cycle.

Moreover, since C is Cohen-Macaulay and C is generically reduced along $C \setminus \{p_0, \cdots, p_{s-1}\}$, it follows that C is reduced over $C \setminus \{p_0, \cdots, p_{s-1}\}$. In particular, $C_{\mathrm{sing}} = \{p_0, \cdots, p_{s-1}\}$. Note that we have $D_{i,\mathrm{sing}} \subset C_{\mathrm{sing}}$ for $i = 1$ and 2. Moreover, at each p_i, at least one of D_1 and D_2 is smooth at p_i since $\deg(\delta_{p_i}) = 1$. Take a general element D in the pencil $\langle D_1, D_2 \rangle \subset |L|$ spanned by D_1 and D_2. Then we have

$$D \cap D_1 = D_2 \cap D_1 = C.$$

This implies

$$D_{\mathrm{sing}} \subset C_{\mathrm{sing}} = \{p_0, \cdots, p_{r-1}\}.$$

Note that the subset of elements in $\langle D_1, D_2 \rangle$ which is smooth at p_i is a nonempty open subset and the set $\{p_0, \cdots, p_{r-1}\}$ is finite. Hence, there exists an element $D \in |L|$ which is smooth. □

In the case $r \geq 3$, our argument above can be applied to recover Kollár's result.

7.3.12. Proposition. *Let X be smooth Moishezon threefold such that* $\mathrm{Pic}(X) \cong \mathbb{Z}L$ *for same big line bundle L. If $-K_X = rL$ for some $r \geq 3$, then X is projective.*

Proof. By Corollary 7.2.9 and Corollary 7.2.11, we get $h^0(X, L) \geq 4$ if $r \geq 3$. Denote by Y the closure of the image of the rational map $\Phi_{|L|}$ defined by $|L|$. Then we have $\dim(Y) \geq 2$ by Lemma 7.3.10. Let D_1 and D_2 be two general elements in $|L|$. Then the intersection curve $C = D_1 \cap D_2$ decomposes as

$$C = A_1 + \cdots + A_d + B_1 + \cdots + B_s,$$

where A_i are the moving components and B_j's are contained in $\mathrm{Bs}\,|L|$. By Lemma 7.3.7, the curves A_i's and $B_{j,\mathrm{red}}$'s are smooth rational curves. On the other hand, if $L \cdot A_i = 0$ for some $1 \leq i \leq d$, then A_i is disjoint from $\mathrm{Bs}\,|L|$ as A_i is not contained in $\mathrm{Bs}\,|L|$. This implies $C = A_i$ since C is connected and

C is smooth outside $\mathrm{Bs}\,|L|$. Hence, $\mathrm{Bs}\,|L| = \emptyset$ and $C = A_i$. As a consequence, we get $L^3 = 0$. This contradicts that L is big. Thus we have $L \cdot A_i > 0$ for all $1 \leq i \leq d$.

Case 1. $r = 4$. By Lemma 7.3.9, since $L \cdot A_1 > 0$, it follows that A_1 is disjoint from the other components of C. Since C is connected, this implies $C = A_1$. As a consequence, $|L|$ is basepoint free by Lemma 7.3.6. Since $L^3 = 1$, it follows that $\dim(Y) = 3$ and $\deg(\Phi_{|L|}) = \deg(Y) = 1$. This shows that $Y \cong \mathbb{P}^3$ and $\Phi_{|L|}$ is an isomorphism ; that is, $X \cong \mathbb{P}^3$.

Case 2. $r = 3$. By Lemma 7.3.9, if $L \cdot A_i \geq 2$ for some $1 \leq i \leq d$, then $C = A_i$. It follows by Lemma 7.3.6 that $|L|$ is basepoint free. This implies $\dim(Y) = 3$ and $\deg(Y) \leq 2$. However, if $r = 3$, Corollary 7.2.9 and Corollary 7.2.11 imply $h^0(X, L) \geq 5$. Therefore, $\deg(Y) = 2$ and $\Phi_{|L|}$ is birational. In particular, Y is a quadric hypersurface. Moreover, since every element in $|L|$ is irreducible, Y is smooth. As a consequence, $\Phi_{|L|}$ is an isomorphism ; that is, $X \cong Q^3$ for some smooth quadric hypersurface Q^3.

Now we consider the case $L \cdot A_i = 1$ for all $1 \leq i \leq d$. By our argument above, we may assume that $\mathrm{Bs}\,|L| \neq \emptyset$ and $\dim(Y) = 2$. If $\dim(\mathrm{Bs}\,|L|) = 0$, as $L^3 = 2$ by Lemma 7.2.11, it follows $d = 2$. This is impossible since the degree of Y is bounded from below by $1 + \mathrm{codim}(Y) \geq 3$. Therefore, we must have $\dim(\mathrm{Bs}\,|L|) = 1$. Moreover, since A_i are disjoint from each other outside $\mathrm{Bs}\,|L|$, it follows that every A_i meets $\mathrm{Bs}\,|L|$ in exactly one point since $L \cdot A_i = 1$ and C is connected. This implies also that $\mathrm{Bs}\,|L|$ does not contain isolated points and it is connected. Therefore, we have $\mathrm{Bs}\,|L| = \mathrm{Supp}(B)$, where B is the union $B_1 \cup \cdots \cup B_s$. Let p_i be the point $A_i \cap B$. Then $\mathcal{I}_{C,p_i} = (x, yz)$ by Lemma 7.3.9. Define a structure sheaf $\mathcal{O}_B$ of B as follows. If $b \in B$ is different from all p_i, we set $\mathcal{O}_{b,B} = \mathcal{O}_{b,C}$. If $b = p_i$, then we set $\mathcal{O}_{b,B}$ to the induced reduced local ring. Then we have a natural exact sequence of sheaves

$$0 \to \mathcal{O}_C \to \bigoplus_{i=1}^{d} \iota_{i*}\mathcal{O}_{A_i} \oplus \iota_*\mathcal{O}_B \to \bigoplus_{i=1}^{d} \mathcal{Q}_{p_i} \to 0. \tag{7.3}$$

Then we have $\dim(\mathcal{Q}_{p_i}) = 1$. Now we tensors the short exact sequence (7.3) with L, we get the exact sequence

$$0 \to L|_C \to \bigoplus_{i=1}^{d} (\iota_{i*}\mathcal{O}_{A_i} \otimes L) \oplus (\iota_*\mathcal{O}_B \otimes L) \to \bigoplus_{i=1}^{d} \mathcal{Q}_{p_i} \to 0.$$

Since the sheaf $\mathcal{Q} = \oplus \mathcal{Q}_{p_i}$ is supported on a finite number of points, we obtain

$$\begin{aligned} h^0(C, L|_C) &\geq \sum_{i=1}^{d} h^0(C, \iota_{i*}\mathcal{O}_{A_i} \otimes L) + h^0(C, \iota_*\mathcal{O}_B \otimes L) - h^0(C, \mathcal{Q}) \\ &\geq \sum_{i=1}^{d} h^0(A_i, \iota_i^* L) + h^0(B, \iota^* L) - h^0(C, \mathcal{Q}) \\ &\geq \sum_{i=1}^{d} h^0(A_i, \iota_i^* L) - d. \end{aligned}$$

The second inequality follows from the projection formula $\iota_{i*}\iota_i^* L \cong \iota_{i*}\mathcal{O}_{A_i} \otimes L$. Moreover, as $L \cdot A_i = 1$, we conclude

$$h^0(X, L) - 2 = h^0(C, L|_C) \geq 2d - d \geq d.$$

Nevertheless, note that we have

$$d = \deg(Y) \geq 1 + \mathrm{codim}(Y) = 1 + h^0(X, L) - 1 - 2 = h^0(C, L|_C) \geq d. \tag{7.4}$$

If the equality holds, then $Y \subset \mathbb{P}^{h^0(X,L)-1}$ is a surface of minimal degree. Moreover, note that there exists an open subset $U \subset Y$ such that every hyperplane section of U is irreducible. Then Y is indeed the projective space $\mathbb{P}^2$ by the classification of varieties of minimal degree given in [EH87]. However, as $r = 3$, we have $h^0(X, L) \geq 5$ by Proposition 7.2.9. This is impossible. Hence, Y is not a surface of minimal degree and we get a contradiction in (7.4). $\square$

In the following result, we show that the condition in Theorem 7.3.11 is automatically satisfied if $h^0(X,L) \geq 5$ by using the classification theorem due to Kollár (cf. Theorem 7.2.13).

7.3.13. Corollary. *Let X be a smooth Moishezon threefold such that* $\mathrm{Pic}(X) = \mathbb{Z}L$ *for some big line bundle L and $-K_X = 2L$. Assume moreover that $h^0(X,L) \geq 5$. Then a general member D of $|L|$ is smooth.*

Proof. If $h^0(X,L) \geq 6$, then $|L|$ is basepoint free by Theorem 7.2.13, so we focus on the case $h^0(X,L) = 5$ and $|L|$ is not basepoint free. Then by Theorem 7.2.13 again, the rational map

$$\Phi_{|L|} \colon X \dashrightarrow \mathbb{P}^4$$

is bimeromorphic to its image and the closure of $\Phi_{|L|}(X)$ is a smooth quadric hypersurface. We set $\Phi = \Phi_{|L|}$. Since X and Q^3 are of different index, then there exists a divisor $Q^2 \subset Q^3$ which is contracted by Φ^{-1}. Let H_1 and H_2 be two general members in $|\mathcal{O}_{Q^3}(1)|$. Then the intersection $C = H_1 \cap H_2$ intersects Q^2 at least in two points. Moreover, we may assume also that Φ^{-1} is well-defined over the generic point of C. Let D_1 and D_2 be two divisors in $|L|$ corresponding to H_1 and H_2, respectively. Then $\Phi_*^{-1}C$ is a mobile component of $D_1 \cap D_2$ and Φ^{-1} is an isomorphism over the generic point of C. If Q^2 is contracted to a point, then $\Phi_*^{-1}C$ is not smooth, which contradicts to Lemma 7.3.7. Thus, Q^2 is contracted to a curve B and $\Phi_*^{-1}C$ intersects B in at least two points. Then we can conclude by Theorem 7.3.11. □

7.3.14. Remark. So far I do not know any of X which does not satisfy the assumptions in Theorem 7.3.11.

7.3.4 An alternative argument for $r = 4$

In this subsection, we give another different argument of Theorem 7.3.12 for $r = 4$. This strategy is essentially the same as that given by Peternell in [Pet89].

7.3.15. Theorem [Rei94, Proposition 2.3]. *Let S be an irreducible and reduced surface such that ω_S is locally free, and denote by $n \colon \widetilde{S} \to S$ its normalization. Then there exists an effective divisor C on $\widetilde{S}$ such that*

$$n^*\omega_S \cong \omega_{\widetilde{S}} \otimes \mathcal{O}_X(C).$$

7.3.16. Definition. *Let S be an irreducible normal complex space of dimension 2. A log-resolution $f \colon \widehat{S} \to S$ is called minimal if $Ex(f)$ does not contain any (-1)-curve.*

7.3.17. Proposition. *Let X be a smooth Moishezon threefold such that* $\mathrm{Pic}(X) \cong \mathbb{Z}L$ *for some big line bundle L. Assume moreover that $-K_X = 4L$. Then $X \cong \mathbb{P}^3$.*

Proof. Thanks to Corollary 7.2.9, we have $h^0(X,L) \geq 4$. Let $S \in |L|$ be a general member. Then we have $\omega_S^\vee \cong \mathcal{O}_S(3D)$ by adjunction formula, where $D \in |L|_S|$. Denote by $n \colon \widetilde{S} \to S$ the normalization of S and denote by $f \colon \widehat{S} \to \widetilde{S}$ the minimal resolution of $\widetilde{S}$. Then there exist an effective divisor $C \subset \widetilde{S}$ and $E \subset \widehat{S}$ such that

$$-K_{\widehat{S}} = f^*n^*(3D) + f_*^{-1}C + E. \tag{7.5}$$

In particular, $-K_{\widehat{S}}$ is big and $\widehat{S}$ is uniruled. If $\widehat{S}$ is not isomorphic to $\mathbb{P}^2$, then there is a fibration by rational curves $\pi \colon \widehat{S} \to B$ over a smooth rational curve B. Let F be a general fiber of π. Then we have $-K_{\widehat{S}} \cdot F = 2$ by adjunction formula. However, by (7.5), we see $-K_{\widehat{S}} \cdot F \geq 3$ since f^*n^*D is big. We get a contradiction. Hence $\widehat{S} \cong \mathbb{P}^2$ and f^*n^*D is ample. This implies that we have moreover $C = E = 0$, i.e., $\widehat{S} \cong S$. In particular, by Lemma 7.3.6, $|L|$ is basepoint free. Then $\deg(\Phi_{|L|}) = 1$ and $\mathrm{im}(\Phi_{|L|}) \cong \mathbb{P}^3$ as $L^3 = 1$. As a consequence, Φ is an isomorphism. □

7.3.18. Remark. The same argument shows that if $r = 3$, then $f_*^{-1}C$ and E are both contained in the fibers of π.

7.4 Global deformation of Fano manifolds

In this section, we collect some basic materials about the global deformation of prime Fano manifolds. For convenience, we will make the following assumption.

7.4.1. Assumption. Let $\pi: X \to \Delta$ be a smooth holomorphic family of compact complex manifolds parametrized by the unit disk Δ in $\mathbb{C}$. Assume moreover that for every nonzero t, the fiber $X_t: = \pi^{-1}(t)$ is biholomorphic to a Fano manifold with Picard number one and index r.

7.4.2. Proposition. *Under the Assumption 7.4.1, then, after shrinking Δ if necessary, there exists a line bundle over X such that* $\mathrm{Pic}(X_t) = \mathbb{Z}L_t$ *and* $-K_{X_t} = rL_t$ *for every* $t \in \Delta$*, here L_t denotes the restriction of L over X_t. In particular, L_0 is big and X_0 is a Moishezon manifold.*

Proof. By adjunction $\omega_{X_t} \cong \omega_X|_{X_t}$, then the upper semicontinuity theorem (see [Uen75, Theorem 1.4]) imlies that for $t \neq 0$ we have

$$h^0(X_0, \omega_{X_0}^{\otimes -m}) \geq h^0(X_t, \omega_{X_t}^{\otimes -m}) \geq cm^{\dim(X_t)}.$$

The last inequality follows from the fact that $\omega_{X_t}^{-1}$ is ample for $t \neq 0$. In particular, $\omega_{X_0}^{-1}$ is a big line bundle and X_0 is a Moishezon manifold. By upper semi-continuity [GR84, Theorem, p. 210], we have $h^{p,q}(X_0) \leq h^{p,q}(X_t)$ for t small enough. On the other hand, since Hodge decomposition holds for X (cf. [Uen75, Corollary 9.3]), it follows that we have

$$b_k(X_0) = \sum_{p+q=k} h^{p,q}(X_0) \geq \sum_{p+q=k} h^{p,q}(X_t) = b_k(X_t).$$

Since X_0 is diffeomorphic to X_t for t small enough by Ehresmann's theorem, we get $h^{p,q}(X_0) = h^{p,q}(X_t)$ for t small enough. Since X_t is Fano if $t \neq 0$, we have $h^{0,q}(X_t) = 0$ for $t \neq 0$ and $q \geq 1$. It follows that $H^q(X_0, \mathcal{O}_{X_0}) = 0$ for $q \geq 1$. As a consequence, $R^p\pi_*\mathcal{O}_X = 0$ for $p \geq 1$ by base change theorem (see [GPR94, III, Theorem 4.3]). Moreover, as Δ is Stein, it follows from Cartan's Theorem B that $H^q(\Delta, \mathcal{O}_\Delta) = 0$ for any $q \geq 1$. Then Leray spectral sequence implies

$$H^p(X, \mathcal{O}_X) \cong H^p(\Delta, \pi_*\mathcal{O}_X) = H^p(\Delta, \mathcal{O}_\Delta) = 0, \ p \geq 1.$$

Considering the exponential sequences on X and X_t ($t \in \Delta$), we obtain a commutative diagram

$$\begin{tikzcd}
H^1(X, \mathcal{O}_X) \arrow[r] \arrow[d] & H^1(X, \mathcal{O}_X^*) \arrow[r] \arrow[d] & H^2(X, \mathbb{Z}) \arrow[r] \arrow[d] & H^2(X, \mathcal{O}_X) \arrow[d] \\
H^1(X_t, \mathcal{O}_{X_t}) \arrow[r] & H^1(X_t, \mathcal{O}_{X_t}^*) \arrow[r] & H^2(X_t, \mathbb{Z}) \arrow[r] & H^2(X_t, \mathcal{O}_{X_t})
\end{tikzcd}$$

As $H^q(X_t, \mathcal{O}_{X_t}) = 0$ for $t \in \Delta$ and $q > 0$, it follows that we have

$$\mathbb{Z} \cong H^1(X_t, \mathcal{O}_{X_t}^*) \cong H^2(X_t, \mathbb{Z}).$$

Since $H^2(X, \mathbb{Z}) \to H^2(X_t, \mathbb{Z})$ is an isomorphism for all $t \in \Delta$, we conclude that the map

$$H^1(X, \mathcal{O}_X^*) \to H^1(X_t, \mathcal{O}_{X_t}^*)$$

is an isomorphism for all $t \in \Delta$. Thus, if we pick up a generator of $\mathrm{Pic}(X)$, then the restriction $L_t: = L|_{X_t}$ is a generator of $\mathrm{Pic}(X_t)$. Moreover, after replacing L by its dual bundle, we can assume that L_t is ample for some $t \neq 0$. Then L_t is big for all $t \in \Delta$ by the upper semi-continuity. Moreover, as $\omega_{X_t}^{-1} \cong L_t^{\otimes r}$ for $t \neq 0$ by assumption, we get

$$\omega_{X_0}^{-1} \cong \omega_X^{-1}|_{X_0} \cong L^{\otimes r}|_{X_0} \cong L_0^{\otimes r}.$$

The second equality follows from $\omega_{X_t} \cong \omega_X|_{X_t}$ for $t \neq 0$. $\square$

7.4.3. **Remark.** In general, the algebraic dimension is not an upper semi-continuous function, a counter-example was constructed by Fujiki and Pontecorvo in [FP10]. Popovich proved that the limit of projective manifolds is Moishezon under the assumption that $h^{0,1}$ is constant near the center fiber (see [Pop13]) and Barlet proved a general upper semi-continuous theorem for the algebraic dimension of a family of weak Kähler manifolds (see [Bar16]).

Combining these results with the existence of good divisor proved in the last section, we get the following result.

7.4.4. **Proposition.** *Under the Assumption 7.4.1, let L be the line bundle provided in Proposition 7.4.2. Assume moreover that* $\dim(X) = 4$. *If $r \geq 3$ or if $r \geq 2$ and $L_t^3 \geq 3$ for $t \neq 0$, then there exists a surface $S \in |L_0|$ such that S is smooth.*

Proof. By Semi-continuity theorem, we have $h^0(X_0, L_0) \geq h^0(X_t, L_t)$. If $r \geq 3$ or if $r \geq 2$ and $L_t^3 \geq 3$, the we have $h^0(X_t, L_t) \geq 5$ (cf. [IP99, §12.1]). Then the result follows immediately from Proposition 7.3.13. $\square$

7.4.5. **Remark.** This proposition is a direct consequence of Theorem 7.1.11. However, it may be interesting to give another direct proof of Theorem 7.1.11 by studying the geometry of S and we hope that this strategy can be also applied to higher dimensional case.

7.5 Non-projective Moishezon manifolds with Picard number one

In this section, we try to understand the structure of the "simplest" non-projective Moishezon manifold. A similar question was also investigated in [Bon96].

7.5.1 Setup and Mori's theory

We begin with some results on the extremal contractions over projective manifolds. Let X be a projective manifold such that K_X is not nef.

7.5.1. **Theorem [Wiś91b].** *If X is projective manifold, then no small contraction of X has fibers of dimension one.*

In view of this theorem, we can restate a theorem which is due to Ando.

7.5.2. **Theorem [And85].** *Let $f\colon X \to Y$ be a contraction of an extremal ray R of a projective manifold X. If every fiber of f is of dimension at most one, then Y is smooth and one of the following cases holds :*

(1) $f\colon X \to Y$ is a conic bundle.
(2) $f\colon X \to Y$ is a blow-up of the manifold Y along a smooth subvariety Z of codimension 2.

Let us recall that $f\colon X \to Y$ is a conic bundle if there exists a rank-3 vector bundle $\mathcal{E}$ over Y such that its projectivization $\bar{f}\colon \mathbb{P}(\mathcal{E}) \to Y$ contains X embedded over Y as a divisor whose restriction to any fiber of $\bar{f}$ is an element of $\mathcal{O}_{\mathbb{P}^2}(2)$.

Setup : Let X be a non-projective Moishezon manifold of dimension n with $\mathrm{Pic}(X) \cong \mathbb{Z}L$. Assume that there exists a smooth curve C in X such that the blow-up $\pi\colon \widehat{X} \to X$ along C is a projective manifold. As $\rho(\widehat{X}) = 2$, we denote by R_1 and R_2 the generators of $\overline{\mathrm{NE}}(\widehat{X})$. Note that $K_{\widehat{X}}$ is not nef, so Mori's cone theorem (cf. Théorème 0.5.3) implies that there exists an extremal contraction $f\colon \widehat{X} \to Y$.

$$\begin{tikzcd}
E \arrow[d, "\pi_E"'] & \subset & \widehat{X} \arrow[r, "f"] \arrow[d, "\pi"'] & Y \\
C & \subset & X \arrow[ur, dashed, "\phi"'] &
\end{tikzcd}$$

7.5.3. **Notation**. *We denote by r_X the index of $c_1(X)$ in* Pic(X)*. Let H be the generator of* Pic(Y)*. Moreover, Y is a $\mathbb{Q}$-factorial variety with $\rho(Y) = 1$. The exceptional divisor of π will be denoted by E. The induced rational map $X \dashrightarrow Y$ is denoted by ϕ.*

From now on we investigate the behaviors of $\overline{\mathrm{NE}}(\widehat{X})$ and f. We start with a simple but useful observation.

7.5.4. **Lemma**. *Let $B \subset X$ be an irreducible curve. Then $L \cdot B \leq 0$ if and only if $B = C$.*

Proof. Take a general very ample divisor A on $\widehat{X}$. Since $\mathrm{Pic}(X) = \mathbb{Z}L$, we have $\pi_* A \simeq kL$ for some positive integer k. It follows that there exists some positive r such that

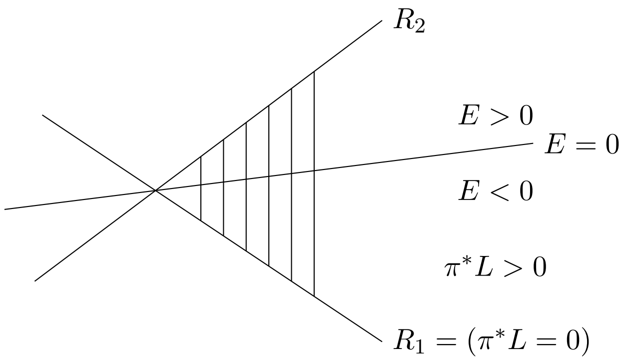
$$\pi^*(kL) = \pi^*\pi_*(A) = A + rE.$$

Let B be an irreducible curve in X other than C and $\widetilde{B}$ the strict transform of B in $\widehat{X}$. Then we have $kL \cdot B = \pi^*(kL) \cdot \widetilde{B} = (A + rE) \cdot \widetilde{B} > 0$.

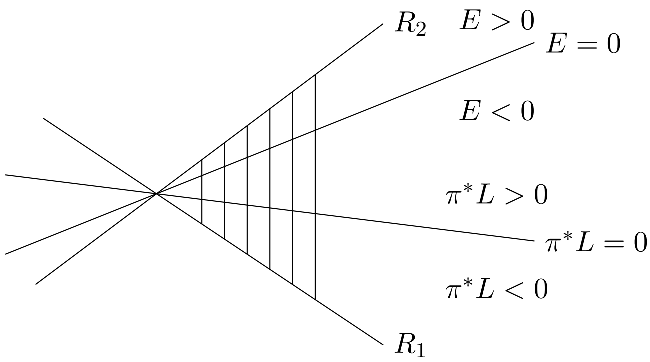
Conversely, if $L \cdot C > 0$, [Kol91b, Theorem 1.4.4] implies that there exists a finite morphism $\Phi \colon X \to \bar{X}$ such that $\bar{X}$ is projective and $L = \Phi^* \bar{L}$ for some ample $\bar{L}$ on $\bar{X}$, but this means that L is also ample, a contradiction. □

Due to the lemma above, we can describe the behavior of R_1 and R_2 with respect to E and π^*L, from which we can deduce the possible behaviors of $K_{\widehat{X}}$. Consequently, the possible positions of the line over which $K_{\widehat{X}}$ is zero are determined, this is the first and crucial step towards Theorem 4.1.26. There are two different cases according to that L is nef or not.

– If L is nef, i.e. $L \cdot C = 0$.



– If L is not nef, i.e. $L \cdot C < 0$.



7.5.5. **Remarks**.

(1) An irreducible curve B in $\widehat{X}$ satisfies $E \cdot B < 0$ if and only if $B \subset E$. We only need to show that $E \cdot B < 0$ for every irreducible curve $B \subset E$. In fact, since E is the exceptional divisor of a blow-up,

we may assume that C is not contained in the fiber of the blow-up π, i.e. $\pi_* B = C$. As in the proof of Lemma 7.5.4, we take a general very ample divisor A on $\widehat{X}$ such that $\pi^*(kL) = A + rE$ for some k and r positive. Then we get $rE \cdot B < \pi^*(kL) \cdot B = kL \cdot C \cdot \deg(\pi|_B) \leq 0$. As r is positive, this implies $E \cdot B < 0$.

(2) If L is nef, the position of the line over which $\pi^* L$ is zero is determined by the fact that $\pi^* L$ is nef and is numerically trivial over E. In particular, for an irreducible curve $B \subset \widehat{X}$, we have $[B] \in R_1$ if and only if $B \subset E$.

(3) If L is not nef, the position of the line over which $\pi^* L$ is zero is given by observing that $\pi^* L$ is not nef on E and it is trivial on the fibres of blow-up π over which the divisor E is negative.

7.5.2 Structure of the extremal contraction

The aim of this subsection is to prove Theorem 4.1.26. The following result is a consequence of basepoint free theorem.

7.5.6. Proposition. *If L is nef, then $\dim(X) = 3$ and $r_X \leq 2$. The map ϕ is actually a morphism contracting C to the only ordinary node singularity point P of Y and it is an isomorphism outside C.*

Proof. Since L is nef and $K_{\widehat{X}} = \pi^*(-r_X L) + (n-2)E$, by the first graph above, we may take R_1 to be our extremal ray. Then Remark 7.5.5 (2) implies that f contracts E to a point. In particular, the induced map ϕ is a morphism and contracts C to the only singular point $f(E) = P$ of Y. Moreover, note that we have

$$K_E = (K_{\widehat{X}} + E)|_E = (\pi^*(-r_X L) + (n-1)E)|_E \equiv (n-1)E|_E.$$

We claim that $-K_E$ is ample. In fact, denote by $\overline{\mathrm{NE}}(E, X)$ the image of $\overline{\mathrm{NE}}(E)$ in $\overline{\mathrm{NE}}(X)$. Then $\overline{\mathrm{NE}}(E, X)$ is contained in R_1. In particular, $-E$ is strictly positive over $\overline{\mathrm{NE}}(E, X) \setminus \{0\}$. This implies that $-K_E$ is strictly positive over $\overline{\mathrm{NE}}(E) \setminus \{0\}$. Then Kleiman's ampleness criterion implies that $-K_E$ is ample. In particular, the vector bundle $N_{C/X}$ is ample. Note that $L \cdot C = 0$ and $C \cong \mathbb{P}^1$, we have $\deg N_{C/X} = \deg K_C = -2$. It follows that $n = 3$ and $N_{C/X} \cong \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$. Then Proposition 7.3.12 implies $r_X \leq 2$ since X is not projective. ☐

7.5.7. Proposition. *If L is not nef, then every fiber of f is of dimension at most one.*

Proof. Let F be a nontrivial fiber of f. By Remark 7.5.5 (1), if the extremal ray of f is R_1, then F is contained in E. Note that in this case ϕ does not contract any curves contained in the fiber of the blow-up π, since such curves lie in the line $\{\pi^* L = 0\}$. Hence the fiber of f is of dimension ≤ 1 in this case. If the extremal ray of f is R_2, then $F \cap E \neq \emptyset$ since E is positive over R_2. Note that E is negative on the curves contained in E, it follows $\dim(F \cap E) = 0$. Hence F is also of dimension at most one. ☐

7.5.8. Theorem. *Notation and assumption as in setup (cf. §7.5.1).*

(1) If L is nef or K_X is big, then $\dim(X) = 3$ and the induced map $\phi\colon X \dashrightarrow Y$ is a birational morphism such that ϕ contracts C to the only ordinary node point P of Y.

(2) If $K_X = 0$ and L is not nef, then $\dim(X) = 3$ and X is obtained by a flop from a projective manifold of dimension 3 with trivial canonical bundle and Picard number one.

(3) If $-K_X$ is big but not nef, and if f is not birational, then f induces a conic bundle over a Fano manifold of Picard number one.

(4) If $-K_X$ is big but not nef, and if f is birational, then f is the blow-up of a projective Fano manifold of dimension n along a submanifold with codimension 2.

Proof. By Theorem 7.5.9 below, if K_X is big, then L is nef. Thus (1) follows immediately from Proposition 7.5.6. We may assume that L is not nef in the sequel, by Theorem 7.5.9, it follows that K_X is not big, i.e. $r_X \geq 0$.

Case 1. $r_X = 0$. Then we can also chose R_1 as the extremal ray of f as $K_{\widehat{X}} = (n-2)E$. Note that f is birational in this case and the fibers of f is of dimension ≤ 1. By Theorem 7.5.2, it follows that f is the blow-up of a projective manifold Y of dimension n along a submanifold of codimension 2 such that the exceptional divisor of f is E. Then we have

$$(n-2)E = K_{\widehat{X}} = f^*K_Y + E.$$

This clearly shows that $n = 3$ and $K_Y = 0$. It follows that the rational map ϕ is a flop.

Case 2. $r_X > 0$. We saw that every fiber of f is of dimension ≤ 1, then f is either a conic bundle over the projective manifold Y or the blow-up of the projective manifold Y along a submanifold Z of codimension 2. In the former case, the morphism $f_E\colon E \to Y$ is surjective and finite. By the ramification formula, there exists an effective divisor R on E such that

$$(K_{\widehat{X}} + E)|_E = K_E = f_E^*K_Y + R.$$

Note that K_E is negative on the fibers of the blow-up π while R is positive. It follows that $f_E^*K_Y$ is negative on the fibers of the blow-up π, therefore $-K_Y$ is ample. We conclude that Y is a Fano manifold of $\rho(Y) = 1$. In the later case, let D be the exceptional divisor of f. Then we have

$$\pi^*K_X + (n-2)E = K_{\widehat{X}} = f^*K_Y + D.$$

If $D \neq E$, since $K_{\widehat{X}}$ is negative on the fibers of the blow-up π, it follows that f^*K_Y is negative on the fibers of the blow-up π. Hence $-K_Y$ is ample. If $D = E$, then C is a rational curve. By pushforward, we have

$$f_*\pi^*K_X = f_*f^*K_Y = K_Y.$$

Since $r_X > 0$, it follows that Y is a Fano manifold with $\rho(Y) = 1$. □

In the proof above, we used the following result due to Bonavero. For the case $\dim(X) = 3$, another proof given by using the theory of deformation of rational curves can be found in [Kol91b, Theorem 5.3.2].

7.5.9. Theorem [Bon96, Théorème 2]. *Let X be a non projective Moishezon manifold of dimension $n \geq 3$ with $\mathrm{Pic}(X) = \mathbb{Z}L$. Assume that K_X is big and there exists a submanifold $Z \subset X$ such that the blow-up of X along Z defines a projective manifold. If K_X is not nef, then we have*

$$\mathrm{codim} Z < \frac{1}{2}(n+1).$$

In particular, if Z is a smooth curve, then K_X and L are both nef.

7.5.3 Application to birational case

Let X be a non-projective Moishezon manifold such that it becomes projective after blow-up along a smooth curve C and $\mathrm{Pic}(X) = \mathbb{Z}L$ for some big line bundle L. By Theorem 7.5.8, we see that if K_X is big or trivial, or if L is nef, then such manifolds exist only in dimension 3. In this section, we are going to see what happens in the birational case under some additional assumptions. In this case, Y is a Fano manifold. We denote by r_Y the index of Y. Recall that E is the exceptional divisor of the blow-up $\pi\colon \widehat{X} \to X$ and D is the exceptional divisor of the contraction $f\colon \widehat{X} \to Y$.

(7.5.3.1) Case $D = E$

In this subsection, we consider the case $D = E$. The next proposition shows that such a manifold exists only in dimension at least 4.

7.5.10. Proposition. *Assumptions and notation as in Theorem 7.5.8. If L is not nef and f is birational such that $D = E$, then $r_X = r_Y = r$ is a factor of $n-3$ and $N_{C/X} \cong \mathcal{O}(-1)^{\oplus(n-1)}$.*

Proof. As $D = E$, it follows that C is a rational curve. Since the rational map ϕ is an isomorphism in codimension 1, we have $r_X = r_Y = r$. Moreover, the exceptional divisor E admits two different contractions

$$\begin{tikzcd} E \arrow[r, "f|_E"] \arrow[d, "\pi|_E"] & \mathbb{P}^{n-2} \\ C \cong \mathbb{P}^1 & \end{tikzcd}$$

Let R_1' and R_2' be the generator rays of $\overline{\mathrm{NE}}(E)$. Then it is easy to see that R_1' and R_2' are contracted by $f|_E$ and $\pi|_E$, respectively. In particular, the adjunction formula

$$(\pi^* K_X + (n-1)E)|_E = K_E = (f^* K_Y + 2E)|_E$$

shows that R_1' and R_2' are both extremal rays of $\overline{\mathrm{NE}}(E)$. As $-K_E$ is positive over $R_1' \setminus \{0\}$ and $R_2' \setminus \{0\}$. Therefore, E is Fano by Kleiman's ampleness criterion. The same argument shows that $-E|_E$ is an ample divisor. This implies that $N_{C/X}$ is an ample vector over C and we can write $N_{C/X}$ as $\oplus \mathcal{O}(-a_i)$ for some $a_i > 0$, $1 \leq i \leq n-1$. Let B be a general fiber of the blow-up f. Then $\pi\colon B \to C$ is surjective and finite. Since $f^* K_Y$ is trivial over B, we get

$$\pi^* K_X \cdot B = -(n-3) E \cdot B = n-3.$$

This shows that $r = r_X$ is a factor of $n-3$. Note that we also have

$$\sum_{i=1}^{n-1} a_i = -\deg(N_{C/X}) = K_X \cdot C - \deg K_C = \frac{1}{\deg(\pi|_B)} \pi^* K_X \cdot B + 2 \leq n-1.$$

It follows $a_i = \deg(\pi|_B) = 1$ for $i = 1, \cdots, n-1$ and the proof is complete. □

(7.5.3.2) Case $D \neq E$

In this subsection, we are going to see what happens under the additional assumption that $f_* \pi^*$ induces an isomorphism between the Picard groups of Y and X. To be more precise, we make the following assumptions in this subsection.

Assumption ($\star$). Throughout this subsection, we always assume that $-K_X$ is big but not nef, f is birational and $f^* H = \pi^* L - aE$ for some $a \in \mathbb{Z}$.

Note that under the assumption ($\star$) and if $D \neq E$, then we must have $a > 0$ since $f^* H$ is positive over the fibers of the blow-up π.

$$\begin{tikzcd} & & D \arrow[r, "f_D"] \arrow[d, phantom, "\cap" description] & f(D) \arrow[d, phantom, "\cap" description] \\ E \arrow[r, phantom, "\subset"] \arrow[d, "\pi_E"'] & & \widehat{X} \arrow[r, "f"] \arrow[d, "\pi"'] & Y \\ C \arrow[rr, phantom, "\subset"] & & X \arrow[ur, dashed, "\phi"'] & \end{tikzcd}$$

The aim of this section is to prove that such a case happens only in dimension 3 and the image of E under f is smooth. Then we show that these manifolds are exactly some examples constructed in §(7.2.3.3). Note that under the assumption ($\star$) the extremal ray of f is R_2 and $f\colon E \to f(E)$ is finite. Moreover, as $D \neq E$, f is an isomorphism over the generic point of E. As a consequence, $f\colon E \to f(E)$ is birational, hence it coincides with the normalization. For simplicity of notation, we write $\widetilde{E}$ instead of $f(E)$.

7.5.11. **Theorem**. *Under the assumption ($\star$). If $D \neq E$, then $\widetilde{E}$ is smooth and* $\dim(X) = 3$.

Proof. Since $\pi^* K_X + (n-2)E = f^* K_Y + D$, by assumption $(\star)$, we obtain

$$r_X f^* H + D = r_Y f^* H + (n - 2 - a r_X)E.$$

It follows

$$(a r_X - n + 2)\widetilde{E} \sim (r_Y - r_X)H. \tag{7.6}$$

Note that by pushforward we have $K_X \sim -r_Y \pi_* f^* H + \pi_* D$. According to our assumption $(\star)$, it follows $(r_Y - r_X)L \sim \pi_* D$. In particular, we obtain $r_Y > r_X$ and $a r_X \geq n-1$.

We claim that the dualising sheaf of $\widetilde{E}$ is anti-ample. In fact, since $\widetilde{E}$ is Cartier, the dualising sheaf of $\widetilde{E}$ is $\omega_Y \otimes \mathcal{O}(\widetilde{E})|_{\widetilde{E}}$. By (7.6), it follows that the dualising sheaf $\omega_{\widetilde{E}}$ is isomorphic to

$$\mathcal{O}_Y(-r_X H - (a r_X - n + 1)\widetilde{E})|_{\widetilde{E}}.$$

Note that r_X is positive and $a r_X \geq n-1$, we conclude that $\omega_{\widetilde{E}}$ is anti-ample.

Since $f\colon E \to \widetilde{E}$ is the normalization, by subadjunction (cf. [Rei94, Proposition 2.3]), there exists an effective Weil divisor Δ on E such that $\omega_E \otimes \mathcal{O}(\Delta) = f^*\omega_{\widetilde{E}}$. Let ℓ be a line contained in the fiber of blow-up π. Combining subadjunction and assumption $(\star)$ gives

$$-(n-1) + \Delta \cdot \ell = -a r_X - a s(a r_X - n + 1)$$

where s is the positive integer such that $\widetilde{E} \sim sH$. However, we see above that $a r_X \geq n-1$ and $\Delta \cdot \ell \geq 0$. Hence, it follows that $a r_X = n-1$ and $\Delta \cdot \ell = 0$. This implies that $\omega_{\widetilde{E}} \cong \mathcal{O}_Y(-r_X H)|_E$. On the other hand, we have

$$K_E = (K_{\widehat{X}} + E) = (-r_X \pi^* L + (n-1)E)|_E = (-r_X f^* H - a r_X E + (n-1)E)|_E = -r_X f^* H|_E.$$

This shows $\Delta = 0$. Therefore $\widetilde{E}$ is smooth and the morphism $f\colon E \to \widetilde{E}$ is an isomorphism. We obtain $\rho(\widetilde{E}) = \rho(E) = 2$. However, by Lefschetz hyperplane theorem [Laz04, Theorem 3.1.17], this cannot happen if $\dim(X) \geq 4$ as $\rho(Y) = 1$. $\square$

7.5.12. Remark. If we drop the assumption $(\star)$, by Lefschetz hyperplane theorem, we can conclude : if $n \geq 4$, then $\widetilde{E}$ is not normal and the non normal locus of $\widetilde{E}$ is $f(D)$. However, now $\widetilde{E}$ may be not Fano.

7.5.13. Corollary. *Under the assumption $(\star)$. If $D \neq E$ and $L^3 = m$, then X is isomorphic to $B_{(3,6-m)}(\mathbb{P}^3)$ $(m \leq 2)$ or $B_{(2,6-m)}(Q^3)$ $(m \leq 3)$.*

Proof. By the proof of Theorem 7.5.11, we see that $\widetilde{E}$ is a smooth del Pezzo surface, $r_Y > r_X$ and $a r_X = 2$. It follows $r_X = 1$ or 2.

First we show that $r_X = 1$ cannot happen. Otherwise, we have $K_{\widetilde{E}} = -H$ and $a = 2$. If $r_Y = 4$, then $\widetilde{E}$ is a smooth cubic surface. It is well known that a smooth cubic surface is of Picard number seven, which is impossible. If $r_Y = 3$, then $H|_{\widetilde{E}} = (1,1)$, which contradicts $a = 2$. If $r_Y = 2$, then $\widetilde{E} \in |H|$. Since $\rho(\widetilde{E}) = 2$, it follows that $\widetilde{E}$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ or $\mathbb{F}_1$. However, in these two cases, we have $8 = K_{\widetilde{E}}^2 = (-H|_{\widetilde{E}})^2 = H^3$, this contradicts the fact $1 \leq H^3 \leq 5$ for $r_Y = 2$ (cf. [IP99, §12.1]).

Now assume $r_X = 2$. As $r_Y > r_X = 2$, it follows that Y is $\mathbb{P}^3$ $(r_Y = 4)$ or Q^3 $(r_Y = 3)$. By the proof of Theorem 7.5.11, it follows that $a = 1$ and $\widetilde{E} = f_* E \in |(r_Y - 2)H|$. Thus, $\widetilde{E}$ is an irreducible smooth quadric surface. We have $E|_E \in |-e_1 + b e_2|$ for some $b \in \mathbb{Z}$, where e_1 and e_2 are the two rulings of quadric and e_2 is the fiber of the blow up π. Since $D + E = f^*(r_Y - 2)H$, we have

$$D|_E = |(r_Y - 1)e_1 + (r_Y - 2 - b)e_2|.$$

In addition,

$$m = (\pi^* L)^3 = (f^* H + E)^3 = 8 - r_Y + b.$$

Therefore $b = m + r_Y - 8$ and $D|_E \in |(r_Y - 1)e_1 + (6-m)e_2|$. Hence X is isomorphic to $B_{(3,6-m)}(\mathbb{P}^3)$ or $B_{(2,6-m)}(Q^3)$. Furthermore, note that $-K_X$ is not nef if and only if $L \cdot C < 0$, this is equivalent to

$$\pi^* L \cdot e_1 = (E + f^* H) \cdot e_1 < 0,$$

which means $m \leq 6 - r_Y$. The proof is complete. $\square$

According to our results in this section, we propose the following questions.

7.5.14. **Question.**

(1) *Is there a non-projective Moishezon n-fold X with $b_2(X) = 1$ as in setup such that the induced morphism f is a conic bundle ?*
(2) *Is there a non-projective Moishezon n-fold X with $b_2(X) = 1$ as in setup such that $-K_X$ is big, f is birational and $D = E$?*
(3) *Is there a non projective Moishezon n-fold X with $b_2(X) = 1$ as in setup such that $-K_X$ is big, f is birational and the induced morphism $\pi_* f^*$ is not an isomorphism ?*

Appendices

A Higher dimensional Fano manifolds with $\rho \geq 2$

In [Wiś91b], Wiśniewski proved that if X is a n-dimensional Fano manifold of index $r \geq 2$ such that $n \leq 2r-2$, then $\rho(X) = 1$ unless $X \cong \mathbb{P}^{r-1} \times \mathbb{P}^{r-1}$. Moreover, a complete classification of Fano manifolds satisfying $n \leq 2r$ was given in [Wiś91b] and [Wiś94].

A.1 Fano manifolds with large index

Let X be a n-dimensional Fano manifolds with index r such that $n = 2r-1$ and $\rho(X) \geq 2$. In [Wiś91a], Wiśniewski shows that there exists a surjective morphism $p\colon X \to Y$ to a Fano manifold Y such that the coherent sheaf $\mathcal{E} = p_*\mathcal{O}_X(H)$ is locally free and $X \cong \mathbb{P}(\mathcal{E})$. The possiblities of the pair $(Y, \mathcal{E})$ are listed in the following table.

Y	$\mathcal{E}$	$h^0(X,H)$
Q^r	$\mathcal{O}(1)^{\oplus r}$	$r(r+2)$
$\mathbb{P}^r$	$T_{\mathbb{P}^r}$	$r(r+2)$
$\mathbb{P}^r$	$\mathcal{O}(2) \oplus \mathcal{O}(1)^{\oplus(r-1)}$	$\frac{3}{2}r(r+1)$

A.2 Fano manifolds with middle index

Let X be a n-dimensional Fano manifold with index r such that $n = 2r$ and $\rho(X) \geq 2$. Let H be the fundamental divisor of X. If $n = 4$, then X is a Mukai manifold and they are classified by Mukai (see [Muk89]). If $n \geq 6$ and X is not isomorphic to $\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2$, then $\rho(X) = 2$ and there exists an elementary contraction $p\colon X \to Y$ such that $\dim(Y) < \dim(X)$ and all fibers of p are of dimension $\leq r$ ([Wiś94]). Let us denote the torsion-free coherent sheaf $p_*\mathcal{O}_X(H)$ by $\mathcal{E}$. Then $X \cong \mathbb{P}(\mathcal{E})$ or X is a smooth divisor of relative degree two in $\mathbb{P}(\mathcal{E})$. We list the possibilities of $(X, \mathcal{E})$ in the following tables. For more details about the classification of these manifolds, we refer to the article [Wiś94].

(1) Y is smooth, $\mathcal{E}$ is locally free and $X \cong \mathbb{P}(\mathcal{E})$. Then the possibilities of the pair $(Y, \mathcal{E})$ and its corresponding $h^0(X,H)$ are listed in the following table.

Y	$\mathcal{E}$	$h^0(X,H)$
V_d	$\mathcal{O}_{V_d}(1)^{\oplus r}$	$r(r+d-1)$
$\mathbb{P}^{r+1}$	$\mathcal{O}(2)^{\oplus 2} \oplus \mathcal{O}(1)^{\oplus(r-2)}$	$(r+2)(2r+1)$
$\mathbb{P}^{r+1}$	$\mathcal{O}(3) \oplus \mathcal{O}(1)^{\oplus(r-1)}$	$\frac{1}{6}(r+2)(r^2+13r+6)$
Q^{r+1}	$\mathcal{O}(2) \oplus \mathcal{O}(1)^{\oplus(r-1)}$	$\frac{1}{2}(3r^2+11r+4)$
Q^4	$\mathbf{E}(1) \oplus \mathcal{O}(1)$	10

The variety V_d is a $(r+1)$-dimensional del Pezzo manifold of degree d and $1 \leq d \leq 5$. and $\mathbf{E}$ is the spinor bundle over Q^4. If we identify Q^4 with the Grassmannian $G(2,4)$, then $\mathbf{E}$ is isomorphic to the quotient bundle or the dual of the tautological bundle, particularly we have $h^0(Q^4, \mathbf{E}(1)) = 4$ [Ott88].

(2) Y is smooth, $\mathcal{E}$ is locally free and X is smooth divisor of relative degree two in $\mathbb{P}(\mathcal{E})$ over Y, namely

$$X \in |\mathcal{O}_{\mathbb{P}(\mathcal{E})}(2) \otimes \bar{p}^*\mathcal{O}_Y(-K_Y - \det(\mathcal{E}))|$$

where $\bar{p}\colon \mathbb{P}(\mathcal{E}) \to Y$ is the projection from the projective bundle.

Y	$\mathcal{E}$	$h^0(X,H)$
Q^r	$\mathcal{O}(1)^{\oplus(r+2)}$	$(r+2)^2$
$\mathbb{P}^r$	$\mathcal{O}(1)^{\oplus(r+2)}$	$(r+1)(r+2)$
$\mathbb{P}^r$	$\mathcal{O}(2) \oplus \mathcal{O}(1)^{\oplus(r+1)}$	$\frac{1}{2}(r+1)(3r+4)$
$\mathbb{P}^r$	$T_{\mathbb{P}^r} \oplus \mathcal{O}(1)^{\oplus 2}$	r^2+4r+2
$\mathbb{P}^r$	$\mathcal{O} \oplus \mathcal{O}(1)^{\oplus(r+1)}$	r^2+2r+2

(3) Y is smooth, $\mathcal{E}$ is not locally free and $X \cong \mathbb{P}(\mathcal{E})$. In this case, there exists an extension of $\mathcal{E}$ by $\mathcal{O}_Y$, that is, we have a sequence

$$0 \to \mathcal{O}_Y \to \mathcal{F} \to \mathcal{E} \to 0.$$

Since Y is a Fano manifold, we get $h^0(Y,\mathcal{E}) = h^0(Y,\mathcal{F}) - h^0(Y,\mathcal{O}_Y)$.

Y	$\mathcal{F}$	$h^0(X,H)$
Q^{r+1}	$\mathcal{O}(1)^{\oplus(r+1)}$	r^2+4r+2
$\mathbb{P}^{r+1}$	$\mathcal{G}$	$(r+2)(r+1)$
$\mathbb{P}^{r+1}$	$\mathcal{O}(2) \oplus \mathcal{O}(1)^{\oplus r}$	$\frac{1}{2}(3r^2+9r+4)$
$\mathbb{P}^{r+1}$	$T_{\mathbb{P}^{r+1}}$	r^2+4r+2

The sheaf $\mathcal{G}$ is a spanned locally free sheaf. Moreover, when $\mathcal{F} = \mathcal{G}$, X is a divisor of bidegree $(2,1)$ in $\mathbb{P}^{r+1} \times \mathbb{P}^r$ [BW96, Theorem 6.8].

B Smooth Fano threefolds with $\rho \geq 3$

Smooth Fano threefolds with $\rho \geq 2$ were given in Mori and Mukai in [MM81] and [MM03], and the general principal of how to classify such threefolds were explained in [MM83] and [MM86]. In the following, we collect the list of Fano threefolds with $\rho \geq 3$ which is used in §6.5.3 and we follow the notation and numbering in [MM81] and [MM03].

B.1 Fano threefolds which are not of type I

The following list contains the Fano threefolds mentioned in §(6.5.3.1) - §(6.5.3.5). We recall that $\mathbb{F}_1$ is the first Hirzebruch surface $\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(-1))$ which is a $\mathbb{P}^1$-bundle over $\mathbb{P}^1$ and it can be obtained by blowing-up a point on $\mathbb{P}^2$. Moreover, by S_r we denote a del Pezzo surface of degree r and we have $\rho(X) \geq 5$ in Table 5.

n^o	X	Section

Table 3		
1	a double cover of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ whose branch locus is a divisor of tridegree $(2,2,2)$	§(6.5.3.1)
2	a member of $\lvert L^{\otimes 2} \otimes \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(2,3)\rvert$ on the $\mathbb{P}^2$-bundle $\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(-1,-1)^{\oplus 2})$ over $\mathbb{P}^1 \times \mathbb{P}^1$, where L is the tautological bundle	§(6.5.3.2)
3	a smooth divisor on $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^2$ of tridegree $(1,1,2)$	§(6.5.3.2)
8	a member of the linear system $p_1^* g^* \mathcal{O}(1) \otimes p_2^* \mathcal{O}(2)$ on $\mathbb{F}_1 \times \mathbb{P}^2$, where p_i is the projection to the i-th factor and $g \colon \mathbb{F}_1 \to \mathbb{P}^2$ is the blow-up	§(6.5.3.2)
17	a smooth divisor on $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^2$ of tridegree $(1,1,1)$	§(6.5.3.2)
19	blow-up of $Q^3 \subset \mathbb{P}^4$ with center two points p and q on it which are not colinear	§(6.5.3.4)
27	$\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$	§(6.5.3.3)
28	$\mathbb{P}^1 \times \mathbb{F}_1$	§(6.5.3.3)
31	the $\mathbb{P}^1$-bundle $\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(1,1))$ over $\mathbb{P}^1 \times \mathbb{P}^1$	§(6.5.3.5)
Table 4		
1	a smooth divisor on $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ of multidegree $(1,1,1,1)$	§(6.5.3.2)
10	$\mathbb{P}^1 \times S_7$	§(6.5.3.3)
Table 5		
$3-8$	$\mathbb{P}^1 \times S_{11-\rho(X)}$	§(6.5.3.3)

B.2 Fano threefolds of Type I : complete intersection

In this appendix, we list the Fano threefolds of type I discussed in Proposition 6.5.14. The details were given in [MM86, §7].

n^o	Y	L	K_S^2	Type
Table 3				
4	$f \colon Y \to \mathbb{P}^1 \times \mathbb{P}^2$ is a double cover whose branch locus is a divisor of bidegree $(2,2)$.	$f^* p_2^* \mathcal{O}_{\mathbb{P}^2}(1)$, where $p_2 \colon \mathbb{P}^1 \times \mathbb{P}^2 \to \mathbb{P}^2$ is the projection to the second factor	4	I_1
7	$W \subset \mathbb{P}^2 \times \mathbb{P}^2$ a smooth divisor of bidegree $(1,1)$	$-\frac{1}{2}K_W$	6	I_1
11	V_7 is the $\mathbb{P}^1$-bundle $\pi \colon \mathbb{P}(\mathcal{E}) \to \mathbb{P}^2$ over $\mathbb{P}^2$ where $\mathcal{E} = \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(1)$.	$-\frac{1}{2}K_{V_7}$	7	I_1
24	$W \subset \mathbb{P}^2 \times \mathbb{P}^2$ a smooth divisor of bidegree $(1,1)$	$\mathcal{O}_W \otimes \mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^2}(0,1)$	8	I_1
26	$\pi \colon V_7 \to \mathbb{P}^3$ is the blow-up of $\mathbb{P}^3$ at a point p with the exceptional divisor E	$\pi^* \mathcal{O}_{\mathbb{P}^3}(1)$	9	I_1
Table 4				
4	$\pi \colon Y \to Q^3$ is the blow-up of $Q^3 \subset \mathbb{P}^4$ with center two points x_1 and x_2 on it which are not colinear with exceptional divisors E_1 and E_2.	$\pi^* \mathcal{O}_{Q^3}(1) \otimes \mathcal{O}_Y(-E_1 - E_2)$	6	I_1

9	$f\colon Y \to \mathbb{P}^3$ is obtained by first blowing up along a line ℓ and then blowing-up an exceptional line of the first blowing-up	$f^*\mathcal{O}_{\mathbb{P}^3}(1)$	8	I_1
Table 5				
1	$\pi\colon Y \to Q^3$ is the blow-up of $Q^3 \subset \mathbb{P}^4$ three points x_i on a conic on it with exceptional divisors E_i ($1 \leq i \leq 3$).	$\pi^*\mathcal{O}_{Q^3}(1) \otimes \mathcal{O}_Y(-E_1 - E_2 - E_3)$	5	I_1

The details of the calculations are as follows.

(1) $n^{\rm o}\,4$ in Table 3. Denote by H_1 and H_2 the line bundles $f^*\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^2}(1,0)$ and $f^*\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^2}(0,1)$, respectively. Then $L = H_2$ and by ramification formula, we have $K_Y = -H_1 - 2H_2$. It follows

$$(K_Y + L)^2 \cdot L = (-H_1 - H_2)^2 \cdot H_2 = 2H_1 \cdot H_2^2 = 4.$$

(2) $n^{\rm o}\,7$ in Table 3. Denote by H_1 and H_2 the line bundles $\mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^2}(1,0)$ and $\mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^2}(0,1)$, respectively. Then $W = H_1 + H_2$ and $K_W = (-2H_1 - 2H_2)|_W$. It follows

$$(K_W + L)^2 \cdot L = (-H_1 - H_2)^2(H_1 + H_2)(H_1 + H_2) = 6H_1^2H_2^2 = 6.$$

(3) $n^{\rm o}\,11$ in Table 3. Denote by ξ a Weil divisor associated to $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ and denote by H a Weil divisor associated to $\pi^*\mathcal{O}_{\mathbb{P}^2}(1)$. Then we have $K_{V_7} = -2\xi - 2H$. On the other hand, we have the following equality

$$\pi^*c_0(\mathcal{E}) \cdot \xi^2 - \pi^*c_1(\mathcal{E}) \cdot \xi - \pi^*c_2(\mathcal{E}) = 0.$$

This implies that $\xi^2 = H \cdot \xi + \pi^*c_2(\mathcal{E})$. As $\mathcal{E} = \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(1)$, we get $c_2(\mathcal{E}) = 0$. It follows

$$(K_{V_7} + L)^2 \cdot L = (\xi + H)^3 = \xi^2(\xi + 3H) + 3\xi \cdot H^2 = 7\xi \cdot H^2 = 7.$$

In fact, V_7 is the Fano manifold given in $n^{\rm o}\,35$ in Table 2 in [MM81]. In particular, we can also derive the same result by the fact $(-K_{V_7})^3 = 56$.

(4) $n^{\rm o}\,24$ in Table 3.enote by H_1 and H_2 the line bundles $\mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^2}(1,0)$ and $\mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^2}(0,1)$, respectively. Then $W = H_1 + H_2$ and $K_W = (-2H_1 - 2H_2)|_W$. It follows

$$(K_W + L)^{\cdot} L = (-2H_1 - 2H_2 + H_2)^2 \cdot H_2 \cdot (H_1 + H_2) = 8H_1^2H_2^2 = 8.$$

(5) $n^{\rm o}\,26$ in Table 3. Denote by H a Weil divisor associated to $L = \pi^*\mathcal{O}_{\mathbb{P}^3}(1)$. Then we have $K_{V_7} = -4H + 2E$. It follows

$$(K_{V_7} + L)^2 \cdot L = (-3H + 2E)^2 \cdot H = 9H^3 = 9.$$

(6) $n^{\rm o}\,4$ in Table 4. Denote by H a Weil divisor associated to $\pi^*\mathcal{O}_{Q^3}(1)$. Note that we have $K_Y = -3H + 2E_1 + 2E_2$. It follows

$$(K_Y + L)^2 \cdot L = (-2H + E_1 + E_2)^2(H - E_1 - E_2) = 4H^3 - E_1^3 - E_2^3 = 6.$$

(7) $n^{\rm o}\,9$ in Table 4. Denote by E_1 the strict transform of the exceptional divisor of the first blowing-up and denote by E_2 the exceptional divisor of the second blowing-up. Then we have $K_Y = -4H + E_1 + 2E_2$, where H is a Weil divisor associated to L. It follows

$$(K_Y + L)^2 \cdot L = (-3H + E_1 + 2E_2)^2 \cdot H = 9H^3 + H \cdot E_1^2 = 8.$$

(8) $n^{\rm o}\,1$ in Table 5. Let H be a Weil divisor associated to $\pi^*\mathcal{O}_{Q^3}(1)$. Then by adjunction formula we

have $K_Y = -3H + 2E_1 + 2E_2 + 2E_3$. It follows

$$(K_Y + L)^2 \cdot L = (-2H + E_1 + E_2 + E_3)^2(H - E_1 - E_2 - E_3) = 4H^3 - E_1^3 - E_2^3 - E_3^3 = 5.$$

B.3 Fano threefolds of Type I : not complete intersection

In this appendix, we list the Fano threefolds of type I such that C is not a complete intersection of two members of $|L|$. The details were given in [MM86, §7]. For n^o 13 in Table 4, we refer to the proof of Theorem 6.4.4 for the construction of free splitting of $-K_X$.

n^o	Y	C	L	Type
Table 3				
5	$\mathbb{P}^2 \times \mathbb{P}^1$	the intersection of two divisors $D_1 \in \lvert\mathcal{O}(2,0)\rvert$ and $D_2 \in \lvert\mathcal{O}(3,1)\rvert$	$\mathcal{O}(3,1)$	I_3
6	$\mathbb{P}^3$	disjoint union of a line and an elliptic curve of degree 4	$\mathcal{O}_{\mathbb{P}^3}(3)$	I_1
9	X is the $\mathbb{P}^1$-bundle $\mathbb{P}(\mathcal{E}) \to \mathbb{P}^2$ over $\mathbb{P}^2$, where $\mathcal{E} = \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(2)$.	the intersection of two divisors $D_1 \in \lvert\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)\rvert$ and $D_2 \in \lvert\mathcal{O}_{\mathbb{P}(\mathcal{E})}(2)\rvert$	$\mathcal{O}_{\mathbb{P}(\mathcal{E})}(2)$	I_2
10	$Q^3 \subset \mathbb{P}^4$	disjoint union of two conics	$\mathcal{O}_{Q^3}(2)$	I_1
12	$\mathbb{P}^3$	disjoint union of a line and a twisted cubic	$\mathcal{O}_{\mathbb{P}^3}(3)$	I_1
13	$W \subset \mathbb{P}^2 \times \mathbb{P}^2$ is a smooth divisor of bidegree $(1,1)$.	the intersection of two divisors $D_1 \in \lvert\mathcal{O}(1,0)\rvert$ and $D_2 \in \lvert\mathcal{O}(0,2)\rvert$	$\mathcal{O}_W \otimes \mathcal{O}(1,2)$	I_2
14	V_7 is the $\mathbb{P}^1$-bundle $\pi\colon \mathbb{P}(\mathcal{E}) \to \mathbb{P}^2$ over $\mathbb{P}^2$, where $\mathcal{E} = \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(1)$.	an intersection of two divisors $D_1 \in \lvert\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)\rvert$ and $D_2 \in \lvert\mathcal{O}_{\mathbb{P}(\mathcal{E})}(2) \otimes \pi^*\mathcal{O}_{\mathbb{P}^2}(1)\rvert$	$\mathcal{O}_{\mathbb{P}(\mathcal{E})}(2) \otimes \pi^*\mathcal{O}_{\mathbb{P}^2}(1)$	I_2
15	$Q^3 \subset \mathbb{P}^4$	disjoint union of a line and a conic	$\mathcal{O}_{Q^3}(2)$	I_1
16	$\pi\colon V_7 \to \mathbb{P}^3$ is the blow-up of $\mathbb{P}^3$ at a point p with the exceptional divisor E.	strict transform of a twisted cubic passing through p	$\pi^*\mathcal{O}_{\mathbb{P}^3}(2) \otimes \mathcal{O}_{V_7}(-E)$	I_1
18	$\mathbb{P}^3$	disjoint union of a line and a conic	$\mathcal{O}_{\mathbb{P}^3}(3)$	I_1
20	$Q^3 \subset \mathbb{P}^4$	disjoint union of two lines	$\mathcal{O}_{Q^3}(2)$	I_1
21	$\mathbb{P}^1 \times \mathbb{P}^2$	an intersection of two divisors $D_1 \in \lvert\mathcal{O}(0,1)\rvert$ and $D_2 \in \lvert\mathcal{O}(1,2)\rvert$	$\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^2}(1,2)$	I_1
22	$\mathbb{P}^1 \times \mathbb{P}^2$	a conic in $t \times \mathbb{P}^2$ $(t \in \mathbb{P}^1)$	$\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^2}(1,2)$	I_1
23	$\pi\colon V_7 \to \mathbb{P}^3$ is the blow-up of $\mathbb{P}^3$ at a point p with the exceptional divisor E	strict transform of a conic passing through p	$\pi^*\mathcal{O}_{\mathbb{P}^3}(2) \otimes \mathcal{O}_{V_7}(-E)$	I_1

25	$\mathbb{P}^3$	disjoint union of two lines	$\mathcal{O}_{\mathbb{P}^3}(2)$	I_1
29	$\pi\colon V_7 \to \mathbb{P}^3$ is the blow-up of $\mathbb{P}^3$ at a point p with the exceptional divisor E.	a complete intersection of D and a divisor $H \in \lvert\pi^*\mathcal{O}_{\mathbb{P}^3}(1) \otimes \mathcal{O}_{V_7}(-E)\rvert$	$\pi^*\mathcal{O}_{\mathbb{P}^3}(2) \otimes \mathcal{O}_{V_7}(-E)$	I_1
30	$\pi\colon V_7 \to \mathbb{P}^3$ is the blow-up of $\mathbb{P}^3$ at a point p with the exceptional divisor E.	strict transform of a line passing through p	$\pi^*\mathcal{O}_{\mathbb{P}^3}(2) \otimes \mathcal{O}_{V_7}(-E)$	I_1
Table 4				
2	Y is the $\mathbb{P}^1$-bundle $\pi\colon \mathbb{P}(\mathcal{E}) \to \mathbb{P}^1 \times \mathbb{P}^1$ over $\mathbb{P}^1 \times \mathbb{P}^1$, where $\mathcal{E} = \mathcal{O} \oplus \mathcal{O}(1,1)$.	an intersection of two divisors $D_1 \in \lvert\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)\rvert$ and $D_2 \in \lvert\mathcal{O}_{\mathbb{P}(\mathcal{E})}(2)\rvert$	$\mathcal{O}_{\mathbb{P}(\mathcal{E})}(2)$	I_2
3	$\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$	an intersection of two divisors $D_1 \in \lvert\mathcal{O}(1,1,0)\rvert$ and $D_2 \in \lvert\mathcal{O}(1,1,1)\rvert$	$\mathcal{O}(1,1,1)$	I_1
5	$\mathbb{P}^1 \times \mathbb{P}^2$	two disjoint curves C_1 and C_2 such that C_1 is an intersection of two divisors $D_1 \in \lvert\mathcal{O}(1,2)\rvert$ and $D_2 \in \lvert\mathcal{O}(0,1)\rvert$ and C_2 is a complete intersection of two members of $\lvert\mathcal{O}(0,1)\rvert$	$\mathcal{O}(1,2)$	I_1
6	$\mathbb{P}^3$	disjoint union of three lines	$\mathcal{O}_{\mathbb{P}^3}(3)$	I_1
7	$W \subset \mathbb{P}^2 \times \mathbb{P}^2$ is a smooth divisor of bidegree $(1,1)$	two disjoint curves on it of bidegree $(1,0)$ and $(0,1)$	$\mathcal{O}_W \otimes \mathcal{O}(1,1)$	I_1
8	$\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$	a smooth curve of tridegree $(0,1,1)$	$\mathcal{O}(1,1,1)$	I_1
11	$\mathbb{P}^1 \times \mathbb{F}_1$	$t \times E$, $t \in \mathbb{P}^1$ and E is an exceptional curve of the first kind on $\mathbb{F}_1$	$t \times \mathbb{F}_1 + \mathbb{P}^1 \times (E+F)$ [1]	I_1
12	$f\colon Y \to \mathbb{P}^3$ is obtained by first blowing up $\mathbb{P}^3$ at a point p and then blowing up along the proper transform of line ℓ passing through p.	the fiber of f over a point $q \neq p$	$f^*\mathcal{O}_{\mathbb{P}^3}(1)$	I_1
13	$\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$	an intersection of two divisors $D_1 \in \lvert\mathcal{O}(1,1,0)$ and $D_2 \in \lvert\mathcal{O}(2,1,1)\rvert$.	$\mathcal{O}(2,1,1)$	I_2
Table 5				
2	$\phi\colon Y \to \mathbb{P}^3$ is the blow-up along two disjoint lines.	two exceptional lines ℓ and ℓ' of ϕ such that ℓ_1 and ℓ_2 lie on the same exceptional divisor of ϕ.	$\phi^*\mathcal{O}_{\mathbb{P}^3}(2)$	I_2

1. F is a fiber of $\mathbb{F}_1 \to \mathbb{P}^1$.

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