PLURI-CANONICAL SYSTEMS FOR SURFACES OF GENERAL TYPE

AFTER ZARISKI, MUMFORD, KODAIRA, BOMBIERI, REIDER

1. The results

Let S be a compact complex surface of general type, S_{min} its minimal model gotten by contracting the finite number of (-1)-curves. It is a smooth surface with nef canonical line bundle $K_{S_{min}}$. It then can be shown that S_{min} has at most b_2 curves on which the canonical line bundle $K_{S_{min}}$ restricts to a non-ample bundle (i.e. whose intersection number with $K_{S_{min}}$ is non positive). These are (-2) smooth rational curves. Artin has shown that there is a normal surface S^* with a finite number of rational double points gotten by contracting those curves.

We will sketch two proofs of the following part of the results of Kodaira, Bombieri and Reider.

Theorem 1.1.
For $m \geq 4$, the m-pluricanonical map $\Phi_m : S_{min} \to \mathbb{P}(|mK_{S_{min}}|)$ is a morphism.

This in particular implies that the canonical line bundle of S_{min} is in fact semi-ample (a result previously obtained by Mumford using the works of Artin on S^* and the work of Zariski on base points of linear systems). Using the finite generation of the algebra associated with $\mathcal{O}_{\mathbb{P}^n}(1)$ on $\mathbb{P}^n$ and the morphism Φ_4 , we infer that the canonical ring $R(S_{min}) := \oplus_m H^0(S_{min}, K^m_{S_{min}})$ of S_{min} is finitely generated. Its projective spectrum $proj(R(S_{min})) =: S_{can}$ is the abstract canonical model of S. It contains no (-2)-curves but it is in general singular.

The second theorem of Kodaira and Bombieri is

Theorem 1.2.
For $m \geq 5$, the m-pluricanonical map $\Phi_m : S_{can} \to \mathbb{P}(|mK_{S_{min}}|)$ is an embedding.

The proof is based on the same kind of arguments.

2. An example

We describe an example due to Bombieri showing the sharpness of the bound in the theorem 1.1.

Start with Fermat quintic S' in $\mathbb{P}^3$ given in homogeneous coordinates by $x_1^5 + x_2^5 + x_3^5 + x_4^5 = 0$. The group $\mathbb{Z}_5$ acts freely by $\epsilon \cdot (x_1, x_2, x_3, x_4) = (x_1, \epsilon x_2, \epsilon^2 x_3, \epsilon^3 x_4)$ and the quotient S is a smooth surface. On the open set $U = \{x_1 \neq 0\}$, with affine coordinates x, y, z, with the equation $f(x, y, z) = 1 + x^2 + y^2 + z^2 = 0$ for S', the m-pluricanonical forms on S' are given in the form

$$w' = \frac{Q_m(x, y, z)}{(\frac{\partial f}{\partial z})^m}(dx \wedge dy)^m$$

where Q_m is a polynomial of degree at most m. A form on S is exactly a form on S' that is invariant under the group action. Note that $\epsilon \cdot dx \wedge dy = \epsilon^3 dx \wedge dy$, $\epsilon \cdot \frac{\partial}{\partial z} = 1/\epsilon^3 \frac{\partial}{\partial z}$. This shows that Q_m

homogenized has to contain only monomials of the form $x_{i_1}x_{i_2}x_{i_3}....x_{i_m}$ where $6m + \sum(i_k - 1) \equiv 0 \; mod 5$ i.e. $\sum i_k \equiv 0 \; mod 5$.

For $m = 3$, these are $x_1x_2^2, x_1^2x_3, x_2x_4^2, x_3^2x_4$.

For $m = 4$, these are $x_1x_3^3, x_1^2x_4^2, x_1^3x_2, x_1x_2x_3x_4, x_2^2x_3^2, x_2^3x_4, x_3x_4^3$.

Hence, $3K_S$ has two base points, whereas $4K_S$ is free.

3. Kodaira' s proof

From now on S will be a minimal surface of general type. Its canonical line bundle $K = K_S$ is nef and $K^2 > 0$.

3.1. **The main lines.** Choose a point x in S outside the locus of the (-2)-curves for simplicity (Otherwise the divisor of (-2)-curves has to be included in the coming discussion on connectedness). Denote by $\mathcal{O}(mK - x)$ the sheaf of local holomorphic sections of K^m that vanishes at x. From the long exact sequence associated with

$$0 \to \mathcal{O}(mK - x) \to \mathcal{O}(mK) \to \mathbb{C}_x \to 0$$

we find that the surjectivity of the evaluation map of m-forms at x would follow from the inequality

$$h^1(\mathcal{O}(mK - x)) \quad = \quad h^1(\mathcal{O}(mK)). \tag{3.1}$$

Let e be an integer greater than 1 such that $dim|eK| \geq 1$. Kodaira shows that

(i) If $m \geq e + 2$, then $h^1(\mathcal{O}(mK - x)) \leq h^1(\mathcal{O}((m - e)K))$.
(ii) If $m \geq e + 2$, then $h^1(\mathcal{O}((m - e)K)) \geq h^1(\mathcal{O}(mK))$.

From (ii), (the m arithmetic subsequences of dimensions are non increasing) there exists an integer m_0 from which the dimensions $h^1(\mathcal{O}((m - e)K))$ and $h^1(\mathcal{O}(mK))$ are equal. For $m \geq e + 2$ and $m \geq m_0$, $|mK|$ is base point free. Therefore, an easy vanishing theorem gives for $p \geq 2$, the equality $h^1(\mathcal{O}(pK)) = h^1(\mathcal{O}(K) \otimes \mathcal{O}((p - 1)K)) = 0$. One can choose $m_0 = e + 2$. (We could have instead argued with Kawamata-Viehweg vanishing)

Now, Riemann-Roch formula for $m \geq 2$ reads

$$P_m := h^0(\mathcal{O}(mK)) = \frac{m(m-1)}{2} K^2 + \chi(S).$$

Kodaira shows that in fact $P_2 \geq 2$ and that finally $e = 2$ and $m_0 = 4$ suits our purpose.

The proof of (ii) is like that of (i) actually easier. We will not give it. The rough idea for (i) is taken from the sequence

$$0 \to H^1(mK - D - x) \to H^1(mK - x) \to H^1(D, mK - x) \to 0$$

where D is a curve in $|eK|$ that passes through x so that $\mathcal{O}(D - x) = \mathcal{O}(D)$. and $H^1(mK - D - x) = H^1((m - e)K)$. The proof of (ii) hence reduces to a vanishing on the curve D. The difficulty is that in general D is neither irreducible nor smooth.

3.2. **Vanishing theorem.** This step is a careful examination of conditions needed on an irreducible curve to infer vanishing theorems from vanishing on smooth curves.

Theorem 3.1. *Let C be an irreducible curve in S, $L \to S$ a holomorphic line bundle, and x point of multiplicity μ in C. Let k be a positive integer.*
If $\deg_C(L-(K+C)) > (k-\mu+1)^+\mu$, then $H^1(C, L-kx) = 0$.

Note that if the point x has a large multiplicity in C, the vanishing is true for large k.

Proof. Consider $\eta : \tilde{C} \to C$ the normalization of the curve C. For a point x in C, $\eta^{-1}(x)$ can be written as a divisor $\sum_\lambda \mu_\lambda p_\lambda$ where λ runs through the set Λ of irreducible components of the germ (C, x).

We define the conductor

$$\delta = \delta_x^k := (k-\mu+1)^+ \sum_{\lambda\in\Lambda} \mu_\lambda p_\lambda.$$

Its degree is $(k-\mu+1)^+\mu$. Its main feature is that it provides an inclusion, where ι is the natural inclusion of C in S

$$\eta_\star\left(\mathcal{O}_{\tilde{C}}\left(K_{\tilde{C}} + (\iota\circ\eta)^\star(L-(K+C)) - \delta\right)\right) \subset \mathcal{O}_C(L-kx) \tag{3.2}$$

with a co-kernel M supported on non-simple points of C. Take it for granted for a moment. The condition on the degree stated in the theorem exactly amounts to assume the ampleness of $(\iota\circ\eta)^\star(L-(K+C)) - \delta$. The vanishing of $H^1(\tilde{C}, \mathcal{O}_{\tilde{C}}(K_{\tilde{C}} + (\iota\circ\eta)^\star(L-(K+C)) - \delta))$ follows. Taking the vanishing of $H^1(C, M)$ into account, this ends the proof.

The proof of the inclusion (3.2) is local and reduces to

$$\eta_\star(\mathcal{O}_{\tilde{C}}(K_{\tilde{C}} - (\iota\circ\eta)^\star(K+C) - \delta)) \subset \mathcal{O}_C(-kx) \tag{3.3}$$

Assume for simplicity that the curve C is locally given in $\mathbb{C}^2_{(w,z)}$ by the equation $R(w,z) = w^m - z^q = 0$ with $q \geq m$ (to insure multiplicity m at x) and $q \wedge m = 1$ (to ensure local irreducibility of (C, x)). Fix (a, b) such that $aq + bm = 1$. With a local coordinate t on $\tilde{C}$, the normalization map η is given by (t^q, t^m).

The idea is to take a function Φ in $\mathcal{O}_{\tilde{C}}$, to write it as $\Phi = \eta^\star\phi$ for a function ϕ in the fraction field of $\mathcal{O}_C$, and check that Φ being in the ideal $\mathcal{O}_{\tilde{C}}(K_{\tilde{C}} - (\iota\circ\eta)^\star(K+C) - \delta)$ makes it possible to choose ϕ in $\mathcal{O}_C(-kx)$.

From the equation $w^m - z^q = 0$, one sees that the function ϕ can be chosen as a polynomial of degree strictly less than m in w. More explicitly, factor $R(w,z)$ as $\prod_{i=1}^m (w - w_i(z))$. The $w_i(z) = \epsilon^i z^{q/m}$ are the m-th root of z^q. Note that $\frac{w^m - z^q}{w - w_i} = \sum_{l=0}^{m-1} w^l w_i^{m-1-l}$ and that $\phi(w_i, z) = \phi(\epsilon^i z^{q/m}, z) = \phi((\epsilon^{ai} z^{1/m})^q, (\epsilon^{ai} z^{1/m})^m) = \Phi(\epsilon^{ai} z^{1/m})$.

Then, applying Lagrange formula and Cauchy computation of power series coefficients (on $\tilde{C}$ for a function on C),

$$\begin{aligned}\phi(w,z) &= \sum_i \frac{R(w,z)}{w-w_i}\frac{\phi(w_i,z)}{\partial_w R(w_i,z)} = \sum_{l=0}^{m-1} w^l \sum_i w_i^{m-1-l}\frac{\Phi(\epsilon^{ai}z^{1/m})}{\partial_w R(w_i,z)} \\ &= \frac{1}{2\sqrt{-1}\pi}\sum_{l=0}^{m-1} w^l \sum_{n=0}^{+\infty}(t^m)^n \int_{|t|=c}\sum_i (\epsilon^i t^q)^{m-1-l}\frac{\Phi(\epsilon^{ai}t)}{\partial_w R(\epsilon^i t^q, t^m)}\frac{d(t^m)}{(t^m)^{n+1}} \\ &= \frac{1}{2\sqrt{-1}\pi}\sum_{l=0}^{m-1} w^l \sum_{n=0}^{+\infty} z^n \int_{|t|=c}\sum_i (\epsilon^i t^q)^{m-1-l}\frac{\Phi(\epsilon^{ai}t)}{(t^m)^{n+1}}\eta^\star\left(\frac{dz}{\partial_w R(w,z)}\right)\end{aligned}$$

If

$$m(n+1) + pole(\eta^\star\left(\frac{dz}{\partial_w R(w,z)}\right)) \leq q(m-1-l) + zero(\Phi)$$

the last integral vanishes. Hence, if we assume that

$$zero(\Phi) \geq pole(\eta^\star\left(\frac{dz}{\partial_w R(w,z)}\right)) + (k-m+1)m$$

the last integral vanishes for all the indices (l,n) with $n+1 \leq (m-1-l)+(k-m+1)$ (i.e. $n+l \leq k-1$). ☐

3.3. **Connectedness of pluricanonical divisors.** Using Hodge index theorem on surfaces, Kodaira shows a connectedness property of pluricanonical divisors. This will help to order the irreducible components of pluricanonical divisors in a way suitable for applying vanishing theorems.

Lemma 3.2. *Every decomposition $D = X + Y$ of a divisor $D \in |eK|$ into a sum of two non-numerically zero effective divisors fulfills $XY \geq 1$.*

Proof. Write the orthogonal decompositions $X = rK + \alpha$ and $Y = sK + \beta$, where $K\alpha = K\beta = 0$. The class $\alpha + \beta$ is $X + Y - (r+s)K = (e-r-s)K$. By orthogonality, $\alpha + \beta \sim 0$. We are interested in $XY = rsK^2 - \alpha^2$. For K is nef, the coefficients r and s are non-negative. For K is big, $K^2 > 0$. If $\alpha \sim 0$, then r and s are positive, hence the result. Otherwise, Hodge index theorem ($K^2 > 0$, $K\alpha = 0$, $\alpha \not\sim 0$) implies $\alpha^2 < 0$. ☐

Lemma 3.3. *One can order the irreducible components of a pluricanonical divisor $D = C_1 + C_2 + \cdots + C_n \in |eK|$ in such a way that*

$$K \cdot C_1 \geq 1 \quad (C_1 + C_2 + \cdots + C_{i-1}) \cdot C_i \geq 1.$$

3.4. **End of the proof.** Choose a point $x \in S$ not in the locus the (-2)-curves. For $|eK|$ is of positive dimension, there is a divisor D in $|eK|$ passing through x. Order its irreducible components thanks to the previous lemma. Set $\Theta_i := \mathcal{O}(mK - (C_i + C_{i+1} + \cdots + C_n) - x)$. Define the integer h to be the greater i such that x belongs to C_i. Then,

- $\forall i \leq h$, $\Theta_i = \mathcal{O}(mK - (C_i + C_{i+1} + \cdots + C_n))$
- $\Theta_1 = \mathcal{O}((m-e)K) \subset \Theta_2 \subset \cdots \subset \Theta_{n+1} = \mathcal{O}(mK - x)$

- $\Theta_{i+1}/\Theta_i = \mathcal{O}_{C_i}(mK - (C_{i+1} + \cdots + C_n) - \delta_{ih}x)$

To apply the vanishing theorem to Θ_{i+1}/Θ_i (for $i \neq h$), one has compute the degree of $mK - (C_{i+1} + \cdots + C_n) - (K + C_i) = (m - e - 1)K + (C_1 + C_2 + \cdots + C_{i-1})$ on C_i. It is positive for K is nef and by the connectedness property. For $i = h$, $K \cdot C_h$ is also positive, because x is not in the locus the (-2)-curves. Therefore, for $m \geq e + 2$, the cohomology groups $H^1(\Theta_{i+1}/\Theta_i)$ vanishes and

$$h^1(\mathcal{O}(mK - x)) = h^1(\Theta_{n+1}) \leq h^1(\Theta_1) = h^1(\mathcal{O}((m - e)K)$$

as required in (i).

4. Two words on Reider's proof

Let S be a surface, $L \to S$ a nef line bundle and x a base point for the linear system $|K_S + L|$. In others words, x fails to impose conditions on $|K_S + L|$. By a converse to the residue theorem due to Griffiths and Harris, there exists a rank 2 vector bundle $E \to S$ with determinant equal to L and a section $s \in H^0(X, L)$ whose zero locus is exactly x. Write its Kozsul complex

$$0 \to detE^\star \to E^\star \xrightarrow{s} \mathcal{M}_x \to 0.$$

It follows that $c_2(E) = 1$. Hence, if $L^2 = c_1(E)^2 \geq 5$ then $c_1(E)^2 > 4c_2(E)$ and E is unstable in the sense of Bogomolov. We have the following diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & & \\
 & & A \arrow[d] & & \\
0 \arrow[r] & \mathcal{O} \arrow[r, "s"] \arrow[dr, "t"'] & E \arrow[r] \arrow[d] & L \otimes \mathcal{M}_x \arrow[r] & 0 \\
 & & B \otimes \mathcal{I}_Z \arrow[d] & & \\
 & & 0 & &
\end{tikzcd}$$

with $(A - B)^2 > 0$ and $(A - B) \cdot H > 0$ for every ample divisor H (this in particular implies that A is destabilizing E, $A \cdot H > \frac{c_1(E) \cdot H}{2}$).

One can how that t provides a non-zero section of B that vanishes at x. Its divisor C is a curve containing x. For $t = 0$ on the irreducible components of C, there is a map $\mathcal{O}_C \to A_{|C}$ so that $A \cdot C = (L - C) \cdot C \geq 0$. On the other hand $1 = c_2(E) = A \cdot B + degZ \geq (L - C) \cdot C$. With a little bit more care, one can show that either ($L \cdot C = 0$ and $C^2 = -1$) or ($L \cdot C = 1$ and $C^2 = 0$).

Applied for $L = 3K$, we infer that on a minimal surface of general type ($L^2 \geq 5$) the linear surface $|4K|$ is base point free.

References

[Artin] Artin, Michael *Some numerical criteria for contractability of curves on algebraic surfaces.* Amer. J. Math. 84 1962 485–496.

[BHPV] Barth, Wolf P.; Hulek, Klaus; Peters, Chris A. M.; Van de Ven, Antonius *Compact complex surfaces.* Second edition. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge.

[Kodaira] Kodaira, Kunihiko *Pluricanonical systems on algebraic surfaces of general type.* J. Math. Soc. Japan 20 1968 170–192.

On compact complex analytic surfaces. I. Ann. of Math. (2) 71 1960 111–152.

[Mumford] In [O. Zariski, Ann. of Math. (2) 76 (1962), 560–615], appendix by D. Mumford, pages 612–615;

[Reider] Reider, Igor *Vector bundles of rank* 2 *and linear systems on algebraic surfaces.* Ann. of Math. (2) 127 (1988), no. 2, 309–316.

[Zariski] Zariski, Oscar *The theorem of Riemann-Roch for high multiples of an effective divisor on an algebraic surface.* Ann. of Math. (2) 76 1962 560–615.