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REMARKS ON THE EXTENSION OF TWISTED HODGE METRICS

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1. INTRODUCTION

The aims of this text are to announce the result in a paper [MT3], to give proofs of some special cases of it, and to make comments and remarks for the proof given there. Because the full proof in [MT3] is much more involved and technical, we shall give a technical introduction and proofs for weaker statements in this text (see Theorem 1.5 and 1.6). This text is basically independent from [MT3].

1.1. **Result in** [MT3]. Our main concern is the positivity of direct image sheaves of adjoint bundles $R^qf_*(K_{X/Y}\otimes E)$, for a Kähler morphism $f:X\longrightarrow Y$ endowed with a Nakano semi-positive holomorphic vector bundle (E,h) on X. In our previous paper [MT2], generalizing a result [B] in case $q=0$, we obtained the Nakano semi-positivity of $R^qf_*(K_{X/Y}\otimes E)$ with respect to the Hodge metric, under the assumption that $f:X\longrightarrow Y$ is smooth. However the smoothness assumption on f is rather restrictive, and it is desirable to remove it.

To state our result precisely, let us fix notations and recall basic facts. Let $f:X\longrightarrow Y$ be a holomorphic map of complex manifolds. A real d-closed $(1,1)$-form ω on X is said to be *a relative Kähler form* for f, if for every point $y\in Y$, there exists an open neighbourhood W of y and a smooth plurisubharmonic function ψ on W such that $\omega+f^*(\sqrt{-1}\partial\bar\partial\psi)$ is a Kähler form on $f^{-1}(W)$. A morphism f is said to be *Kähler*, if there exists a relative Kähler form for f ([Tk, 6.1]), and $f:X\longrightarrow Y$ is said to be a *Kähler fiber space*, if f is proper, Kähler, and surjective with connected fibers,

Set up 1.1. (1) Let X and Y be complex manifolds of $\dim X=n+m$ and $\dim Y=m$, and let $f:X\longrightarrow Y$ be a Kähler fiber space. We do not fix a relative Kähler form for f, unless otherwise stated. The *discriminant locus* of $f:X\longrightarrow Y$ is the minimum closed analytic subset $\Delta\subset Y$ such that f is smooth over $Y\setminus\Delta$.

(2) Let (E,h) be a Nakano semi-positive holomorphic vector bundle on X. Let q be an integer with $0\le q\le n$. By Kollár [Kol] and Takegoshi [Tk], $R^qf_*(K_{X/Y}\otimes E)$ is torsion free on Y, and moreover it is locally free on $Y\setminus\Delta$ ([MT2, 4.9]). In particular

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we can let $S_q \subset \Delta$ be the minimum closed analytic subset of $\mathrm{codim}_Y S_q \geq 2$ such that $R^q f_*(K_{X/Y} \otimes E)$ is locally free on $Y \setminus S_q$. Let $\pi : \mathbb{P}(R^q f_*(K_{X/Y} \otimes E)|_{Y \setminus S_q}) \longrightarrow Y \setminus S_q$ be the projective space bundle, and let $\pi^*(R^q f_*(K_{X/Y} \otimes E)|_{Y \setminus S_q}) \longrightarrow \mathcal{O}(1)$ be the universal quotient line bundle.

(3) Let ω_f be a relative Kähler form for f. Then we have the Hodge metric g on the vector bundle $R^q f_*(K_{X/Y} \otimes E)|_{Y \setminus \Delta}$ with respect to ω_f and h ([MT2, §5.1]). By the quotient $\pi^*(R^q f_*(K_{X/Y} \otimes E)|_{Y \setminus \Delta}) \longrightarrow \mathcal{O}(1)|_{\pi^{-1}(Y \setminus \Delta)}$, the metric $\pi^* g$ gives the quotient metric $g^{\circ}_{\mathcal{O}(1)}$ on $\mathcal{O}(1)|_{\pi^{-1}(Y \setminus \Delta)}$. The Nakano, even weaker Griffiths, semi-positivity of g (by [B, 1.2] for $q = 0$, and by [MT2, 1.1] for q general) implies that $g^{\circ}_{\mathcal{O}(1)}$ has a semi-positive curvature. $\square$

In these notations, the main result in [MT3] is as follows.

Theorem 1.2. *Let $f : X \longrightarrow Y$, (E, h) and $0 \leq q \leq n$ be as in Set up 1.1.*

(1) Unpolarized case. Then, for every relatively compact open subset $Y_0 \subset Y$, the line bundle $\mathcal{O}(1)|_{\pi^{-1}(Y_0 \setminus S_q)}$ on $\mathbb{P}(R^q f_*(K_{X/Y} \otimes E)|_{Y_0 \setminus S_q})$ has a singular Hermitian metric with semi-positive curvature, and which is smooth on $\pi^{-1}(Y_0 \setminus \Delta)$.

(2) Polarized case. Let ω_f be a relative Kähler form for f. Assume that there exists a closed analytic set $Z \subset \Delta$ of $\mathrm{codim}_Y Z \geq 2$ *such that $f^{-1}(\Delta)|_{X \setminus f^{-1}(Z)}$ is a divisor and has a simple normal crossing support (or empty). Then the Hermitian metric $g^{\circ}_{\mathcal{O}(1)}$ on $\mathcal{O}(1)|_{\pi^{-1}(Y \setminus \Delta)}$ can be extended as a singular Hermitian metric $g_{\mathcal{O}(1)}$ with semi-positive curvature of $\mathcal{O}(1)$ on $\mathbb{P}(R^q f_*(K_{X/Y} \otimes E)|_{Y \setminus S_q})$.*

Theorem 1.2 (1) is reduced to Theorem 1.2 (2) for $f' = f \circ \mu : X' \longrightarrow Y$ after a modification $\mu : X' \longrightarrow X$. Then however the induced map $f' : X' \longrightarrow Y$ is only locally Kähler in general. Hence we need to restrict everything on relatively compact subsets of Y in Theorem 1.2 (1).

If in particular in Theorem 1.2, $R^q f_*(K_{X/Y} \otimes E)$ is locally free and Y is a smooth projective variety, then the vector bundle $R^q f_*(K_{X/Y} \otimes E)$ is pseudo-effective in the sense of [DPS, §6]. This notion [DPS, §6] is a natural generalization of the fact that on a smooth projective variety, a divisor D is pseudo-effective (i.e., a limit of effective divisors) if and only if the associated line bundle $\mathcal{O}(D)$ admits a singular Hermitian metric with semi-positive curvature. The above curvature property of $\mathcal{O}(1)$ leads to the following algebraic positivity of $R^q f_*(K_{X/Y} \otimes E)$.

Theorem 1.3. *Let $f : X \longrightarrow Y$ be a surjective morphism with connected fibers between smooth projective varieties, and let (E, h) be a Nakano semi-positive holomorphic vector bundle on X. Then the torsion free sheaf $R^q f_*(K_{X/Y} \otimes E)$ is weakly positive over $Y \setminus \Delta$ (the smooth locus of f), in the sense of Viehweg* [Vi2, 2.13].

See [MT3, §1] for further introduction.

1.2. **Statement in this text.** We would like to explain the proofs of the following two theorems in this text. Because there is no essential limitations of the number of pages, we may repeat some arguments and make comments repetitiously.

Set up 1.4. (General set up.) Let $f : X \longrightarrow Y$ be a holomorphic map of complex manifolds, which is proper, Kähler, surjective with connected fibers, and f is smooth over the complement $Y \setminus \Delta$ of a closed analytic subset $\Delta \subset Y$. Let ω_f be a relative Kähler form for f, and let (E, h) be a Nakano semi-positive holomorphic vector bundle on X. Let q be a non-negative integer.

It is known by Kollár [Kol] and Takegoshi [Tk] that $R^q f_*(K_{X/Y} \otimes E)$ is torsion free, and moreover it is locally free where f is smooth ([MT2, 4.9]). In particular we can let $S_q \subset \Delta$ be the minimum closed analytic subset of $\operatorname{codim}_Y S_q \geq 2$ such that $R^q f_*(K_{X/Y} \otimes E)$ is locally free on $Y \setminus S_q$. Once we take a relative Kähler form ω_f for f, we then have the Hodge metric g on the vector bundle $R^q f_*(K_{X/Y} \otimes E)|_{Y \setminus \Delta}$ with respect to ω_f and h ([MT2, §5.1] or Remark 2.6). □

Theorem 1.5. *In Set up 1.4, assume further that* $\dim Y = 1$. *Let L be a quotient holomorphic line bundle of $R^q f_*(K_{X/Y} \otimes E)$. Then L has a singular Hermitian metric with semi-positive curvature, whose restriction on $Y \setminus \Delta$ is the quotient metric of the Hodge metric g on $R^q f_*(K_{X/Y} \otimes E)|_{Y \setminus \Delta}$.*

Theorem 1.6. *In Set up 1.4, assume further that f has reduced fibers in codimension 1 on Y, i.e., there exists a closed analytic set $Z \subset \Delta$ of* $\operatorname{codim}_Y Z \geq 2$ *such that every fiber of $y \in Y \setminus Z$ is reduced. Let L be a holomorphic line bundle on Y with a surjection $R^q f_*(K_{X/Y} \otimes E)|_{Y \setminus Z} \longrightarrow L|_{Y \setminus Z}$. Then L has a singular Hermitian metric with semi-positive curvature, whose restriction on $Y \setminus \Delta$ is the quotient metric of the Hodge metric g on $R^q f_*(K_{X/Y} \otimes E)|_{Y \setminus \Delta}$.*

The above assumptions: $\dim Y = 1$, and/or with reduced fibers, or even fibers are semi-stable, are quite usual in algebraic geometry. In this sense, the assumptions in Theorem 1.5 and 1.6 are not so artificial.

1.3. **Complement.** Here is a comment on the relation between the statements in §1.1 and those in §1.2. Although we will not give proofs, we can pursue the method of proof of Theorem 1.5 and 1.6 to show the following two statements, as we show Theorem 1.2 in [MT3].

Theorem 1.7. *In Set up 1.4, assume further that* $\dim Y = 1$. *Then the line bundle $\mathcal{O}(1)$ for $\pi : \mathbb{P}(R^q f_*(K_{X/Y} \otimes E)) \longrightarrow Y$ has a singular Hermitian metric $g_{\mathcal{O}(1)}$ with semi-positive* *curvature, and whose restriction on* $\pi^{-1}(Y \setminus \Delta)$ *is the quotient metric* $g^\circ_{\mathcal{O}(1)}$ *of* $\pi^* g$*, where* g *is the Hodge metric with respect to* ω_f *and* h*.*

Theorem 1.8. *In Set up 1.4, assume further that* f *has reduced fibers in codimension 1 on* Y*. Then the line bundle* $\mathcal{O}(1)$ *for* $\pi : \mathbb{P}(R^q f_*(K_{X/Y} \otimes E)|_{Y \setminus S_q}) \longrightarrow Y \setminus S_q$ *has a singular Hermitian metric* $g_{\mathcal{O}(1)}$ *with semi-positive curvature, and whose restriction on* $\pi^{-1}(Y \setminus \Delta)$ *is the quotient metric* $g^\circ_{\mathcal{O}(1)}$ *of* $\pi^* g$*, where* g *is the Hodge metric with respect to* ω_f *and* h*.*

One clear difference between §1.1 and §1.2 is geometric conditions on $f : X \longrightarrow Y$. Another is about line bundles to be considered, namely $\mathcal{O}(1)$ or L. For example, Theorem 1.7 (or 1.2) concerns all rank 1 quotient of $R^q f_*(K_{X/Y} \otimes E)$, while Theorem 1.5 concerns a rank 1 quotient of $R^q f_*(K_{X/Y} \otimes E)$, hence Theorem 1.7 is naturally stronger than Theorem 1.5. In fact Theorem 1.7 implies Theorem 1.5 by a standard argument ([MT3, §6.2]). The proof of Theorem 1.7 (as well as Theorem 1.2) requires another uniform estimate which does not depend on rank 1 quotients L of $R^q f_*(K_{X/Y} \otimes E)$, other than the uniform estimate given in Lemma 3.3 of the proof of Theorem 1.5.

2. Preliminary Arguments

2.1. **Localization.** As the next lemma shows, to see our theorems, we can neglect codimension 2 analytic subsets of Y.

Lemma 2.1. *Let* Y *be a complex manifold, and* Z *a closed analytic subset of* Y *with* $\operatorname{codim}_Y Z \geq 2$*. Let* L *be a holomorphic line bundle on* Y *with a singular Hermitian metric* h *on* $L|_{Y \setminus Z}$ *with semi-positive curvature. Then* h *extends as a singular Hermitian metric on* L *with semi-positive curvature.*

Proof. Let W be a small open subset of Y with a nowhere vanishing section $e \in H^0(W, L)$. Then a function $h(e,e)$ on $W \setminus Z$ can be written as $h(e,e) = e^{-\varphi}$ with a plurisubharmonic function φ on $W \setminus Z$. By Hartogs type extension for plurisubharmonic functions, φ can be extended uniquely as a plurisubhamonic function $\widetilde{\varphi}$ on W. Then $e^{-\widetilde{\varphi}}$ gives the desired extension of h on W. □

In particular, we can neglect the set S_q (resp. Z) in Set up 1.4 (resp. in Theorem 1.6), and only consider codimension 1 part of the discriminant locus Δ. Once we obtain the Hodge metric g of $R^q f_*(K_{X/Y} \otimes E)|_{Y \setminus \Delta}$ or the quotient metric g°_L of $L|_{Y \setminus \Delta}$, the extension property of g°_L is a local question. Hence we can further reduce our situation to the following

Set up 2.2. (Generic local set up.) Let Y be (a complex manifold which is biholomorphic to) a unit ball in $\mathbb{C}^m$ with coordinates $t = (t_1, \ldots, t_m)$, X a complex manifold of $\dim X = n + m$ with a Kähler form ω. Let $f : X \longrightarrow Y$ be a proper surjective holomorphic map with connected fibers. Let (E, h) be a Nakano semi-positive holomorphic vector bundle on X, and let q be an integer with $0 \leq q \leq n$. Let $K_Y \cong \mathcal{O}_Y$ be a trivialization by a nowhere vanishing section $dt = dt_1 \wedge \ldots \wedge dt_m \in H^0(Y, K_Y)$. Let g be the Hodge metric on $R^q f_*(K_{X/Y} \otimes E)|_{Y \setminus \Delta}$ with respect to ω and h. Let us assume the following:

(1) f is flat, and the discriminant locus $\Delta \subset Y$ is $\Delta = \{t_m = 0\}$.

(2) $R^q f_*(K_{X/Y} \otimes E) \cong \mathcal{O}_Y^{\oplus r}$, i.e., globally free and trivialized of rank r.

(3) Let $f^*\Delta = \sum b_i B_i$ be the prime decomposition. For every B_i, the induced morphism $f : \operatorname{Reg} B_i \longrightarrow \Delta$ is surjective and smooth. Here $\operatorname{Reg} B_i$ is the smooth locus of B_i. If $B_i \neq B_j$, the intersection $B_i \cap B_j$ does not contain any fiber of f.

We may replace Y by slightly smaller balls, or may assume everything is defined over a slightly larger ball. □

Remark 2.3. (1) For this moment, in Set up 2.2, we do not assume that $\dim Y = 1$, nor that f has reduced fibers.

(2) Set up 2.2 (3) is automatically satisfied in case $\dim Y = 1$.

(3) Refer [MT2, 5.2] for the replacement of a relative Kähler form ω_f by a Kähler form ω. □

Notation 2.4. (1) For a non-negative integer d, we set $c_d = \sqrt{-1}^{d^2}$.

(2) Let $f : X \longrightarrow Y$ be as in Set up 2.2. We set $\Omega^p_{X/Y} = \bigwedge^p \Omega^1_{X/Y}$ rather formally, because we will only deal $\Omega^p_{X/Y}$ on which f is smooth. For an open subset $U \subset X$ where f is smooth, and for a differentiable form $\sigma \in A^{p,0}(U, E)$, we say σ is relatively holomorphic and write $[\sigma] \in H^0(U, \Omega^p_{X/Y} \otimes E)$, if $\sigma \wedge f^*dt \in H^0(U, \Omega^{p+m}_X \otimes E)$. □

2.2. **Relative hard Lefschetz type theorem.** We discuss in Set up 2.2.

One fundamental ingredient, even in the definition of Hodge metrics, is the following proposition. In case $q = 0$, this is quite elementary.

Proposition 2.5. [Tk, 5.2]. *There exist $H^0(Y, \mathcal{O}_Y)$-module homomorphisms*

$$* \circ \mathcal{H} : H^0(Y, R^q f_*(K_{X/Y} \otimes E)) \longrightarrow H^0(X, \Omega_X^{n+m-q} \otimes E),$$
$$L^q : H^0(X, \Omega_X^{n+m-q} \otimes E) \longrightarrow H^0(Y, R^q f_*(K_{X/Y} \otimes E))$$

such that (1) $(c_{n+m-q}/q!) L^q \circ (* \circ \mathcal{H}) = id$, and (2) for every $u \in H^0(Y, R^q f_*(K_{X/Y} \otimes E))$, there exists a relative holomorphic form $[\sigma_u] \in H^0(X \setminus f^{-1}(\Delta), \Omega^{n-q}_{X/Y} \otimes E)$ such that

$$(* \circ \mathcal{H}(u))|_{X \setminus f^{-1}(\Delta)} = \sigma_u \wedge f^*dt.$$

Proof. We take a smooth strictly plurisubhamonic exhaustion function ψ on Y, for example $\|t\|^2$. Recalling $R^qf_*(K_{X/Y} \otimes E) = K_Y^{\otimes(-1)} \otimes R^qf_*(K_X \otimes E)$, the trivialization $K_Y \cong \mathcal{O}_Y$ by dt gives an isomorphism $R^qf_*(K_{X/Y} \otimes E) \cong R^qf_*(K_X \otimes E)$. Since Y is Stein, we have also a natural isomorphism $H^0(Y, R^qf_*(K_X \otimes E)) \cong H^q(X, K_X \otimes E)$. We denote by α^q the composed isomorphism

$$\alpha^q : H^0(Y, R^qf_*(K_{X/Y} \otimes E)) \xrightarrow{\sim} H^q(X, K_X \otimes E).$$

With respect to the Kähler form ω on X, we denote by $*$ the Hodge $*$-operator, and by

$$L^q : H^0(X, \Omega_X^{n+m-q} \otimes E) \longrightarrow H^q(X, K_X \otimes E)$$

the Lefschetz homomorphism induced from $\omega^q \wedge \bullet$. Also with respect to ω and h, we set $\mathcal{H}^{n+m,q}(X, E, f^*\psi) = \{u \in A^{n+m,q}(X, E);\ \overline{\partial}u = \vartheta_h u = 0,\ e(\overline{\partial}(f^*\psi))^*u = 0\}$. (We do not explain what this space of harmonic forms is, because the definition is not important in this text.) By [Tk, 5.2.i], $\mathcal{H}^{n+m,q}(X, E, f^*\psi)$ represents $H^q(X, K_X \otimes E)$ as an $H^0(Y, \mathcal{O}_Y)$-module, and hence there exists a natural isomorphism

$$\iota : \mathcal{H}^{n+m,q}(X, E, f^*\psi) \xrightarrow{\sim} H^q(X, K_X \otimes E)$$

given by taking the Dolbeault cohomology class. We have an isomorphism

$$\mathcal{H} = \iota^{-1} \circ \alpha^q : H^0(Y, R^qf_*(K_{X/Y} \otimes E)) \xrightarrow{\sim} \mathcal{H}^{n+m,q}(X, E, f^*\psi).$$

Also by [Tk, 5.2.i], the Hodge $*$-operator gives an injective homomorphism

$$* : \mathcal{H}^{n+m,q}(X, E, f^*\psi) \longrightarrow H^0(X, \Omega_X^{n+m-q} \otimes E),$$

and induces a splitting $* \circ \iota^{-1} : H^q(X, K_X \otimes E) \longrightarrow H^0(X, \Omega_X^{n+m-q} \otimes E)$ for the Lefschetz homomorphism L^q such that $(c_{n+m-q}/q!)L^q \circ * \circ \iota^{-1} = id$. (The homomorphism δ^q in [Tk, 5.2.i] with respect to ω and h is $* \circ \iota^{-1}$ times a universal constant.) In particular

$$(c_{n+m-q}/q!)((\alpha^q)^{-1} \circ L^q) \circ (* \circ \mathcal{H}) = id.$$

All homomorphisms $\alpha^q, *, L^q, \iota, \mathcal{H}$ are as $H^0(Y, \mathcal{O}_Y)$-modules.

Let $u \in H^0(Y, R^qf_*(K_{X/Y} \otimes E))$. Then we have $* \circ \mathcal{H}(u) \in H^0(X, \Omega_X^{n+m-q} \otimes E)$, and then by [Tk, 5.2.ii]

$$(* \circ \mathcal{H}(u))|_{X \setminus f^{-1}(\Delta)} = \sigma_u \wedge f^*dt$$

for some $[\sigma_u] \in H^0(X \setminus f^{-1}(\Delta), \Omega_{X/Y}^{n-q} \otimes E)$. It is not difficult to see $[\sigma_u] \in H^0(X \setminus f^{-1}(\Delta), \Omega_{X/Y}^{n-q} \otimes E)$ does not depend on the particular choice of a global frame dt of K_Y. □

Remark 2.6. We recall the definition of the Hodge metric g of $R^qf_*(K_{X/Y} \otimes E)|_{Y \setminus \Delta}$ with respect to ω and h [MT2, 5.1]. We only mention it for a global section $u \in$

$H^0(Y, R^q f_*(K_{X/Y} \otimes E))$. It is given by

$$g(u,u)(t) = \int_{X_t} (c_{n-q}/q!)(\omega^q \wedge \sigma_u \wedge h\overline{\sigma_u})|_{X_t}$$

at $t \in Y \setminus \Delta$. In the relation

$$(* \circ \mathcal{H}(u))|_{X \setminus f^{-1}(\Delta)} = \sigma_u \wedge f^* dt,$$

the left hand side is holomorphically extendable across $f^{-1}(\Delta)$, and is non-vanishing if u is, in an appropriate sense. In the right hand side, f^*dt may only have zero along $f^{-1}(\Delta)$, that is "Jacobian" of f, and hence σ_u may only have "pole" along $f^{-1}(\Delta)$. This is the main reason why $g(u,u)(t)$ has a positive lower bound on $Y \setminus \Delta$, and which is fundamental for the extension of positivity (see (5) of the proof of Proposition 2.7 below). The importance of the role of the Jacobian of f is already observed by Fujita [Ft]. □

2.3. **Non-uniform estimate.** Here we state a weak extension property. This is a basic reason for all extension of positivity of direct image sheaves of relative canonical bundles, for example in [Ft], [Ka1], [Vi1], and so on. However this is not enough to conclude the results in §1.

Proposition 2.7. *In Set up 1.4, let $W \subset Y$ be an open subset, and let $u \in H^0(W \setminus S_q, R^q f_*(K_{X/Y} \otimes E))$ which is nowhere vanishing on $W \setminus S_q$. Then the smooth plurisubharmonic function $-\log g(u,u)$ on $W \setminus \Delta$ can be extended as a plurisubharmonic function on W.*

Proof. We may assume $W = Y$. Moreover it is enough to consider in Set up 2.2 as before. In particular $S_q = \emptyset$ and $\Delta = \{t_m = 0\}$. We shall discuss the extension property at the origin $t = 0 \in Y$, and hence we replace Y by a small ball centered at $t = 0$.

(1) By Proposition 2.5, we have $* \circ \mathcal{H}(u) \in H^0(X, \Omega_X^{n+m-q} \otimes E)$. This $* \circ \mathcal{H}(u)$ does not vanish identically along $\Delta = \{t_m = 0\} \subset Y$ as an element of $H^0(Y, \mathcal{O}_Y)$-module $H^0(X, \Omega_X^{n+m-q} \otimes E)$. This is saying that there exists at least one component B_j in $f^*\Delta = \sum b_i B_i$ such that $* \circ \mathcal{H}(u)$ does not vanish of order greater than or equal to b_j along B_j. We take one such B_j and denote by

$$B = B_j \text{ and } b = b_j.$$

(2) We take a general point $x_0 \in B \cap f^{-1}(0)$ so that x_0 is a smooth point on $(f^*\Delta)_{red}$, and take local coordinates $(U; z = (z_1, \dots, z_{n+m}))$ centered at $x_0 \in X$. We may assume $f(U) = Y$ and $t = f(z) = (z_{n+1}, \dots, z_{n+m-1}, z_{n+m}^b)$ on U.

Over U, the bundle E is also trivialized, i.e., $E|_U \cong U \times \mathbb{C}^{r(E)}$, where $r(E)$ is the rank of E. Using the local trivializations on U, we have a constant $a > 0$ such that (i) $\omega \geq a\omega_{eu}$ on U, where $\omega_{eu} = \sqrt{-1}/2 \sum_{i=1}^{n+m} dz_i \wedge d\overline{z_i}$ is the standard complex euclidean

Kähler form, and (ii) $h \geq a\mathrm{Id}$ on U as Hemitian matrixes. Here we regard $h|_U(x)$ as a positive definite Hermitian matrix at each $x \in U$ in terms of $E|_U \cong U \times \mathbb{C}^{r(E)}$, and here Id is the $r(E) \times r(E)$ identity matrix.

(3) By Proposition 2.5, we can write as $(* \circ \mathcal{H}(u))|_{X \setminus f^{-1}(\Delta)} = \sigma_u \wedge f^*dt$ for some $\sigma_u \in A^{n-q,0}(X \setminus f^{-1}(\Delta), E)$. We write $\sigma_u = \sum_{I \in I_{n-q}} \sigma_I dz_I + R$ on $U \setminus B$. Here I_{n-q} is the set of all multi-indexes $1 \leq i_1 < \ldots < i_{n-q} \leq n$ of length $n-q$ (not including $n+1, \ldots, n+m$), $\sigma_I = {}^t(\sigma_{I,1}, \ldots, \sigma_{I,r(E)})$ is a vector valued holomorphic function with $\sigma_{I,i} \in H^0(U \setminus B, \mathcal{O}_X)$, and here $R = \sum_{k=1}^m R_k \wedge dz_{n+k} \in A^{n-q,0}(U \setminus B, E)$. Now

$$\sigma_u \wedge f^*dt = bz_{n+m}^{b-1}\left(\sum_{I \in I_{n-q}} \sigma_I dz_I\right) \wedge dz_{n+1} \wedge \ldots \wedge dz_{n+m}$$

on $U \setminus B$. Since $\sigma_u \wedge f^*dt = (* \circ \mathcal{H}(u))|_{X \setminus f^{-1}(\Delta)}$ and $* \circ \mathcal{H}(u) \in H^0(X, \Omega_X^{n+m-q} \otimes E)$, all $z_{n+m}^{b-1}\sigma_I$ can be extended holomorphically on U. By the non-vanishing property of $* \circ \mathcal{H}(u)$ along bB, we have at least one $\sigma_{J_0,i_0} \in H^0(U \setminus B, \mathcal{O}_X)$ whose divisor is

$$\mathrm{div}\,(\sigma_{J_0,i_0}) = -pB|_U + D$$

with some integer $0 \leq p \leq b-1$, and an effective divisor D on U not containing $B|_U$. We take such

$$J_0 \in I_{n-q} \text{ and } i_0 \in \{1, \ldots, r(E)\}.$$

(Now $\mathrm{div}\,(\sigma_{J_0,i_0}) = -pB|_U + D$ is fixed.) We set

$$Z_u = \{y \in \Delta;\ D \text{ contains } B|_U \cap f^{-1}(y)\}.$$

We can see that Z_u is not Zariski dense in Δ, because otherwise D contains $B|_U$, and also that Z_u is Zariski closed of $\mathrm{codim}_Y Z_u \geq 2$ (particularly using f is flat).

(4) We take any point $y_1 \in \Delta \setminus Z_u$, and a point $x_1 \in B|_U \cap f^{-1}(y_1)$ such that $x_1 \notin D$. Let $0 < \varepsilon \ll 1$ be a sufficiently small number so that, on the ε-polydisc neighbourhood $U(x_1, \varepsilon) = \{z = (z_1, \ldots, z_{n+m}) \in U;\ |z_i - z_i(x_1)| < \varepsilon \text{ for any } 1 \leq i \leq n+m\}$, we have

$$A := \inf\{|\sigma_{J_0,i_0}(z)|;\ z \in U(x_1, \varepsilon) \setminus B\} > 0.$$

We should note that σ_{J_0,i_0} may have a pole along B, but no zeros on $U(x_1, \varepsilon)$. We set $Y' := f(U(x_1, \varepsilon))$ which is an open neighbourhood of $y_1 \in Y$, since f is flat (in particular it is an open mapping). Then for any $t \in Y' \setminus \Delta$, we have

$$\begin{aligned}\int_{X_t} (c_{n-q}/q!)(\omega^q \wedge \sigma_u \wedge h\overline{\sigma_u})|_{X_t} &\geq a \int_{X_t \cap U} (c_{n-q}/q!)(\omega^q \wedge \sigma_u \wedge \overline{\sigma_u})|_{X_t \cap U} \\ &= a^{q+1} \int_{z \in X_t \cap U} \sum_{I \in I_{n-q}} \sum_{i=1}^{r} |\sigma_{I,i}(z)|^2 dV_n \\ &\geq a^{q+1} \int_{z \in X_t \cap U(x_1, \varepsilon)} A^2 \, dV_n \\ &= a^{q+1} A^2 (\pi \varepsilon^2)^n.\end{aligned}$$

Here $dV_n = (\sqrt{-1}/2)^n \bigwedge_{i=1}^n dz_i \wedge d\overline{z_i}$ is the standard euclidean volume form in $\mathbb{C}^n$. Namely we have $g(u,u)(t) \geq a^{q+1} A^2 (\pi\varepsilon^2)^n$ for any $t \in Y' \setminus \Delta$.

(5) We proved that $-\log g(u,u)$ is bounded from above around every point of $\Delta \setminus Z_u$. This means that a plurisubharmonic function $-\log g(u,u)$ on $Y \setminus \Delta$ can be extended as a plurisubharmonic function on $Y \setminus Z_u$ by Riemann type extension, and hence as a plurisubharmonic function on Y by Hartogs type extension. □

Remark 2.8. Here are some remarks when we try to generalize the proof above to obtain Theorem 1.5 and 1.6. The point is the set Z_u above depends on u. This is the main difficulty when we consider an extension property of quotient metrics. In that case, we need to obtain a uniform estimate of $g(u_s, u_s)$ for a family $\{u_s\}$. If s moves, then Z_{u_s} also may move and cover a larger subset of Δ, which may not be negligible for the extension of plurisubharmonic functions.

The intersection $B|_U \cap D$ is a set of indeterminacies. If (a part of) a fiber $f^{-1}(y)$ is contained in $B|_U \cap D$, the analysis of the behavior of $g(u,u)$ around such y is quite hard and in fact indeterminate. This is why we do not want to touch Z_u. In some geometric setting as below, we can avoid such phenomena. We can delete one of two in the right hand side of $\operatorname{div}(\sigma_{I,i}) = -pB|_U + D$.

(i) In case $\dim Y = 1$, we can take $D = 0$. This is because, if a prime divisor Γ on U contains $B|_U \cap f^{-1}(y)$, then $\Gamma = B|_U$. In case when $\dim Y = 1$, $q = 0$ and $E = \mathcal{O}_X$, a uniform estimate is cleared by Fujita [Ft, 1.11] (as we will see below). This will lead Theorem 1.5.

(ii) In case the fibers of f are reduced, we can take $p = 0$ (cf. $0 \leq p \leq b-1$ in (3) of the proof above). This will lead Theorem 1.6.

To deal with a general case in [MT3], we use a semi-stable reduction for f. A computation of Hodge metrics is a kind of an estimation of integrals, which usually can be done only after a good choice of local coordinates. A semi-stable reduction can be seen as a resolution of singularities of a map $f : X \longrightarrow Y$. Then the crucial point is to compair two Hodge metrics: the original one and the one after taking a semi-stable reduction. □

3. Proof of Theorems

3.1. **Quotient metric.** We discuss in Set up 2.2.

We denote by $F = R^q f_*(K_{X/Y} \otimes E)$ which is locally free on Y, and by r the rank of F. We have a smooth Hermitian metric g defined on $Y \setminus \Delta$ (not on Y). Let $F \longrightarrow L$ be a quotient line bundle with the kernel M: $0 \longrightarrow M \longrightarrow F \longrightarrow L \longrightarrow 0$ (exact). We first describe the quotient metric on $L|_{Y\setminus\Delta}$. We take a frame $e_1, \dots, e_r \in H^0(Y, F)$ over Y such that $e_1, \dots, e_{r-1}$ generate M. Then the image

$$\widehat{e}_r \in H^0(Y, L)$$

of e_r under $F \longrightarrow L$ generates L. We represent the Hodge metric g on $Y \setminus \Delta$ in terms of this frame as $g_{i\bar{j}} = g(e_i, e_j) \in A^0(Y \setminus \Delta, \mathbb{C})$. At each point $t \in Y \setminus \Delta$, $(g_{i\bar{j}}(t))_{1\le i,j\le r}$ is a positive definite Hermitian matrix, in particular, $(g_{i\bar{j}}(t))_{1\le i,j\le r-1}$ is also positive definite. We let $(g^{\bar{i}j}(t))_{1\le i,j\le r-1}$ be the inverse matrix. Then the pointwise orthogonal projection of e_r to $(M|_{Y\setminus\Delta})^{\perp}$ with respect to g is given by

$$P(e_r) = e_r - \sum_{i=1}^{r-1}\sum_{j=1}^{r-1} e_i g^{\bar{i}j} g_{j\bar{r}} \;\in A^0(Y \setminus \Delta, F).$$

We have in fact $P(e_r) - e_r \in A^0(Y \setminus \Delta, M)$ and $g(P(e_r), s) = 0$ for any $s \in A^0(Y \setminus \Delta, M)$. Then the quotient metric on $L|_{Y\setminus\Delta}$ is defined by

$$g_L^\circ(\widehat{e}_r, \widehat{e}_r) = g(P(e_r), P(e_r))$$

on $Y \setminus \Delta$.

It is well-known after Griffiths that the curvature does not decrease by a quotient. In our setting, the Nakano semi-positivity of $(F|_{Y\setminus\Delta}, g)$ [MT2, 1.1], or even weaker the Griffiths semi-positivity implies that $(L|_{Y\setminus\Delta}, g_L^\circ)$ is semi-positive. In particular if we write $g_L^\circ(\widehat{e}_r, \widehat{e}_r) = e^{-\varphi}$ with $\varphi \in A^0(Y \setminus \Delta, \mathbb{R})$, this φ is plurisubharmonic on $Y \setminus \Delta$. If we can show φ is extended as a plurisubharmonic function on Y, then g_L° extends as a singular Hermitian metric on L over Y with semi-positive curvature. By virtue of Riemann type extension for plurisubharmonic functions, it is enough to show that φ is bounded from above (i.e., $g_L^\circ(\widehat{e}_r, \widehat{e}_r)$ is bounded from below by a positive constant) around every point $y \in \Delta$. In the next two subsections, we shall prove the following

Lemma 3.1. *In Set up 2.2 and the notations above, assume further that* $\dim Y = 1$, *or that f has reduced fibers. Let $y \in \Delta$. Then there exists a neighbourhood Y' of $y \in Y$ and a positive number N such that $g_L^\circ(\widehat{e}_r, \widehat{e}_r)(t) \ge N$ for any $t \in Y' \setminus \Delta$.*

Corollary 3.2. *Theorem 1.5 and Theorem 1.6 hold true.*

We introduce the following notations for the following arguments. For $s = (s_1, \dots, s_r) \in \mathbb{C}^r$, we let $u_s = \sum_{i=1}^r s_i e_i \in H^0(Y, F)$. We note that u_s is nowhere vanishing on Y as soon as $s \neq 0$. We also note that, with respect to the standard topology of $\mathbb{C}^r$ and the topology of uniform convergence on compact sets for $H^0(X, \Omega_X^{n+m-q} \otimes E)$, the map $\mathbb{C}^r \longrightarrow H^0(X, \Omega_X^{n+m-q} \otimes E)$ given by $s \mapsto u_s \mapsto * \circ \mathcal{H}(u_s) = \sum_{i=1}^r s_i(* \circ \mathcal{H}(e_i))$, is continuous. Let $S^{2r-1} = \{s \in \mathbb{C}^r;\ |s| = (\sum |s_i|^2)^{1/2} = 1\}$.

3.2. **Over curves.** We shall prove Theorem 1.5 by showing Lemma 3.1 in this case. It is enough to consider in Set up 2.2 with $\dim Y = 1$. In particular $Y = \{t \in \mathbb{C};\ |t| < 1\}$ a unit disc, and $\Delta = 0 \in Y$ the origin. We will use both $\Delta \subset Y$ and $t = 0 \in Y$ to compair our argument here with a general case. Let $F = R^q f_*(K_{X/Y} \otimes E) \longrightarrow L$ be a quotient line bundle, and use the same notation in §3.1, in particular we have a frame $e_1, \dots, e_r \in H^0(Y, F)$, $\widehat{e_r} \in H^0(Y, L)$ generates L and so on. We use $u_s = \sum_{i=1}^r s_i e_i \in H^0(Y, F)$ for $s = (s_1, \dots, s_r) \in \mathbb{C}^r$. The key is to obtain the following uniform bound.

Lemma 3.3. *(cf.* [Ft, 1.11]*.) In Set up 2.2 with* $\dim Y = 1$ *and the notation above, let* $s_0 \in S^{2r-1}$*. Then there exist a neighbourhood* $S(s_0)$ *of* s_0 *in* S^{2r-1}*, a neighbourhood* Y' *of* $0 \in Y$ *and a positive number* N *such that* $g(u_s, u_s)(t) \geq N$ *for any* $s \in S(s_0)$ *and any* $t \in Y' \setminus \Delta$.

Proof. We denote by $f^*\Delta = \sum b_i B_i$.

(1) By Proposition 2.5, we have $*\circ\mathcal{H}(u_{s_0}) \in H^0(X, \Omega_X^{n+1-q} \otimes E)$. This $*\circ\mathcal{H}(u_{s_0})$ does not vanish at $t = 0$ as an element of $H^0(Y, \mathcal{O}_Y)$-module $H^0(X, \Omega_X^{n+1-q} \otimes E)$. Then there exists a component B_j in $f^*\Delta = \sum b_i B_i$ such that $(* \circ \mathcal{H})(u_{s_0})$ does not vanish of order greater than or equal to b_j along B_j. We take one such B_j and denote by $B = B_j$ and $b = b_j$.

(2) We take a general point $x_0 \in B$ so that x_0 is a smooth point on $(f^*\Delta)_{red} = f^{-1}(0)$, and take local coordinates $(U; z = (z_1, \dots, z_{n+1}))$ centered at $x_0 \in X$ such that $t = f(z) = z_{n+1}^b$ on U. Over U, the bundle E is also trivialized. Using the local trivializations on U, we have a constant $a > 0$ such that (i) $\omega \geq a\omega_{eu}$ on U, where $\omega_{eu} = \sqrt{-1}/2 \sum_{i=1}^{n+1} dz_i \wedge d\overline{z_i}$, and (ii) $h \geq a\mathrm{Id}$ on U as Hemitian matrixes, as in the proof of Proposition 2.7.

(3) Let $s \in S^{2r-1}$. By Proposition 2.5, we can write as $(* \circ \mathcal{H}(u_s))|_{X \setminus f^{-1}(\Delta)} = \sigma_s \wedge f^*dt$ for some $\sigma_s \in A^{n-q,0}(X \setminus f^{-1}(\Delta), E)$. We write $\sigma_s = \sum_{I \in I_{n-q}} \sigma_{sI} dz_I + R_s \wedge dz_{n+1}$ on $U \setminus B$. Here I_{n-q} is the set of all multi-indexes $1 \leq i_1 < \dots < i_{n-q} \leq n$ of length $n - q$, $\sigma_{sI} = {}^t(\sigma_{sI,1}, \dots, \sigma_{sI,r(E)})$ with $\sigma_{sI,i} \in H^0(U \setminus B, \mathcal{O}_X)$, and here $R_s \wedge dz_{n+1} \in A^{n-q,0}(U \setminus B, E)$. Now

$$\sigma_s \wedge f^*dt = bz_{n+1}^{b-1}\left(\sum_{I \in I_{n-q}} \sigma_{sI} dz_I\right) \wedge dz_{n+1}$$

on $U \setminus B$. Since $\sigma_s \wedge f^*dt = (* \circ \mathcal{H}(u_s))|_{X \setminus f^{-1}(\Delta)}$ and $* \circ \mathcal{H}(u_s) \in H^0(X, \Omega_X^{n+1-q} \otimes E)$, all $z_{n+1}^{b-1}\sigma_{sI}$ can be extended holomorphically on U.

At the point $s_0 \in S^{2r-1}$, by the non-vanishing property of $* \circ \mathcal{H}(u_{s_0})$ along bB, we have at least one $\sigma_{s_0 J_0, i_0} \in H^0(U \setminus B, \mathcal{O}_X)$ whose divisor is $\operatorname{div}(\sigma_{s_0 J_0, i_0}) = -p_0 B|_U$ with some integer $0 \le p_0 \le b-1$ (being $x_0 \in B|_U$ general, and U sufficiently small). Here we used $\dim Y = 1$. We take such $J_0 \in I_{n-q}$ and $i_0 \in \{1, \dots, r(E)\}$. By the continuity of $s \mapsto u_s \mapsto * \circ \mathcal{H}(u_s)$, we can take the same J_0 and i_0 for any $s \in S^{2r-1}$ near s_0, so that $\operatorname{div}(\sigma_{s J_0, i_0}) = -p(s) B|_U$ with the order $p(s)$ satisfies $0 \le p(s) \le p_0 = p(s_0)$ for any $s \in S^{2r-1}$ near s_0.

(4) By the continuity of $s \mapsto u_s \mapsto * \circ \mathcal{H}(u_s)$, we can take an ε-polydisc neighbourhood $U(x_0, \varepsilon) = \{z = (z_1, \dots, z_{n+1}) \in U;\ |z_i - z_i(x_0)| < \varepsilon$ for any $1 \le i \le n+1\}$ for some $\varepsilon > 0$, and a neighbourhood $S(s_0)$ of s_0 in S^{2r-1} such that $A := \inf\{|\sigma_{s J_0, i_0}(z)|;\ s \in S(s_0),\ z \in U(x_0, \varepsilon) \setminus B\} > 0$. We should note that $\sigma_{s J_0, i_0}$ may have a pole along B, but no zeros on $U(x_0, \varepsilon)$. We set $Y' := f(U(x_0, \varepsilon))$ which is an open neighbourhood of $0 \in Y$, since f is flat. Then for any $s \in S(s_0)$ and any $t \in Y' \setminus \Delta$, we have $g(u_s, u_s)(t) \ge a^{q+1} A^2 (\pi \varepsilon^2)^n$ as in Proposition 2.7. □

Lemma 3.4. *(cf.* [Ft, 1.12]*.) There exist a neighbourhood Y' of $0 \in Y$ and a positive number N such that $g(u_s, u_s)(t) \ge N$ for any $s \in S^{2r-1}$ and any $t \in Y' \setminus \Delta$.*

Proof. Since S^{2r-1} is compact, this is clear from Lemma 3.3. □

Lemma 3.5. *(cf.* [Ft, 1.13]*.) There exists a neighbourhood Y' of $0 \in Y$ and a positive number N such that $g_L^\circ(\widehat{e}_r, \widehat{e}_r)(t) \ge N$ for any $t \in Y' \setminus \Delta$.*

Proof. We take a neighbourhood Y' of $0 \in Y$ and a positive number N in Lemma 3.4. We may assume Y' is relatively compact in Y. We put $s_i = -\sum_{j=1}^{r-1} g^{\bar{i}j} g_{j\bar{r}} \in A^0(Y \setminus \Delta, \mathbb{C})$ for $1 \le i \le r-1$, and $s_r = 1$. Then $P(e_r) = \sum_{i=1}^r s_i e_i$ on $Y \setminus \Delta$. For every $t \in Y' \setminus \Delta$, we have $s = (s_1, s_2, \dots, s_r) \in \mathbb{C}^r \setminus \{0\}$, and $s(t)/|s(t)| \in S^{2r-1}$. Then for any $t \in Y' \setminus \Delta$, we have $g_L^\circ(\widehat{e}_r, \widehat{e}_r)(t) = g(u_{s(t)}, u_{s(t)})(t) = |s(t)|^2 g(u_{s(t)/|s(t)|}, u_{s(t)/|s(t)|})(t) \ge N$, since $s/|s| \in S^{2r-1}$. □

3.3. **Fiber reduced.** We shall prove Theorem 1.6 by the same strategy in the previous subsection. By Lemma 2.1 we may assume the set Z in Theorem 1.6 is empty. It is enough to consider in Set up 2.2 with $f^*\Delta = \sum B_i$. Let $F = R^q f_*(K_{X/Y} \otimes E) \longrightarrow L$ be a quotient line bundle, and use the same notation in §3.1, in particular we have a frame $e_1, \dots, e_r \in H^0(Y, F)$, $\widehat{e}_r \in H^0(Y, L)$ generates L and so on. We use $u_s = \sum_{i=1}^r s_i e_i \in H^0(Y, F)$ for $s = (s_1, \dots, s_r) \in \mathbb{C}^r$. As we observed in the previous subsection, it is enough to show the following

Lemma 3.6. *(cf.* [Ft. 1.11]*.) In Set up 2.2 and the notation above, let* $s_0 \in S^{2r-1}$*. Then there exist a neighbourhood* $S(s_0)$ *of* s_0 *in* S^{2r-1}*, a neighbourhood* Y' *of* $0 \in Y$ *and a positive number* N *such that* $g(u_s, u_s)(t) \geq N$ *for any* $s \in S(s_0)$ *and any* $t \in Y' \setminus \Delta$.

Proof. (1) By Proposition 2.5, we have $* \circ \mathcal{H}(u_{s_0}) \in H^0(X, \Omega_X^{n+m-q} \otimes E)$. This $* \circ \mathcal{H}(u_{s_0})$ does not vanish at $t = 0$ as an element of $H^0(Y, \mathcal{O}_Y)$-module $H^0(X, \Omega_X^{n+m-q} \otimes E)$. There exists a component B_j in $f^*\Delta = \sum B_i$ such that $* \circ \mathcal{H}(u_{s_0})$ does not vanish identically along $B_j \cap f^{-1}(0)$. Here we used our assumption in Theorem 1.6 that f has reduced fibers. In fact, if $* \circ \mathcal{H}(u_{s_0})$ does vanish identically along all $B_i \cap f^{-1}(0)$ in $f^*\Delta = \sum B_i$, then $* \circ \mathcal{H}(u_{s_0})$ vanishes at $t = 0$ as an element of $H^0(Y, \mathcal{O}_Y)$-module $H^0(X, \Omega_X^{n+m-q} \otimes E)$, and leads a contradiction. We take one such B_j and denote by $B = B_j$ (with $b = b_j = 1$).

(2) We take a point $x_0 \in B \cap f^{-1}(0)$ such that $* \circ \mathcal{H}(u_{s_0})$ does not vanish at x_0, and that $f^*\Delta$ is smooth at x_0. We then take local coordinates $(U; z = (z_1, \dots, z_{n+m}))$ centered at $x_0 \in X$ such that $t = f(z) = (z_{n+1}, \dots, z_{n+m-1}, z_{n+m})$ on U. Over U, the bundle E is also trivialized. Using the local trivializations on U, we have a constant $a > 0$ such that (i) $\omega \geq a\omega_{eu}$ on U, where $\omega_{eu} = \sqrt{-1}/2 \sum_{i=1}^{n+m} dz_i \wedge d\overline{z_i}$, and (ii) $h \geq a\mathrm{Id}$ on U as Hemitian matrixes, as in the proof of Proposition 2.7.

(3) Let $s \in S^{2r-1}$. By Proposition 2.5, we can write as $(* \circ \mathcal{H}(u_s))|_{X \setminus f^{-1}(\Delta)} = \sigma_s \wedge f^*dt$ for some $\sigma_s \in A^{n-q,0}(X \setminus f^{-1}(\Delta), E)$. We write $\sigma_s = \sum_{I \in I_{n-q}} \sigma_{sI} dz_I + R_s$ on $U \setminus B$. Here I_{n-q} is the set of all multi-indexes $1 \leq i_1 < \dots < i_{n-q} \leq n$ of length $n - q$, $\sigma_{sI} = {}^t(\sigma_{sI,1}, \dots, \sigma_{sI,r(E)})$ with $\sigma_{sI,i} \in H^0(U \setminus B, \mathcal{O}_X)$, and here $R_s = \sum_{k=1}^m R_{sk} \wedge dz_{n+k} \in A^{n-q,0}(U \setminus B, E)$. Now

$$\sigma_s \wedge f^*dt = \left(\sum_{I \in I_{n-q}} \sigma_{sI} dz_I \right) \wedge dz_{n+1} \wedge \dots \wedge dz_{n+m}$$

on $U \setminus B$. Since $\sigma_s \wedge f^*dt = (* \circ \mathcal{H}(u_s))|_{X \setminus f^{-1}(\Delta)}$ and $* \circ \mathcal{H}(u_s) \in H^0(X, \Omega_X^{n+m-q} \otimes E)$, all σ_{sI} can be extended holomorphically on U.

At the point $s_0 \in S^{2r-1}$, by the non-vanishing property of $* \circ \mathcal{H}(u_{s_0})$ at x_0, we have at least one $\sigma_{s_0 J_0, i_0} \in H^0(U \setminus B, \mathcal{O}_X)$ whose divisor is $\operatorname{div}(\sigma_{s_0 J_0, i_0}) = D_0$ with some effective divisor D_0 on U not containing x_0. This is because, if all $\sigma_{s_0 I, i}$ vanish at x_0, we see $* \circ \mathcal{H}(u_{s_0}) = \sigma_{s_0} \wedge f^*dt$ (now on U) vanishes at x_0, and we have a contradiction. We take such $J_0 \in I_{n-q}$ and $i_0 \in \{1, \dots, r(E)\}$. By the continuity of $s \mapsto u_s \mapsto * \circ \mathcal{H}(u_s)$, we can take the same J_0 and i_0 for any $s \in S^{2r-1}$ near s_0. By the same token, the divisor $D(s)$ may depend on $s \in S^{2r-1}$, but we can keep the condition that $D(s)$ does not contain $B|_U \cap f^{-1}(0)$ if $s \in S^{2r-1}$ is close to s_0.

(4) Then by the continuity of $s \mapsto u_s \mapsto * \circ \mathcal{H}(u_s)$, we can take an ε-polydisc neighbourhood $U(x_0, \varepsilon) = \{z = (z_1, \dots, z_{n+m}) \in U;\ |z_i - z_i(x_0)| < \varepsilon$ for any $1 \leq i \leq n + m\}$ for some $\varepsilon > 0$, and a neighbourhood $S(s_0)$ of s_0 in S^{2r-1} such that $A := \inf\{|\sigma_{sJ_0,i_0}(z)|;\ s \in$

$S(s_0)$, $z \in U(x_0,\varepsilon) \setminus B\} > 0$. We set $Y' := f(U(x_0,\varepsilon))$ which is an open neighbourhood of $0 \in Y$, since f is flat. Then for any $s \in S(s_0)$ and any $t \in Y' \setminus \Delta$, we have $g(u_s,u_s)(t) \geq a^{q+1}A^2(\pi\varepsilon^2)^n$ as in Proposition 2.7. □

4. Examples

Here are some related examples and counter-examples of the positivity of direct image sheaves, including cases the total space X can be singular. These are due to Wiśniewski and Höring, and taken from [Hö].

Our general set up is as follows. We take a vector bundle E of rank $n+2$ over a smooth projective variety Y. Denote by $p : \mathbb{P}(E) \longrightarrow Y$ the natural (smooth) projection. We take a hypersurface X in $\mathbb{P}(E)$ cut out by a section of $N := \mathcal{O}_E(d) \otimes p^*\lambda$ for $d > 0$ and some line bundle λ on Y. Denote by $f : X \longrightarrow Y$ the induced (non necessary smooth) map of relative dimension n. Because X is a divisor, the sheaf $\omega_{P(E)/Y} \otimes N \otimes \mathcal{O}_X$ equals $\omega_{X/Y}$. We choose a line bundle $L := \mathcal{O}_E(k) \otimes p^*\mu$ with $k > 0$ and with a line bundle μ on Y, and set $L_X := L|_X$ the resrtiction on X. We then consider the exact sequence

$$0 \longrightarrow \omega_{\mathbb{P}(E)/Y} \otimes L \longrightarrow \omega_{\mathbb{P}(E)/Y} \otimes N \otimes L \longrightarrow \omega_{X/Y} \otimes L_X \longrightarrow 0.$$

Note that since L is p-ample, we have $R^1p_*(\omega_{\mathbb{P}(E)/Y} \otimes L) = 0$. We push the sequence forward by p to get the following exact sequence of sheaves on Y:

$$0 \longrightarrow p_*(\omega_{\mathbb{P}(E)/Y} \otimes L) \longrightarrow p_*(\omega_{\mathbb{P}(E)/Y} \otimes N \otimes L) \longrightarrow f_*(\omega_{X/Y} \otimes L_X) \longrightarrow 0.$$

Remember that $\omega_{\mathbb{P}(E)/Y} = \mathcal{O}_E(-n-2) \otimes p^* \det E$, so that $p_*(\omega_{\mathbb{P}(E)/Y} \otimes \mathcal{O}_E(k)) = 0$ for $k < n+2$, and that $p_*(\omega_{\mathbb{P}(E)/Y} \otimes \mathcal{O}_E(k)) = S^{k-n-2}E \otimes \det E$ for $k \geq n+2$.

Example 4.1. ([Hö, 2.C].) Choose $Y = \mathbb{P}^1$, $E = \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^1}$, $N = \mathcal{O}_E(2)$ that is effective and defines X, and $L = \mathcal{O}_E(1) \otimes p^*\mathcal{O}_{\mathbb{P}^1}(1)$ that is semi-positive. The push-forward sequence reads

$$0 \longrightarrow 0 \longrightarrow \mathcal{O}_{\mathbb{P}^1}(-1) \longrightarrow f_*(\omega_{X/Y} \otimes L_X) \longrightarrow 0.$$

Hence $f_*(\omega_{X/Y} \otimes L_X)$ is negative. The point is that here, X is not reduced. □

Example 4.2. ([Hö, 2.D].) Choose $Y = \mathbb{P}^1$ and $E = \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}^{\oplus 3}$. Take $N = \mathcal{O}_E(4)$ whose generic section defines X. This scheme is a 3-fold smooth outside the 1-dimensional base locus $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(-1)) \subset \mathbb{P}(E)$, Gorenstein as a divisor, and normal since smooth in codimension 1. Choose $L = \mathcal{O}_E(k) \otimes p^*\mathcal{O}_{\mathbb{P}^1}(k)$ that is semi-positive. The push-forward sequence shows that for $1 \leq k < 4$, $S^kE \otimes \mathcal{O}_{\mathbb{P}^1}(k-1) = f_*(\omega_{X/Y} \otimes L_X)$ is not nef. For $k \geq 4$, the push-forward sequence reads

$$0 \longrightarrow S^{k-4}E \otimes \mathcal{O}_{\mathbb{P}^1}(k-1) \xrightarrow{\sigma} S^kE \otimes \mathcal{O}_{\mathbb{P}^1}(k-1) \longrightarrow f_*(\omega_{X/Y} \otimes L_X) \longrightarrow 0.$$

Here the map σ is given by the contraction with the section $s \in H^0(\mathbb{P}(E), N) = H^0(Y, S^4E)$ $= H^0(Y, S^4\mathcal{O}_{\mathbb{P}^1}^{\oplus 3})$, whereas the quotient $S^kE/\mathrm{Im}(S^4\mathcal{O}_{\mathbb{P}^1}^{\oplus 3} \otimes S^{k-4}E)$ contains the factor $\mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus k}$. Hence $f_*(\omega_{X/Y} \otimes L_X)$ is not weakly positive. The point here is that the locus of non-rational singularities of X projects onto Y by $f : X \longrightarrow Y$. □

Example 4.3. ([Hö, 2.A].) Choose Y to be $\pi : Y = \mathbb{P}(F) \longrightarrow \mathbb{P}^3$, where $F := \mathcal{O}_{\mathbb{P}^3}(2)^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^3}$ is semi-ample but not ample. Choose E to be $\mathcal{O}_F(1)^{\oplus 2} \oplus (\mathcal{O}_F(1) \otimes \pi^*\mathcal{O}_{\mathbb{P}^3}(1))$. Wiśniewski showed that the linear system $|N| := |\mathcal{O}_E(2) \otimes p^*\pi^*\mathcal{O}_{\mathbb{P}^3}(-2)|$ has a smooth member, that we denote by X. Remark that $L := \mathcal{O}_E(1)$ is semi-positive, but the push-forward sequence shows that

$$\mathcal{O}_F(3) \otimes \pi^*\mathcal{O}_{\mathbb{P}^3}(-1) = f_*(\omega_{X/Y} \otimes L_X)$$

is not nef. The point here is that the conic bundle $f : X \longrightarrow Y$ has some non-reduced fibers, that make the direct image only weakly positive. □

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