Asymptotics of L^2 and Quillen metrics for degenerations of Calabi–Yau manifolds

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Our first motivation was to generalize Kodaira's bundle formula for elliptic surfaces $\pi\colon S \to C$ (a proper surjective relatively minimal Kähler morphism with connected fibres between a smooth complex surface S and a smooth complex curve C with general fibers of genus one) to families of higher dimensional Calabi–Yau manifolds, using a metric point of view. The original formulation of Kodaira [5], giving the contributions of the singular fibres in topological terms,

$$\begin{aligned} K_{S/C} &= \pi^\star\pi_\star(K_{S/C}) \otimes \mathcal{O}_S\left(\sum_{\substack{\pi^{-1}(p) \text{ of} \\ \text{type } m\mathrm{I}_n}} \frac{m-1}{m}\pi^{-1}(p)\right), \\ (\pi_\star K_{S/C})^{\otimes 12} &= \mathcal{O}_S\left(\sum_{p\in C}\chi\big(\pi^{-1}(p)_{\mathrm{red}}\big)p\right), \end{aligned}$$

turns out to be related to Quillen-type metrics, whereas the reformulation by Kawamata in terms of the modular map $J\colon C \to \mathbb{P}^1$ for elliptic fibers and the log canonical threshold $\mathrm{lct}(S, \pi^{-1}(p))$ of the singular fibres (see [6])

$$(K_{S/C})^{12} = \pi^\star J^\star\mathcal{O}_{\mathbb{P}^1}(\infty) \otimes \pi^\star\mathcal{O}_C\left(12\sum_{p\in C}\big(1-\mathrm{lct}(S,\pi^{-1}(p))\big)p\right)$$

turns out to be related to L^2 metrics.

We consider a Calabi–Yau family $\pi\colon \mathcal{X} \to C$, that is a proper surjective Kähler morphism with connected fibres between a connected smooth complex manifold $\mathcal{X}$ and a connected smooth complex curve C, whose smooth fibers have trivial canonical bundle.

By computations of Tian [8] and Todorov [9], the curvature of the L^2 metric on the direct image $\pi_\star(K_{\mathcal{X}/C})$ of the relative canonical bundle on the smooth part of the morphism is given by the Weil–Petersson metric through the classifying map ι to the Kuranishi family

$$\mathrm{c}_1\big(\pi_\star(K_{\mathcal{X}/C}), h_{L^2}\big) = \iota^*\omega_{WP}.$$

This accounts for the non-negativity of the modular part. The asymptotic of this metric around a singular fibre can be estimated by fibre integrals and displays as the coefficient of the dominant term (a log-term) one minus the log canonical threshold of the singular fibre (that vanishes for semi-stable degenerations) and as the coefficient of the sub-dominant term (a log log-term) an integer computed from the weighted incidence graph of the singular fibre when arranged with normal crossings (that reflects the strength of the degeneration). Those two terms can also be given in terms of the limit Hodge structure [7]. Explicit computations can be made for semi-stable degenerations of an elliptic curve : the L^2 norm of the form $\mathrm{d}z$ in the model $\mathbb{C}/(\mathbb{Z}\oplus\tau\mathbb{Z})$ is equal to the volume of the elliptic curve and to the imaginary part $\mathrm{Im}(\tau)$ of the period τ whereas the parameter $q:=\exp(2\pi\mathrm{i}\tau)$ behaves like the regular discriminant function when the period τ tends to $\mathrm{i}\infty$. This leads to the asymptotic

$$-\log\|\mathrm{d}z\|_{L^2}\sim-\log|\log|t||$$

for a local regular parameter t on the base C centred at a singular value of π.

Turning to Quillen type metrics, we consider the BCOV bundle, named after Bershadsky–Cecotti–Ooguri–Vafa [1], together with the BCOV metric defined by Fang–Lu–Yoshikawa [4]. By results of Bismut–Gillet–Soulé [2], its curvature on the smooth part is also related to the Weil–Petersson metric

$$\mathrm{c}_1\big(\lambda_{\mathrm{BCOV}}(\Omega^\bullet_{\mathcal{X}/S}),h_{\mathrm{BCOV}}\big)=\frac{\chi(X_\infty)}{12}\iota^*\omega_{WP}.$$

where X_∞ is any smooth fibre. Building on works of Yoshikawa [10], we show that the dominant term in the asymptotic of this metric displays the vanishing cycles when the relative minimality assumption $K_{\mathcal{X}}=\mathcal{O}_{\mathcal{X}}$ is made: for a local frame σ of the BCOV bundle and a local frame η of the direct image $\pi_\star(K_{\mathcal{X}/C})$, we find

$$-\log\|\widetilde{\sigma}\|^2_{\mathrm{BCOV}}\quad\sim\quad\frac{9n^2+11n+2}{24}\big(\chi(X_\infty)-\chi(X_0)\big)\log|t|^2-\frac{\chi(X_\infty)}{12}\log\|\eta\|^2_{L^2}.$$

Details are written in [3].

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