Exercises on foliations

Exercise 1. (*Intersection of curves*)

a) Let p and q be two different positive integers. Compute the intersection number at $(0,0)$ and at $(1,1)$ of the curves with equation $x=y$ and $x^p=y^q$.
b) What happens when $p=q$?

Exercise 2. (*Germ of curve*)

a) In the complex plane $\mathbb{C}^2$, compute the intersection number at $(0,0)$ of the curve with equation $(y^2=x^3+x^2)$ and, case by case, of the curve with equation
   i) $(x=0)$
   ii) $(y=0)$
   iii) $(x=y)$
   iv) $(x=-y)$
b) Deduce the shape of the curve with equation $(y^2=x^3+x^2)$ near the point $(0,0)$.

Exercise 3. (*Fibration*)

Let $[X_1:Y_1][X_2:Y_2]$ be the homogeneous coordinates on $\mathbb{P}^1\times\mathbb{P}^1$. Consider in $\mathbb{P}^1\times\mathbb{P}^1$ the $\mathcal{F}$ holomorphic foliation by complex curves given by the fibers of the projection p_1 onto the first factor $\mathbb{P}^1$.

a) Recall that the cohomology group $H^2(\mathbb{P}^1\times\mathbb{P}^1,\mathbb{Z})=\mathbb{Z}F_1\oplus\mathbb{Z}F_2$ where F_1 is the fiber of a point by p_1, and F_2 by p_2. Write the intersection form on $H^2(\mathbb{P}^1\times\mathbb{P}^1,\mathbb{Z})$.
b) Determine the normal and tangent bundle of $\mathcal{F}$ by giving definig vector fields on all four canonical charts. Write $c_1(N_{\mathcal{F}})$ as aF_1+bF_2 with explicit a and b.
c) Verify the self-intersection formula of the normal bundle.
d) Compute the number of tangencies of $\mathcal{F}$ with the curve C_1 of equation $X_2=0$ by explicitly determining the points of tangency and their order. Write the Poincaré dual $[C_1]$ as aF_1+bF_2 with explicit a and b.
e) Compute the number of tangencies of $\mathcal{F}$ with the curve of equation $X_1=0$ by explicitly determining the points of tangency and their order.
f) Calculate the number of tangencies of $\mathcal{F}$ with the curve C_2 of equation $X_2^2Y_1=X_1Y_2^2$ by explicitly determining the points of tangency and their order. Write the Poincaré dual $[C_2]$ as aF_1+bF_2 with explicit a and b.
g) For curves C_1 and C_2, check the formula for the degree of $T\mathcal{F}$.
h) Check the formula for the degree of $T\mathcal{F}$ on the curve with equation $X_1=0$.

Exercise 4. (*Chern classes*)

Let ζ be the first Chern class of the tautological quotient line bundle $\mathcal{O}_{\mathbb{P}^3}(1)$ on the complex projective space $\mathbb{P}^3$. Recall the Euler sequence on the complex projective space $\mathbb{P}^3$.

$$0\to\mathcal{O}_{\mathbb{P}^3}\to\mathcal{O}_{\mathbb{P}^4}(1)^{\oplus 4}\to T\mathbb{P}^3\to 0$$

a) Determine the canonical bundle $K_{\mathbb{P}^3}=(\det T\mathbb{P}^3)^\vee$ of the complex projective space $\mathbb{P}^3$.
b) Determine the Chern classes $c_1(\mathbb{P}^3)$ and $c_2(\mathbb{P}^3)$ of the tangent bundle of $\mathbb{P}^4$.

c) Let X be a hypersurface of $\mathbb{P}^3$ of degree 4. Explain the origin of the exact sequence

$$0 \to TX \to T\mathbb{P}^4_{|X} \to \mathcal{O}_{\mathbb{P}^3}(4)_{|X} \to 0$$

.

d) Deduce the canonical bundle of X and the Chern classes of its tangent bundle.

e) What is the nature of the space X (smoothness, dimension, sign of the canonical bundle, classification) ? Take for granted that X is, like $\mathbb{P}^3$, simply connected (Lefschetz theorem)

Exercise 5. (*Chern classes*)

Let ζ be the first Chern class of the tautological quotient line bundle $\mathcal{O}_{\mathbb{P}^4}(1)$ on the complex projective space $\mathbb{P}^4$. Recall the Euler sequence on the complex projective space $\mathbb{P}^4$.

$$0 \to \mathcal{O}_{\mathbb{P}^4} \to \mathcal{O}_{\mathbb{P}^4}(1)^{\oplus 5} \to T\mathbb{P}^4 \to 0$$

.

a) Determine the canonical bundle $K_{\mathbb{P}^4}$ of the complex projective space $\mathbb{P}^4$.

b) Determine the Chern classes $c_1(\mathbb{P}^4)$ and $c_2(\mathbb{P}^4)$ of the tangent bundle of $\mathbb{P}^4$.

c) Let X be a transverse everywhere intersection of a hypersurface H_2 of $\mathbb{P}^4$ of degree 2 and another H_3 of degree 3. Explain the origin of the exact sequence

$$0 \to TX \to T\mathbb{P}^4_{|X} \to \mathcal{O}_{\mathbb{P}^4}(2) \oplus \mathcal{O}_{\mathbb{P}^4}(3)_{|X} \to 0$$

d) What is the nature of space X ? Take for granted that X is, like $\mathbb{P}^4$ and H_2, simply connected (Lefschetz's Theorem).

e) Compute the signature $\sigma(X) = \frac{1}{3}\left(c_1^2(X) - 2c_2(X)\right)$.