Exam on foliations

Recall that if E is a holomorphic vector bundle of rank r and L a holomorphic line bundle then

$$\det(E \otimes L) = \det(E) \otimes L^{\otimes r}.$$

Recall that the first Chern class $c_1(E)$ of a holomorphic vector bundle E is equal to that $c_1(\det E)$ of its determinant

$$c_1(E) = c_1(\det E).$$

The first Chern class $c_1(L \otimes L')$ of the tensor product of two holomorphic line bundles L and L' is the sum

$$c_1(L \otimes L') = c_1(L) + c_1(L').$$

Exercise 1. (*On the course*)

Take a general complex projective surface not of general type. Does it support many holomorphic foliations by curves ? Explain, in no more than ten lines, how we got some results in this direction in the course.

Exercise 2. (*Chern numbers of Hirzebruch surfaces*)

Let k be an integer. Let $E_k = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(k) \to \mathbb{P}^1$ be a rank two vector bundle on $\mathbb{P}^1$. Define the Hirzebruch surface F_k to be the variety of rank one quotients of E_k

$$F_k := \mathbb{P}(E_k) \xrightarrow{\pi} \mathbb{P}^1$$

and $\mathcal{O}_{F_k}(1) \to F_k$ the tautological quotient bundle. The relative tangent bundle $T_{F_k/\mathbb{P}^1}$ is the kernel of the differential of the map π.

$$0 \to T_{F_k/\mathbb{P}^1} \to TF_k \xrightarrow{\pi_\star} \pi^\star T_{\mathbb{P}^1} \to 0.$$

It can be computed by the relative Euler sequence

$$0 \to \mathcal{O}_{F_k} \to \pi^\star E_k^\vee \otimes \mathcal{O}_{F_k}(1) \to T_{F_k/\mathbb{P}^1} \to 0$$

a) Compute the canonical bundle K_{F_k} of F_k in terms of $\pi^\star\mathcal{O}_{\mathbb{P}^1}(1)$ and $\mathcal{O}_{F_k}(1)$.
b) Compute the total Chern class $c(F_k)$ of TF_k in terms of $\alpha := c_1(\pi^\star\mathcal{O}_{\mathbb{P}^1}(1))$ and $\beta := c_1(\mathcal{O}_{F_k}(1))$.
c) On the curve $C = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}) \subset F_k = \mathbb{P}(E_k)$ defined by the rank one quotient $E_k \to \mathcal{O}_{\mathbb{P}^1}$ of E_k, the tautological quotient $\mathcal{O}_{F_k}(1)$ restricts to $\mathcal{O}_C$. Find a and b such that the Poincaré dual of the cycle class of C is $[C] = a\alpha + b\beta \in H^2(F_k, \mathbb{Z})$. You can use the intersection numbers in $H^4(F_k, \mathbb{Z}) \equiv \mathbb{Z}$,

$$\alpha \cdot \alpha = 0\,;\, \alpha \cdot \beta = 1\,;\, \beta^2 = k.$$

Compute the self intersection $[C] \cdot [C]$ of the curve C.

continued the other side

Exercise 3. (*Curves in complex tori of dimension* 2)

Let $S = \mathbb{C}^2/\Lambda$ be a complex torus of dimension 2. Let (z_1, z_2) be affine coordinates on $\mathbb{C}$. Let τ be a complex number and $\omega := dz_1 + \tau dz_2$ be a holomorphic one-form with constant coefficients on $\mathbb{C}^2$ that descends to S. Let $\mathcal{F}$ be the holomorphic foliation on S defined by ω. We recall that if C is a smooth curve then the index $Z(\mathcal{F}, C, p)$ is non negative. We recall that the topological Euler characteristic $\chi(C)$ of a smooth compact curve is

$$\chi(C) = b^0(C) - b^1(C) + b^2(C) = 2 - 2g(C) = \int_C c_1(TC)$$

by Gauss-Bonnet theorem.

a) Show that the tangent bundle TS of S is trivial.
b) What are the singularities of $\mathcal{F}$?
c) Show that the normal and tangent bundles $T\mathcal{F}$ and $N\mathcal{F}$ of $\mathcal{F}$ are trivial.
d) Show that elliptic curves (i.e. of genus 1) in S are either everywhere transverse to the foliation $\mathcal{F}$ or invariant by $\mathcal{F}$.
e) Show that smooth curves of genus strictly greater than 1 are not invariant by $\mathcal{F}$.
f) Show that there are no smooth rational (i.e. of genus 0) curve in S.