Université de Rennes

Exam on foliations

Recall that if E is a holomorphic vector bundle of rank r and L a holomorphic line bundle then

$$\det(E \otimes L) = \det(E) \otimes L^{\otimes r}.$$

Recall that the first Chern class $c_1(E)$ of a holomorphic vector bundle E is equal to that $c_1(\det E)$ of its determinant

$$c_1(E) = c_1(\det E).$$

The first Chern class $c_1(L \otimes L')$ of the tensor product of two holomorphic line bundles L and L' is the sum

$$c_1(L \otimes L') = c_1(L) + c_1(L').$$

Exercise 1. (*On the course*)

Take a general complex projective surface not of general type. Does it support many holomorphic foliations by curves ? Explain, in no more than ten lines, how we got some results in this direction in the course.

Exercise 2. (*Chern numbers of Hirzebruch surfaces*)

Let k be an integer. Let $E_k = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(k) \to \mathbb{P}^1$ be a rank two vector bundle on $\mathbb{P}^1$. Define the Hirzebruch surface F_k to be the variety of rank one quotients of E_k

$$F_k := \mathbb{P}(E_k) \xrightarrow{\pi} \mathbb{P}^1$$

and $\mathcal{O}_{F_k}(1) \to F_k$ the tautological quotient bundle. The relative tangent bundle $T_{F_k/\mathbb{P}^1}$ is the kernel of the differential of the map π.

$$0 \to T_{F_k/\mathbb{P}^1} \to TF_k \xrightarrow{\pi_\star} \pi^\star T_{\mathbb{P}^1} \to 0.$$

It can be computed by the relative Euler sequence

$$0 \to \mathcal{O}_{F_k} \to \pi^\star E_k^\vee \otimes \mathcal{O}_{F_k}(1) \to T_{F_k/\mathbb{P}^1} \to 0$$

a) Compute the canonical bundle K_{F_k} of F_k in terms of $\pi^\star\mathcal{O}_{\mathbb{P}^1}(1)$ and $\mathcal{O}_{F_k}(1)$.
b) Compute the total Chern class $c(F_k)$ of TF_k in terms of $\alpha := c_1(\pi^\star\mathcal{O}_{\mathbb{P}^1}(1))$ and $\beta := c_1(\mathcal{O}_{F_k}(1))$.
c) On the curve $C = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}) \subset F_k = \mathbb{P}(E_k)$ defined by the rank one quotient $E_k \to \mathcal{O}_{\mathbb{P}^1}$ of E_k, the tautological quotient $\mathcal{O}_{F_k}(1)$ restricts to $\mathcal{O}_C$. Find a and b such that the Poincaré dual of the cycle class of C is $[C] = a\alpha + b\beta \in H^2(F_k, \mathbb{Z})$. You can use the intersection numbers in $H^4(F_k, \mathbb{Z}) \equiv \mathbb{Z}$,

$$\alpha \cdot \alpha = 0 \,;\, \alpha \cdot \beta = 1 \,;\, \beta^2 = k.$$

Compute the self intersection $[C] \cdot [C]$ of the curve C.

Answer :

a) The determinant of the first sequence yields

$$\det TF_k = \det T_{F_k/\mathbb{P}^1} \otimes \det \pi^\star T_{\mathbb{P}^1} = T_{F_k/\mathbb{P}^1} \otimes \pi^\star T_{\mathbb{P}^1} = T_{F_k/\mathbb{P}^1} \otimes \pi^\star\mathcal{O}_{\mathbb{P}^1}(2)$$

The determinant of the seecond sequence yields

$$T_{F_k/\mathbb{P}^1} = \det(\pi^\star E_k^\vee \otimes \mathcal{O}_{F_k}(1)) = \pi^\star \det(E_k^\vee) \otimes \mathcal{O}_{F_k}(2) = \pi^\star\mathcal{O}_{\mathbb{P}^1}(-k) \otimes \mathcal{O}_{F_k}(2)$$

It gives $K_{F_k} = \pi^\star\mathcal{O}_{\mathbb{P}^1}(k-2) \otimes \mathcal{O}_{F_k}(-2)$.

b) Whitney formula for the second sequence leads to

$$\begin{aligned} c_1(T_{F_k/\mathbb{P}^1}) &= c_1(\pi^\star E_k^\vee \otimes \mathcal{O}_{F_k}(1)) = c_1(\pi^\star \det E_k^\vee \otimes \mathcal{O}_{F_k}(2)) \\ &= c_1(\pi^\star \det E_k^\vee) + c_1(\mathcal{O}_{F_k}(2)) = -k\alpha + 2\beta \end{aligned}$$

Whitney formula for the first sequence leads to

$$\begin{aligned} c(F_k) &= (1 + c(T_{F_k/\mathbb{P}^1}))(1 + c_1(\pi^\star T_{\mathbb{P}^1})) = (1 - k\alpha + 2\beta)(1 + 2\alpha) \\ &= 1 + (2-k)\alpha + 2\beta + 4\alpha\beta \end{aligned}$$

c)

$$[C] \cdot \alpha = 1 = b \text{ and } [C] \cdot \beta = 0 = a + bk$$

Hence $[C] = -k\alpha + \beta$ and $C \cdot C = (-k\alpha + \beta)^2 = 0 - 2k + k = -k$.

Exercise 3. (*Curves in complex tori of dimension* 2)

Let $S = \mathbb{C}^2/\Lambda$ be a complex torus of dimension 2. Let (z_1, z_2) be affine coordinates on $\mathbb{C}$. Let τ be a complex number and $\omega := dz_1 + \tau dz_2$ be a holomorphic one-form with constant coefficients on $\mathbb{C}^2$ that descends to S. Let $\mathcal{F}$ be the holomorphic foliation on S defined by ω. We recall that if C is a smooth curve then the index $Z(\mathcal{F}, C, p)$ is non negative. We recall that the topological Euler characteristic $\chi(C)$ of a smooth compact curve is

$$\chi(C) = b^0(C) - b^1(C) + b^2(C) = 2 - 2g(C) = \int_C c_1(TC)$$

by Gauss-Bonnet theorem.

a) Show that the tangent bundle TS of S is trivial.
b) What are the singularities of $\mathcal{F}$?
c) Show that the normal and tangent bundles $T\mathcal{F}$ and $N\mathcal{F}$ of $\mathcal{F}$ are trivial.
d) Show that elliptic curves (i.e. of genus 1) in S are either everywhere transverse to the foliation $\mathcal{F}$ or invariant by $\mathcal{F}$.
e) Show that smooth curves of genus strictly greater than 1 are not invariant by $\mathcal{F}$.
f) Show that there are no smooth rational (i.e. of genus 0) curve in S.

Answer :

a) $(\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2})$ gives a global frame for TS. Hence it is a trivial holomorphic vector bundle of rank 2.
b) As ω does not vanish, the holomorphic foliation it defines is regular everywhere.
c) As $\tau\frac{\partial}{\partial z_1} - \frac{\partial}{\partial z_2}$ is a nowhere vanishing section of $T\mathcal{F}$, the latter is trivial. As, $T\mathcal{F} \otimes N\mathcal{F} = \det TS = \mathcal{O}$, we derive that $N\mathcal{F}$ is also trivial.
d) For a non-invariant curve C we have

$$0 = \deg_C N\mathcal{F} = \chi(C) + Tang(\mathcal{F}, C) \geqslant 2 - 2g(C).$$

Hence, $g(C) \geqslant 1$. And if $g(C) = 1$ then there are no tangencies with $\mathcal{F}$.
e) For a smooth invariant curve C,

$$0 = \deg_C T\mathcal{F} = \chi(C) - Z(\mathcal{F}, C) \leqslant 2 - 2g(C).$$

Hence $g(C) \leqslant 1$.
f) Rational curves have to be invariant under every linear foliation. But smooth rational curves have definite tangent direction at every point.