

TOPOLOGY AND DYNAMICS OF LAMINATIONS IN SURFACES OF GENERAL TYPE

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To the memory of Marco Brunella

ABSTRACT. We study Riemann surface laminations in complex algebraic surfaces of general type. We focus on the topology and dynamics of minimal sets of holomorphic foliations and on Levi-flat hypersurfaces. We begin by providing various examples. Then our first result shows that Anosov Levi-flat hypersurfaces do not embed in surfaces of general type. This allows to classify the possible Thurston's geometries carried by Levi-flat hypersurfaces in surfaces of general type. Our second result establishes that minimal sets in surfaces of general type have large Hausdorff dimension as soon as there exists a simply connected leaf. For both results, our methods rely on ergodic theory: we use harmonic measures and Lyapunov exponents.

Key Words: Riemann surfaces laminations, Minimal set, Levi-flat hypersurface, Harmonic measure, Lyapunov exponent, Surface of general type, Thurston's geometry.

2010 AMS: 37F75, 32V40, 37A50, 57R30, 14J29.

1. INTRODUCTION

A *Riemann surface lamination* is a compact metric space M covered by open sets $(U_j)_j$ with the following property, see the survey of Ghys [28]. There exist homeomorphisms $U_j \simeq \mathbb{D} \times \tau_j$, with $\tau_j \subset \mathbb{D}$, such that the transition mappings $\mathbb{D} \times \tau_j \rightarrow \mathbb{D} \times \tau_{j'}$ satisfy on their domain of definition

$$(1) \quad (z_j, u_j) \mapsto (\alpha_{jj'}(z_j, u_j), \beta_{jj'}(u_j))$$

where $\alpha_{jj'}$ is continuous in both variables and holomorphic with respect to the first one. A *plaque* is the preimage in M of a horizontal disk $\mathbb{D} \times \{u_j\} \subset \mathbb{D} \times \tau_j$. The *leaves* are the smallest connected subsets of M which are saturated by the plaques. Every leaf \mathcal{L} is an immersed Riemann surface in M . We denote by \mathcal{F} the foliation induced on M ; it is called *hyperbolic* if the universal cover of every leaf \mathcal{L} is biholomorphic to \mathbb{D} . In this article, a *complex surface* S stands for a compact, connected and smooth 2-dimensional complex manifold. Assuming that M is holomorphically immersed in a complex surface, the function $\beta_{jj'}$ in Equation (1) is Hölder continuous, due to the λ -Lemma of Mañé-Sad-Sullivan. The present work concerns Levi-flat hypersurfaces and minimal sets of holomorphic foliations in complex surfaces. The transverse regularity of these Riemann surface laminations is better than Hölder. Let us specify the definitions.

Definition 1.1. *Let S be a complex surface, TS be its tangent bundle and J be the endomorphism of TS given by the complex structure (satisfying $J^2 =$*

–ld). Let M be a compact 3-manifold of class C^k with $k \geq 2$. Let $p : M \rightarrow S$ be an immersion of class C^k . We say that the couple (M, p) is an immersed Levi-flat hypersurface in S of class C^k if for every open subset $U \subset M$ on which p is injective, the 2-dimensional distribution $Tp(U) \cap JTp(U)$ is integrable in the sense of Frobenius. We denote by \mathcal{F} the foliation on M induced by this distribution, and call it the Cauchy Riemann foliation.

If $M \subset S$ and $p : M \rightarrow S$ is the inclusion map then M is simply called a Levi-flat hypersurface in S . Note that for immersed Levi-flat hypersurfaces, the fact that M is of class C^k with $k \geq 2$ implies that the function $\beta_{jj'}$ in Equation (1) is of class C^k , see [4]. Classical examples of Levi-flat hypersurfaces appear in flat $\mathbb{P}^1(\mathbb{C})$ -bundles whose monodromy is a Fuchsian group, and in singular holomorphic fibrations, see Section 2. Other examples appear in complex tori (linear example of Grauert), in elliptic surfaces and in Kummer surfaces [51, 52]. Levi-flat hypersurfaces also exist in non Kählerian surfaces like Hopf surfaces, see Subsection 2.1, and Inoue surfaces, see [5, Section V.19]. We now define the minimal sets we shall deal with. We refer to Brunella's book [8, Chapter 2] for an introduction to singular holomorphic foliations on complex surfaces.

Definition 1.2. Let S be a complex surface and let $\hat{\mathcal{F}}$ be a non singular holomorphic foliation defined on an open set of S . A minimal set of $\hat{\mathcal{F}}$ is a compact set M included in this open set such that: M is saturated by the leaves of $\hat{\mathcal{F}}$ and every leaf of $\hat{\mathcal{F}}$ in M is dense in M . We denote by \mathcal{F} the restriction of $\hat{\mathcal{F}}$ to M . A minimal set is called exceptional if it does not coincide with S nor a holomorphic curve of S .

Since the ambient foliation $\hat{\mathcal{F}}$ is holomorphic, the function $\beta_{jj'}$ in Equation (1) is the restriction of a holomorphic function $\hat{\beta}_{jj'}$. Minimal sets appear in flat $\mathbb{P}^1(\mathbb{C})$ -bundles whose monodromy is a Kleinian group, see Subsection 2.6. Levi-flat hypersurfaces can be minimal sets, for instance if the Kleinian group is a Fuchsian group. It is still unknown whether every complex surface contains a Levi-flat hypersurface or an exceptional minimal set. It is believed that the projective plane $\mathbb{P}^2(\mathbb{C})$ should not contain such subsets, see [12].

We shall explore the topology and dynamics of immersed Levi-flat hypersurfaces and minimal sets in surfaces of general type. A *surface of general type* is a complex surface S whose Kodaira dimension is equal to 2. This means that its canonical bundle K_S is big: the dimension of the space of holomorphic sections of $K_S^{\otimes n}$ grows like n^2 . This property is stable under blow-ups. The following analytic property will be central: the canonical bundle K_S of a surface of general type has a metric whose curvature is positive outside finitely many curves in S . We refer to [5, Chapter 7] and to [23, Chapter 10] for an introduction to surfaces of general type. Examples of such complex surfaces are smooth hypersurfaces in $\mathbb{P}^3(\mathbb{C})$ of degree ≥ 5 and smooth quotients of the bidisc or of the unit ball of \mathbb{C}^2 by co-compact torsion free subgroups. There are many interesting Levi-flat hypersurfaces and minimal sets of holomorphic foliations in surfaces of general type, e.g.

singular holomorphic fibrations or ramified covers of flat $\mathbb{P}^1(\mathbb{C})$ -bundle with real monodromy, see Subsection 1.3 and Section 2.

1.1. Hyperbolicity and pinching in surfaces of general type. The following propositions are cornerstones of our work.

Proposition 1.3 (Hyperbolicity). *Let S be a surface of general type and let M be a minimal set or an immersed Levi-flat hypersurface of class C^2 in S . Then \mathcal{F} is hyperbolic.*

Let us emphasize that there is no assumption on the minimal set, it can be exceptional or not. The proof needs two ingredients. Let $T_{\mathcal{F}}$ and $N_{\mathcal{F}}$ be the tangent and normal line bundles to \mathcal{F} , see Subsection 3.2. The first ingredient is a theorem due to Candel [13] stating that if \mathcal{F} has a leaf which is not hyperbolic, then there exists a positive and closed $(1, 1)$ current T directed by \mathcal{F} such that $T \cdot T_{\mathcal{F}} \geq 0$ (the intersection product is explained in Subsection 3.6). The second ingredient is a theorem essentially due to Camacho-Lins-Neto-Sad, see [12, Theorem 2], stating that $T \cdot N_{\mathcal{F}} = 0$ for every positive and closed $(1, 1)$ -current T directed by the foliation. The arguments developed in [12] concerns minimal sets, they can be extended to immersed Levi-flat hypersurfaces. To conclude the proof, the leafwise adjunction formula $p^*K_S = N_{\mathcal{F}} - T_{\mathcal{F}}$ yields $T \cdot K_S \leq 0$, which contradicts the positivity properties of the canonical bundle. We note that Proposition 1.3 is a weak form of the Green-Griffiths conjecture, see Subsection 4.3 for more details.

Now, let us fix an immersed Levi-flat hypersurface or a minimal set in a surface of general type. Proposition 1.3 allows to endow $T_{\mathcal{F}}$ with the Poincaré metric of Gaussian curvature -1 , denoted by m_P , coming from the uniformizations of the leaves. This metric is continuous by results due to Verjovsky [63] and Candel [13]. A *harmonic measure* μ is a probability measure on M which is invariant by the leafwise heat diffusion (in our context with respect to the Poincaré metric), this notion was introduced by Garnett [25]. A harmonic measure μ is *ergodic* if every \mathcal{F} -saturated measurable subset of M has μ -measure 0 or 1. Let us define its Lyapunov exponent. For every $x \in M$ we denote by \mathcal{L}_x the leaf containing x and by Γ_x the set $\{\gamma : \mathbb{R}^+ \rightarrow \mathcal{L}_x, \gamma(0) = x\}$ of leafwise continuous paths starting at x . Let also W_x be the Wiener probability measure on Γ_x given by Brownian motion. For every $\gamma \in \Gamma_x$ we denote by $h_{\gamma,t}$ the holonomy map between transversals to \mathcal{F} along the restriction of γ on $[0, t]$. Garnett's random ergodic theorem asserts that there exists $\lambda(\mu) \in \mathbb{R}$ satisfying

$$\lim_{t \rightarrow +\infty} \frac{1}{t} \log |h'_{\gamma,t}(x)| = \lambda(\mu)$$

for μ -almost every $x \in M$ and W_x -almost every $\gamma \in \Gamma_x$. The number $\lambda(\mu)$ is called the *Lyapunov exponent* of μ , see Subsection 5.3 for more details.

Proposition 1.4 (Pinching). *Let S be a surface of general type. Let M be a minimal set or an immersed Levi-flat hypersurface of class C^2 in S . Let us endow $T_{\mathcal{F}}$ with the leafwise Poincaré metric and let μ be an ergodic harmonic measure. Then the Lyapunov exponent of μ satisfies $\lambda(\mu) > -1$.*

The main ingredient is a cohomological formula for $\lambda(\mu)$ established by the first author, see [17, Appendice A]. Indeed, if we denote by T the harmonic current defined as the quotient of the harmonic measure μ by the leafwise Poincaré volume, then

$$\lambda(\mu) = -2\pi T \cdot N_{\mathcal{F}}.$$

The conclusion follows from the leafwise adjunction formula and the positivity properties of K_S (similar tools as for the proof of Proposition 1.3).

1.2. Anosov Levi-flat hypersurfaces. We introduce the following definition. An immersed Levi-flat hypersurface M is *Anosov* if \mathcal{F} is topologically conjugate to the weak unstable foliation of a 3-dimensional Anosov flow on some compact 3-manifold N . Remarkably, to be topological conjugate will be a sufficient condition to obtain theorem 1.5 below. Note that this definition does not involve the complex structure of the leaves: there may be a lot of such structures, see [18, Theorem 1.1]. Classical examples of Anosov flows are the horizontal flow on hyperbolic torus bundles (see Subsection 2.2) and the geodesic flow on the unitary tangent bundle of compact orientable surfaces of genus ≥ 2 (see Subsection 2.3). There are many other examples, for instance on hyperbolic 3-manifolds and graph 3-manifolds, see [2, 22, 32, 33]. One can verify that Anosov Levi-flat hypersurfaces do not have any transverse invariant measure, their foliation \mathcal{F} is therefore hyperbolic, see Subsection 6.1. We prove the following upper bound for the Lyapunov exponent.

Theorem 1.5. *Let S be a complex surface and M be an immersed Anosov Levi-flat hypersurface in S . Let us endow $T_{\mathcal{F}}$ with the leafwise Poincaré metric and let μ be an ergodic harmonic measure. Then the Lyapunov exponent of μ satisfies $\lambda(\mu) \leq -1$.*

The pinching property provided by Proposition 1.4 then implies the

Corollary 1.6. *Let S be a surface of general type and let M be an immersed Levi-flat hypersurface in S . Then M is not Anosov.*

Our method to prove Theorem 1.5 holds more generally for immersed Levi-flat hypersurfaces admitting a *point at infinity* (see Section 6 for a precise definition). To show that Anosov flows satisfy this property, we use that their trajectories in the hyperbolic uniformizations of the leaves are quasigeodesics for the Poincaré metric. The bound ≤ -1 then relies on volume estimates in the spirit of Margulis-Ruelle's inequality. The ingredients involve the shadowing property of geodesics by Brownian paths due to Ancona, see [1, Théorème 7.3, page 103].

1.3. Topology of Levi-flat hypersurfaces. The preceding results allow to classify the possible geometries which can be carried by Levi-flat hypersurfaces in surfaces of general type. We will use the following terminology: a 3-manifold possesses a *geometry* if it admits a complete locally homogeneous metric (homogenous meaning that two different points admit isometric neighborhoods). Thurston classified in eight classes the compact 3-manifolds possessing a geometry, depending on the isometric class of their universal

cover among:

$$(2) \quad \mathbb{S}^3, \mathbb{R}^3, \mathbb{H}^3, \mathbb{S}^2 \times \mathbb{R}, \mathbb{H}^2 \times \mathbb{R}, \text{Nil}, \widetilde{\text{SL}(2, \mathbb{R})}, \text{Sol}.$$

The spaces $\mathbb{S}^p, \mathbb{R}^p$ and \mathbb{H}^p for $p \in \{2, 3\}$ stand for the complete simply connected Riemannian manifolds of dimension p of constant sectional curvature $1, 0, -1$ respectively. The last three models are Lie groups equipped with left invariant metrics. We refer to the article of Scott [58] for a more complete treatment. Let \mathcal{M} be one of the eight simply connected manifold in the list (2). We say that a compact 3-manifold M carries the geometry of \mathcal{M} if M is the quotient of \mathcal{M} by a discrete group of isometries of \mathcal{M} . In Section 2 we show that every geometry is carried by a Levi-flat hypersurface in an algebraic complex surface (apart \mathbb{S}^3 which does not occur on any compact Kähler surface). In particular:

Theorem 1.7. *There exist Levi-flat hypersurfaces in surfaces of general type which carry the geometries $\mathbb{H}^3, \mathbb{H}^2 \times \mathbb{R}$ or $\widetilde{\text{SL}(2, \mathbb{R})}$.*

The two first models are easily obtained by using singular holomorphic fibrations. The construction of Levi-flat hypersurfaces in surfaces of general type which carry the geometry $\widetilde{\text{SL}(2, \mathbb{R})}$ is more involved. This geometry is carried for instance by non-trivial circle bundles over compact orientable surfaces with negative Euler characteristic χ , see [58, Theorem 5.3]. We exhibit Levi-flat circle bundles in surfaces of general type whose ratio $|e/\chi|$ is arbitrary close to $1/5$, where e is the Euler class of the circle bundle.

Our next result shows that the other geometries do not appear in surfaces of general type. We also specify the case of $\widetilde{\text{SL}(2, \mathbb{R})}$ by showing that the quotients of $\text{PSL}(2, \mathbb{R})$ by a co-compact lattice (namely the unitary tangent bundle of orientable compact surfaces of genus ≥ 2) do not occur. We say that these quotients carry the geometry of $\text{PSL}(2, \mathbb{R})$.

Theorem 1.8. *Let S be a surface of general type. Let M be an immersed Levi-flat hypersurface of class C^2 in S . Then M neither carries the geometries $\mathbb{S}^3, \mathbb{R}^3, \mathbb{S}^2 \times \mathbb{R}, \text{Nil}, \text{PSL}(2, \mathbb{R})$ nor Sol.*

The proof of Theorem 1.8 relies on topological and dynamical methods. The first four cases are ruled out by proving that the fundamental group of Levi-flat hypersurfaces has exponential growth: this part uses the hyperbolicity of \mathcal{F} (Proposition 1.3), the absence of Reeb component and Novikov's theorems. To eliminate $\text{PSL}(2, \mathbb{R})$ and Sol, we establish the Anosov property for hypothetical Levi-flat hypersurfaces carrying these geometries, and we use Corollary 1.6 to exclude them. This part relies on deep theorems by Ghys-Sergiescu [30], Matsumoto [46] and Thurston [61].

1.4. Hausdorff dimension of minimal sets. We explain in Subsection 2.6 that there exist minimal sets in flat $\mathbb{P}^1(\mathbb{C})$ -bundles over compact Riemann surfaces of genus $g \geq 2$ with the most general and complicated dynamics that a Kleinian group can produce (Cantor set, quasi-circle, dendrite,...). In particular, their transverse Hausdorff dimension can be arbitrarily small. By performing convenient ramified coverings, we construct in Proposition 2.18 minimal sets in surfaces of general type modelled on Kleinian groups.

However, due to the ramification, the genus of their leaves is infinite. We prove in Theorem 1.10 below that this is a general fact in surfaces of general type: if a minimal set has a simply connected leaf, then its Hausdorff dimension must be large.

For that purpose we establish Theorem 1.9 which provides a lower bound for the transverse Hausdorff dimension of minimal sets, that we denote by $\dim_H M^\natural$. The statement involves Kaimanovich entropy and discreteness of the holonomy pseudogroup of \mathcal{F} , let us specify these notions. Let S be a complex surface and M be a minimal set of a non singular holomorphic foliation $\hat{\mathcal{F}}$. Let m be a continuous metric on $T_{\mathcal{F}}$ (see Subsection 3.3), let $p_m(t, x, y)$ be its leafwise heat kernel and vol_m be its volume form (see Subsection 5.1). We note that m induces a conformal metric on every leaf. Let μ be an ergodic harmonic measure with respect to m . Kaimanovich, see [37, Theorem 1], proved that there exists $h(\mu) \geq 0$ such that for μ -almost every $x \in M$:

$$(3) \quad h(\mu) = \lim_{t \rightarrow +\infty} -\frac{1}{t} \int_{\mathcal{L}_x} p_m(t, x, y) \log p_m(t, x, y) \text{vol}_m(dy).$$

The notion of discreteness for pseudogroups of analytic maps was introduced by Ghys [27] and Nakai [50]. We say that *the holonomy pseudo-group of \mathcal{F} is not discrete* if there exist transversal discs $D' \Subset D$ and a sequence of holonomy maps $h_n : D' \rightarrow D$ (defined using the ambient foliation $\hat{\mathcal{F}}$) which are different from the identity and which converge uniformly to the identity. Otherwise, the holonomy pseudogroup of \mathcal{F} is *discrete*.

Theorem 1.9 below requires for a minimal set the existence of ergodic harmonic measures with negative Lyapunov exponent. This existence is ensured once the foliation has no transverse invariant measure, see the article of Deroin-Kleptsyn [19, Theorem B]. Moreover, if m is transversally Hölder continuous, then the harmonic measure is unique, hence ergodic.

Theorem 1.9. *Let S be a complex surface and let $\hat{\mathcal{F}}$ be a non singular holomorphic foliation on an open set of S . Let M be a minimal set of $\hat{\mathcal{F}}$, let m be a continuous metric on $T_{\mathcal{F}}$ and let μ be an ergodic harmonic measure on M with negative Lyapunov exponent $\lambda(\mu)$. If the holonomy pseudogroup of \mathcal{F} is discrete then*

$$\dim_H M^\natural \geq \frac{h(\mu)}{|\lambda(\mu)|}.$$

In particular, the holonomy pseudogroup of \mathcal{F} is not discrete once the ratio $h(\mu)/|\lambda(\mu)|$ is > 2 . Up to now, the only method to prove non-discreteness consisted in using explicit elements close to identity, see [27, 44, 50]. Theorem 1.9 provides a new approach for this problem. In the context of random walks on linear groups, the dimension of the harmonic measure is equal to the ratio entropy/exponent [39, 40]. Such a ratio also appears for the dimension of invariant measures for conformal dynamical systems, see [45] for the case of rational maps on $\mathbb{P}^1(\mathbb{C})$.

Theorem 1.10. *Let S be a surface of general type, let $\hat{\mathcal{F}}$ be a non singular holomorphic foliation on an open set of S and let M be a minimal set of $\hat{\mathcal{F}}$. Assume that \mathcal{F} has no transverse invariant measure and that \mathcal{F} has a*

simply connected leaf. Then $\dim_H M^\natural \geq 1$. Moreover, the equality holds if and only if M is an analytic Levi-flat hypersurface (in this case the holonomy pseudogroup of \mathcal{F} is not discrete).

The proof is provided in Subsection 8.3. The argument depends on the discreteness of the holonomy pseudogroup of \mathcal{F} . If it is not discrete, the combination of [19, Corollary 1.3] and [44, Section 4] yields holomorphic flows in the closure of the pseudogroup, therefore $\dim_H M^\natural \geq 1$. We show further that if $\dim_H M^\natural = 1$, then M is an analytic Levi-flat hypersurface. This part does not use the general type property for S neither the existence of a simply connected leaf. If the holonomy pseudogroup is discrete, we proceed as follows. On one hand we establish that the presence of a simply connected leaf implies the same property for almost every leaf, this allows to deduce $h(\mu) = 1$. On the other hand the pinching property of Proposition 1.4 yields $\lambda(\mu) > -1$ (S is a surface of general type). Theorem 1.9 then implies $\dim_H M^\natural > 1$, which completes the proof of Theorem 1.10.

1.5. Outline. Section 2 is devoted to examples of Levi-flat hypersurfaces, we review the eight Thurston's geometries and prove Theorem 1.7. Section 3 contains the analytic and geometric tools we shall need (line bundles, harmonic currents, intersection product). Section 4 contains the proof of Proposition 1.3 (hyperbolicity) and the first part of the proof of Theorem 1.8. Harmonic measures and Lyapunov exponents are introduced in Section 5, which also contains the proof of Proposition 1.4 (pinching). Section 6 deals with Anosov Levi-flat hypersurfaces and establishes Theorem 1.5 and Corollary 1.6. The second part of Theorem 1.8 (impossibility of Sol and $\mathrm{PSL}(2, \mathbb{R})$ in surfaces of general type) is established in Section 7. Finally, Section 8 is devoted to the Hausdorff dimension of minimal sets, we prove Theorems 1.9 and 1.10. Some technical proofs are postponed to Section 9.

1.6. Acknowledgements. We thank É. Ghys for sharing his views on the theory of 2-dimensional foliations of 3-manifolds, and S. Boucksom for discussions about surfaces of general type. Part of this work was motivated by questions asked by M. Brunella, we dedicate this work to him. We warmly thank the referee for his very cautious reading and his numerous comments. The exposition of the article has been greatly improved thanks to his advices. The authors are supported by the ANR projects 08-JCJC-0130-01, 09-BLAN-0116 for B.D. and 07-JCJC-0006-01 for C.D. The work was initiated during a common visit of the authors at IMPA, made possible with the help of ANR and France-Brazil cooperation. The first author thanks the Mittag-Leffler Institute for the very nice working conditions offered to us.

2. EXAMPLES OF LAMINATIONS IN COMPLEX SURFACES

We review the eight Thurston's geometries and prove that each of them occurs as a Levi-flat hypersurface in an algebraic surface, with one exception, S^3 , which does not occur in any Kähler surface. We also provide in surfaces of general type examples of Levi-flat hypersurfaces carrying the geometry $\widetilde{\mathrm{SL}(2, \mathbb{R})}$ and examples of minimal sets with small Hausdorff dimension.

2.1. Reeb components, orientability and \mathbb{S}^3 . A Reeb component R is a solid torus foliated by 2-dimensional manifolds such that its torus boundary is a leaf. An example is given by the quotient of $\mathbb{C} \times [0, \infty) \setminus \{(0, 0)\}$ by $(z, u) \mapsto (2z, 2u)$, this solid torus is foliated by the quotient of the horizontal foliation. The standard Reeb foliation on \mathbb{S}^3 is obtained by gluing two Reeb components along their boundary leaf, see [15, Section 8.1]. Remarkably, the Cauchy-Riemann foliation of any Levi-flat hypersurface is never topologically conjugate to this foliation, see [3, Theorem 1]. We refer to [15, Section 9.1] for the following result.

Theorem 2.1 (Novikov). *Let M be a compact orientable 3-manifold endowed with a transversally orientable 2-dimensional foliation \mathcal{F} of class C^2 . The following assertions are equivalent*

- (a) *The foliation \mathcal{F} contains a Reeb component.*
- (b) *There exists a leaf $\mathcal{L} \in \mathcal{F}$ such that the inclusion map $\pi_1(\mathcal{L}) \rightarrow \pi_1(M)$ between the fundamental groups has a non-trivial kernel.*

Moreover, if there exists a closed and homotopically trivial loop transverse to \mathcal{F} , then the foliation \mathcal{F} contains a Reeb component. This occurs in particular when the fundamental group of M is finite.

Orientability assumptions can be fulfilled by using the following Lemma, for which we refer to [11, Section 4.1.2]. The first item uses the fact that the leaves of \mathcal{F} are orientable, since they are endowed with a complex structure.

Lemma 2.2. *Let S be a complex surface and $p : M \rightarrow S$ be an immersed Levi-flat hypersurface.*

- (1) *\mathcal{F} is transversally orientable if and only if M is orientable.*
- (2) *Assume that M is not orientable. Let $\mathcal{D} : \hat{M} \rightarrow M$ be the double covering of orientations. Then \hat{M} is an orientable immersed Levi-flat hypersurface in S for the immersion $\hat{p} = p \circ \mathcal{D}$. The Cauchy-Riemann foliation on \hat{M} is equal to $\hat{p}^*\mathcal{F}$ and is transversally orientable.*

The following proposition was proved in [34, Theorem 1], it discards the geometry of \mathbb{S}^3 for immersed Levi-flat hypersurfaces in Kähler surfaces.

Proposition 2.3. *Let S be a Kähler surface and M be an immersed Levi-flat hypersurface of class C^2 in S . Then M does not carry the geometry of \mathbb{S}^3 .*

Proof. Let $p : M \rightarrow S$ be the immersion and let $q : \mathbb{S}^3 \rightarrow M$ be the quotient map given by the action of the group of isometries. Then $p \circ q : \mathbb{S}^3 \rightarrow S$ defines an immersed Levi-flat hypersurface of class C^2 in S . Its foliation is transversally orientable by Lemma 2.2(1). The conclusion then follows from Theorem 2.1 and Lemma 2.4 below. \square

Lemma 2.4. *Let S be a Kähler surface. Immersed Levi-flat hypersurfaces of class C^2 in S do not contain any Reeb component.*

Proof. Assume that there exists an immersed Levi-flat hypersurface $p : M \rightarrow S$ containing a Reeb component R . Then the pull-back $p^*\omega$ of a Kähler form ω on S is positive on the leaves of \mathcal{F} , in particular on ∂R . The closedness of ω and Stoke's formula yield a contradiction: $0 < \int_{\partial R} p^*\omega = \int_R p^*d\omega = 0$. \square

We observe that there exist Levi-flat hypersurfaces in non Kählerian surfaces containing Reeb components. Consider for that purpose the Hopf surface $S = \mathbb{C}^2 \setminus \{(0, 0)\} / (x, y) \sim (2x, 2y)$. Then the quotient of $\mathbb{C} \times [0, \infty) \setminus \{(0, 0)\}$ is a Levi-flat hypersurface in S .

2.2. Realization of \mathbb{R}^3 , \mathbb{H}^3 , $\mathbb{S}^2 \times \mathbb{R}$, $\mathbb{H}^2 \times \mathbb{R}$, Nil and Sol. The aim of this subsection is to prove Proposition 2.5 below.

First we recall that the geometries Nil, Sol and \mathbb{H}^3 are supported by non trivial surface bundles over the circle. A *surface bundle* is the quotient of $[0, 1] \times \Sigma$ by the relation $(0, x) \sim (1, \Phi(x))$, where Σ is a compact oriented surface and Φ is a diffeomorphism of Σ preserving the orientation.

We shortly denote a surface bundle by $\mathbb{S}^1 \rtimes_{\Phi} \Sigma$. Its *monodromy* is the projection $[\Phi]$ of Φ in the mapping class group $\text{MCG}(\Sigma)$. An element $[\Phi] \in \text{MCG}(\Sigma)$ is called *elliptic* if its order is finite, *reducible* if there is a finite and non-empty collection of pairwise disjoint simple closed curves in Σ whose union is invariant by a diffeomorphism in $[\Phi]$, and *pseudo-Anosov* in the other cases, see [62, Section 2].

If Σ has genus 1, the surface bundle is called a *torus bundle*. The group $\text{SL}(2, \mathbb{Z})$ acts on $\Sigma \simeq \mathbb{R}^2 / \mathbb{Z}^2$ by linear transformations. Every element of $\text{MCG}(\Sigma)$ can be represented by such a linear transformation. An *unipotent torus bundle* is a torus bundle whose monodromy comes from a unipotent matrix in $\text{SL}(2, \mathbb{Z}) \setminus \{\text{Id}\}$ (reducible monodromy), it carries the Nil geometry. A *hyperbolic torus bundle* is a torus bundle whose monodromy comes from a hyperbolic matrix in $\text{SL}(2, \mathbb{Z})$ (pseudo-Anosov monodromy), it carries the Sol geometry. We refer to [58, Section 4] for more details about these geometries.

We shall realize such surface bundles in *singular holomorphic fibrations*. Such a fibration stands for a holomorphic map $f : S \rightarrow B$ where S is a complex surface and B is a compact Riemann surface, see [5, Chapter III, 8]. Here we do not assume that the fibers of f are connected. Let p_1, \dots, p_n be the singular values of f (this set may be empty). A *fibred Levi-flat hypersurface* is a Levi-flat hypersurface of the form $f^{-1}(\gamma)$, where $f : S \rightarrow B$ is a singular holomorphic fibration and $\gamma \subset B \setminus \{p_1, \dots, p_n\}$ is a simple closed path. These examples were already considered by Poincaré, see [56].

Proposition 2.5. *Every geometry \mathbb{R}^3 , \mathbb{H}^3 , $\mathbb{S}^2 \times \mathbb{R}$, $\mathbb{H}^2 \times \mathbb{R}$, Nil or Sol is carried by a fibred Levi-flat hypersurface. Moreover, \mathbb{H}^3 and $\mathbb{H}^2 \times \mathbb{R}$ are carried by fibred Levi-flat hypersurfaces in surfaces of general type.*

The remainder of this subsection is devoted to the proof. It is easy to realize \mathbb{R}^3 , $\mathbb{S}^2 \times \mathbb{R}$ and $\mathbb{H}^2 \times \mathbb{R}$ by using products of compact Riemann surfaces $S = \Sigma \times B$ with Σ of genus 1, 0 or $g \geq 2$ respectively.

To exhibit fibred Levi-flat hypersurfaces with the geometries Nil and Sol, we use the following classical proposition, see [24, Chapter II, Section 2.3]. An elliptic fibration $f : S \rightarrow \mathbb{P}^1(\mathbb{C})$ is called *nodal* if all of its singular fibers are irreducible rational curves with one node. We refer to [24, Chapter I, Section 1.7.1] for the existence of such fibrations.

Proposition 2.6. *Let $f : S \rightarrow \mathbb{P}^1(\mathbb{C})$ be a nodal elliptic fibration. Let p_1, \dots, p_n be the singular values of f , assume that this set is not empty.*

Then the monodromy representation from the fundamental group of $\mathbb{P}^1(\mathbb{C}) \setminus \{p_1, \dots, p_n\}$ to $\mathrm{SL}(2, \mathbb{Z})$ is surjective.

Using this proposition, let γ_1 and γ_2 be closed curves in $\mathbb{P}^1(\mathbb{C}) \setminus \{p_1, \dots, p_n\}$ whose monodromy representations are respectively unipotent and hyperbolic. If γ_1 and γ_2 are embedded closed curves, then $f^{-1}(\gamma_1)$ and $f^{-1}(\gamma_2)$ are respectively Levi-flat hypersurfaces in S carrying the geometries Nil and Sol. Let us deal with the case when for instance γ_1 is not embedded. Then there exist a closed curve $\hat{\gamma}_1$ in $\mathbb{P}^1(\mathbb{C}) \setminus \{p_1, \dots, p_n\}$ homotopic to γ_1 and a ramified covering $c : \mathbb{P}^1(\mathbb{C}) \rightarrow \mathbb{P}^1(\mathbb{C})$ such that $c^{-1}(\hat{\gamma}_1)$ contains an embedded closed curve $\tilde{\gamma}_1$. To find $\hat{\gamma}_1$ and c , let us parametrize γ_1 by the unit circle \mathbb{T} : such a parametrization yields a Fourier series $\sum_{n \in \mathbb{Z}} c_n e^{in\theta}$. Then $c(z) := \sum_{-N}^N c_n z^n$ and $\hat{\gamma}_1 := c(\mathbb{T})$ are convenient for N large enough. We thank D. Cerveau for this argument. Now let us consider the algebraic complex surface $\tilde{S} := \{(x, y) \in S \times \mathbb{P}^1(\mathbb{C}) \mid f(x) = c(y)\}$ and the singular elliptic fibration $\tilde{f} : \tilde{S} \rightarrow \mathbb{P}^1(\mathbb{C})$ defined by $\tilde{f}(x, y) = y$. (the fibers of \tilde{f} may be not connected.) Then $\tilde{f}^{-1}(\hat{\gamma}_1)$ is a Levi-flat hypersurface in \tilde{S} carrying the geometry Nil. Similar arguments for γ_2 yield a Levi-flat hypersurface carrying the geometry Sol.

To realize \mathbb{H}^3 we use Thurston's theorem, see [49, Section 3.4] and [62, Theorem 0.1].

Theorem 2.7 (Thurston). *Let Σ be a compact oriented surface of genus $g \geq 2$. A surface bundle $S^1 \times_{\Phi} \Sigma$ carries the geometry \mathbb{H}^3 if and only if its monodromy $[\Phi]$ is pseudo-Anosov.*

By using the same arguments as before, the following theorem provides fibered Levi-flat hypersurfaces modelled on \mathbb{H}^3 , see [59, Corollary 1].

Theorem 2.8 (Shiga). *Let B be a compact Riemann surface with genus larger than or equal to 2. Let $f : S \rightarrow B$ be a singular holomorphic fibration, such that the generic fiber has genus ≥ 2 and its modulus is not locally constant. Let p_1, \dots, p_n be the critical values of f . Then there exists an immersed simple closed curve γ in $B \setminus \{p_1, \dots, p_n\}$ whose monodromy is pseudo-Anosov.*

Examples of singular holomorphic fibration $f : S \rightarrow B$ satisfying the hypothesis of Theorem 2.8 are provided by Kodaira fibrations, see [5, Chapter V, Section 14]. The surface S is of general type, since the genus of B and of the fibers of f are larger than or equal to 2, see [5, Chapter III, Theorem 18.4]. This completes the proof of Proposition 2.5.

2.3. Realization of $\widetilde{\mathrm{SL}(2, \mathbb{R})}$. The geometry $\widetilde{\mathrm{SL}(2, \mathbb{R})}$ is supported for instance by non-trivial circle bundles over compact oriented surfaces of genus $g \geq 2$, see [58, Theorem 5.3]. We shall construct Levi-flat hypersurfaces with this topology in flat $\mathbb{P}^1(\mathbb{C})$ -bundles over compact Riemann surfaces. Let Σ be a compact Riemann surface of genus $g \geq 2$, $\tilde{\Sigma}$ a universal covering of Σ , and $\pi_1(\Sigma)$ its covering group. Let $\rho : \pi_1(\Sigma) \rightarrow \mathrm{PSL}(2, \mathbb{C})$ be a representation. By definition, the corresponding flat $\mathbb{P}^1(\mathbb{C})$ -bundle over Σ is the

quotient of $\tilde{\Sigma} \times \mathbb{P}^1(\mathbb{C})$ by

$$(4) \quad \forall \gamma \in \pi_1(\Sigma), \quad \gamma \cdot (p, z) = (\gamma(p), \rho(\gamma)z),$$

where the group $\mathrm{PSL}(2, \mathbb{C})$ acts on $\mathbb{P}^1(\mathbb{C})$ by homographies. We denote by $X := \Sigma \times_{\rho} \mathbb{P}^1(\mathbb{C})$ this complex surface. This is an algebraic surface, in particular it admits Kähler metrics, see [5, Section V.4]. The vertical fibration is invariant by the action of $\pi_1(\Sigma)$ defined by Equation (4), let $f : X \rightarrow \Sigma$ be the associated $\mathbb{P}^1(\mathbb{C})$ -fibration. The horizontal foliation of $\mathbb{H} \times \mathbb{P}^1(\mathbb{C})$ is also invariant, we denote by \mathcal{F}_{ρ} its quotient in X . Note that \mathcal{F}_{ρ} defines a non-singular holomorphic foliation on S transverse to the $\mathbb{P}^1(\mathbb{C})$ -fibration.

Now assume that ρ takes its values in $\mathrm{PSL}(2, \mathbb{R})$. Since the circle \mathbb{S}^1 is invariant by $\mathrm{PSL}(2, \mathbb{R})$, the subset $M_{\rho} := \Sigma \times_{\rho} \mathbb{S}^1$ is a Levi-flat hypersurface in X which carries a natural structure of oriented circle bundle over Σ . We denote by e the Euler class of M_{ρ} . This invariant belongs to $H^2(\Sigma, \mathbb{Z})$ and characterizes the circle bundle up to isomorphism, see [15, Chapter 4] and [46, Section 2]. Let us note that M_{ρ} is an example of a Seifert fibre space, we refer to [58, Section 3] for an introduction of such spaces and for the definition of their Euler class. Changing the orientation of the circle bundle changes e into $-e$. Moreover $|e| = 2g - 2$ if and only if ρ is an isomorphism between $\pi_1(\Sigma)$ and a Fuchsian group. In this case M_{ρ} is diffeomorphic to the unitary tangent bundle of Σ , see [64, Proposition 6.2].

Proposition 2.9. *Let Σ be a compact oriented surface of genus $g \geq 2$ and let $e \in \mathbb{Z}$ satisfy $|e| \leq 2g - 2$. There exists a flat $\mathbb{P}^1(\mathbb{C})$ -bundle X over Σ and a Levi-flat hypersurface $M \subset X$ which is diffeomorphic to a circle bundle over Σ with Euler class e .*

Proof. If $|e| \leq 2g - 2$ then there exists a representation $\rho : \pi_1(\Sigma) \rightarrow \mathrm{PSL}(2, \mathbb{R})$ such that M_{ρ} has Euler class e , see [31, Theorems A and B]. \square

In the next subsection we shall use the following remark: if $\varphi : \Sigma \rightarrow \Sigma'$ is a covering between oriented compact surfaces and if $b' \in H^1(\Sigma', \mathbb{S}^1)$ is a circle bundle over Σ' , then the Euler class of $b := \varphi^* b'$ satisfies $e = \varphi^* e'$ in $H^2(\Sigma, \mathbb{Z})$. Making the identifications $H^2(\Sigma, \mathbb{Z}) \simeq \mathbb{Z}$ and $H^2(\Sigma', \mathbb{Z}) \simeq \mathbb{Z}$ by sending the fundamental classes to 1, we obtain $|e| = |\deg \varphi| \cdot |e'|$.

2.4. Realization of $\widetilde{\mathrm{SL}(2, \mathbb{R})}$ in surfaces of general type. We begin with an upper bound on the Euler class of circle bundle Levi-flat hypersurfaces.

Proposition 2.10. *Let S be a surface of general type and M be a Levi-flat hypersurface of class C^2 in S . Assume that M is an oriented circle bundle over a compact oriented surface Σ of genus $g \geq 2$. Then the Euler class of M satisfies $|e| \leq 2g - 2$.*

Proof. We can assume that $e \neq 0$. First we prove that the Cauchy-Riemann foliation \mathcal{F} on M does not contain any compact leaf. Suppose to the contrary that \mathcal{F} contains a compact leaf \mathcal{L} . Since S is a surface of general type, \mathcal{L} has genus $g \geq 2$ by Proposition 1.3 (proved in Subsection 4.1 below). Then the natural map $\pi_1(\mathcal{L}) \rightarrow \pi_1(M)$ is injective by Theorem 2.1 and Lemma 2.4. Let us verify that such an incompressible surface does not exist in a non trivial circle bundle. Observe that the natural map $\pi_1(\mathcal{L}) \rightarrow \pi_1(\Sigma)$ is

injective because $\pi_1(M)$ is a *central* extension $\mathbb{Z} \rightarrow \pi_1(M) \rightarrow \pi_1(\Sigma)$ (in a surface group, the centralizer of any non trivial element is a cyclic subgroup). Moreover the index of $\pi_1(\mathcal{L})$ in $\pi_1(\Sigma)$ is finite. Indeed, if it were false, $\pi_1(\mathcal{L})$ would be the fundamental group of a non compact surface, hence a (non abelian) free group. In particular the projection map $\mathcal{L} \rightarrow \Sigma$ (which is determined up to homotopy by its action on the fundamental group since the universal covers of \mathcal{L} and Σ are contractible) is homotopic to a covering of positive degree d . Now consider the pull-back of the circle bundle M over Σ by the projection map $\mathcal{L} \rightarrow \Sigma$. We get a circle bundle N over \mathcal{L} having a natural section, hence N is the trivial bundle. This contradicts the multiplicativity of Euler class $0 = e(N) = d \cdot e(M) \neq 0$, see the end of Subsection 2.3. All this proves that \mathcal{F} does not contain any compact leaf. The upper bound $|e| \leq 2g - 2$ then follows from the combination of the two results below. We refer to [61] for the first one (see also [42, Theorem page 572]) and to [64] for the second one. \square

Theorem 2.11 (Thurston). *Let M be an oriented circle bundle over a compact oriented surface Σ of genus $g \geq 2$. Assume that \mathcal{F} is an oriented 2-dimensional foliation on M of class C^2 , and that \mathcal{F} does not have any compact leaf. Then there exists a diffeomorphism Ψ of M of class C^2 isotopic to the identity such that $\mathcal{G} := \Psi_*\mathcal{F}$ is transverse to the circle fibration.*

Theorem 2.12 (Milnor-Wood). *Let M be an oriented circle bundle over a compact oriented surface Σ of genus $g \geq 2$. If M supports a transversally oriented 2-dimensional foliation which is transverse to the circle fibration, then its Euler class satisfies $|e| \leq 2g - 2$.*

Our next result is the construction of Levi-flat hypersurfaces in surfaces of general type with a non trivial Euler class. We obtain the following theorem.

Theorem 2.13. *For every $\epsilon > 0$ there exist a surface of general type S_ϵ and a Levi-flat hypersurface $M_\epsilon \subset S_\epsilon$ which is diffeomorphic to an oriented circle bundle M_ϵ over a compact oriented surface Σ_ϵ of genus ≥ 2 and satisfies $|e(M_\epsilon)/\chi(\Sigma_\epsilon)| \in [1/5 - \epsilon, 1/5]$, where we denote by $e(M_\epsilon)$ the Euler class of M_ϵ and by $\chi(\Sigma_\epsilon)$ the Euler characteristic of Σ_ϵ .*

The remainder of this subsection is devoted to the proof of Theorem 2.13. Our method consists in performing ramified coverings to flat $\mathbb{P}^1(\mathbb{C})$ -bundles. Let Σ' be a compact Riemann surface of genus $g' \geq 2$ and $\chi' := 2 - 2g'$ be its Euler characteristic. By the uniformization theorem, there exists a representation $\rho : \pi_1(\Sigma') \rightarrow \mathrm{PSL}(2, \mathbb{R})$ and a ρ -equivariant biholomorphism $D : \tilde{\Sigma}' \rightarrow \mathbb{H}$. Let $X' := \Sigma' \times_{\rho} \mathbb{P}^1(\mathbb{C})$ be the flat $\mathbb{P}^1(\mathbb{C})$ -bundle and $M' := \Sigma' \times_{\rho} \mathbb{S}^1$ be the Levi-flat hypersurface in X' as defined in Subsection 2.3. Observe that M' is the unitary tangent bundle of Σ' , its Euler class e' satisfies $|e'| = 2g' - 2$. We denote by $f : X' \rightarrow \Sigma'$ the $\mathbb{P}^1(\mathbb{C})$ -fibration and by \mathcal{F}_ρ the horizontal foliation on X' . Let σ be the divisor of X' defined as the quotient of $\{(p, D(p)), p \in \tilde{\Sigma}'\}$ by the action of $\pi_1(\Sigma')$ defined by Equation (4), it is transverse to the $\mathbb{P}^1(\mathbb{C})$ -fibration and to the foliation \mathcal{F}_ρ .

We need several definitions and notations, we refer the reader to [5, Section I.6] and to [23, Chapter 1] for more details. The following material will be useful to introduce ramified coverings. Let X be a smooth complex

projective surface. Let $\text{Pic}(X) = H^1(X, \mathcal{O}_X^*)$ be the group of (isomorphism classes of) line bundles on X . We denote by $L_1 + L_2$ the tensor product of two line bundles. We recall that $L \in \text{Pic}(X)$ is *very ample* if

$$(5) \quad x \in X \mapsto [s_0(x) : \dots : s_N(x)] \in \mathbb{P}^N(\mathbb{C})$$

is a holomorphic embedding, where (s_0, \dots, s_N) is a basis of the space of holomorphic sections of L . A line bundle L is *ample* if there exists $n \geq 1$ such that $nL := L^{\otimes n}$ is very ample. A line bundle L is *big* if $N_n \simeq n^2$, where N_n is the dimension of the space of holomorphic sections of nL . We denote by K_X the canonical line bundle of X , its holomorphic sections are the holomorphic 2-forms on X . A *curve* (respectively *smooth curve*) on X is a complex analytic subvariety (respectively submanifold) of X of complex dimension 1. A *divisor* D is a finite formal sum $\sum_j a_j C_j$ where $a_j \in \mathbb{Z}$ and each C_j is an irreducible curve. This divisor is *reduced* if $a_j = 1$ for every j . If s is a meromorphic section of $L \in \text{Pic}(X)$, we set $(s) := (s)_0 - (s)_\infty$, where $(s)_0$ is the zero divisor and $(s)_\infty$ is the polar divisor of s . We denote by $\mathcal{O}_X(D)$ the element of $\text{Pic}(X)$ having a meromorphic section such that $(s) = D$. For every $L \in \text{Pic}(X)$ we denote by $[L] \in H^2(X, \mathbb{Z})$ the Chern class of L . If D is a divisor on X , we define $[D] := [\mathcal{O}_X(D)]$. We shall use the intersection pairing on $H^2(X, \mathbb{Z})$ coming from the intersection pairing on $H_2(X, \mathbb{Z})$ and Poincaré duality. Let $\text{Pic}^0(X) \subset \text{Pic}(X)$ be the kernel of the Chern class morphism, $\text{Pic}^0(X)$ is an abelian variety [5, Section I.13]. To prove Theorem 2.13 we need the following classical proposition, see [5, Section I.17].

Proposition 2.14. *Let $X' = \Sigma' \times_{\rho} \mathbb{P}^1(\mathbb{C})$ as before. Let $L_1 \in \text{Pic}(X')$. Assume that $2L_1$ has a holomorphic section s such that $s^*(0) \subset X'$ is a reduced smooth curve.*

- (1) *There exist a complex surface S and a ramified covering $\Theta : S \rightarrow X'$ of degree 2 such that $s^{-1}(0)$ is the ramification locus of Θ .*
- (2) *$K_S = \Theta^*(K_{X'} + L_1)$. In particular S is a surface of general type if $K_{X'} + L_1$ is a big line bundle on X' .*
- (3) *Let M' be a Levi-flat hypersurface in X' . If the foliation \mathcal{F} is transverse to $s^{-1}(0)$, then $\Theta^{-1}(M')$ is a Levi-flat hypersurface in S .*

Proof. Let (z, w, ξ) be local coordinates for $2L_1$, where $(z, w) \in S$ and $\xi \in \mathbb{C}$. Since $s^*(0)$ is smooth and reduced, one can assume that $s(z, w) = z$ near points where s vanishes and $s(z, w) = 1$ elsewhere. Let $q : L_1 \rightarrow 2L_1$ locally defined by $q(z, w, \xi) = (z, w, \xi^2)$. We define $S := q^{-1}(s(X'))$. This is a complex surface in L_1 , its smoothness comes from its local definitions $\{(z, w, \xi) \in L_1, \xi^2 = z\}$ or $\{(z, w, \xi) \in L_1, \xi^2 = 1\}$. We define $\Theta : S \rightarrow X'$ as the restriction to S of the projection $L_1 \rightarrow X'$. This proves the first item. We refer to [5, Section I.17, Lemma 17.1] for the second one. For the third item we assume for sake of simplicity that M' is locally equal to $\{\Im(w) = 0\}$, the plaques are therefore the level sets of $\Re(w)$, which are transverse to $s^{-1}(0)$. The general case follows the same idea. Since the derivatives of $\Im(w)$ and $\Re(w)$ are both independent from the derivatives of $\xi^2 - z$ and $\xi^2 - 1$, we obtain that $\Theta^{-1}(M')$ is a smooth compact 3-manifold foliated by Riemann surfaces. \square

To construct Levi-flat hypersurfaces in surfaces of general type, it suffices, in view of Proposition 2.14, to exhibit $L_1 \in \text{Pic}(X')$ such that $K_{X'} + L_1$ is ample (hence big) and such that $2L_1$ has a holomorphic section whose divisor is a smooth reduced curve transverse to \mathcal{F}_ρ . We refer for instance to [5, Section 5.4], [6, Section III.17] and [43, Section 2] for the following properties. The self-intersection of the divisor σ (defined after Theorem 2.13) satisfies $[\sigma] \cdot [\sigma] = \chi'$: this is due to the fact that σ is transverse to the $\mathbb{P}^1(\mathbb{C})$ -fibration, the tangent bundle and the normal bundle of σ are thus isomorphic. If φ is any fiber of the $\mathbb{P}^1(\mathbb{C})$ -fibration, we have $H^2(X', \mathbb{Z}) = \mathbb{Z}[\sigma] + \mathbb{Z}[\varphi]$, with $[\sigma] \cdot [\varphi] = 1$ and $[\varphi] \cdot [\varphi] = 0$. Adjunction formula yields $[K_{X'}] = -2[\sigma]$. Now we establish the

Proposition 2.15. *Let $X' = \Sigma' \times_{\rho} \mathbb{P}^1(\mathbb{C})$ be a flat $\mathbb{P}^1(\mathbb{C})$ -bundle and $n' := 4(1 - 2\chi')$. Let $(\varphi_i)_i$ be n' pairwise disjoint fibers of the $\mathbb{P}^1(\mathbb{C})$ -fibration f , and let $L_2 := \mathcal{O}_{X'}(6\sigma + \sum_1^{n'} \varphi_i)$. There exists $L_1 \in \text{Pic}(X')$ such that $L_2 = 2L_1$. Moreover, $K_{X'} + L_1$ is ample and L_2 is very ample.*

Proof. Let φ be a fixed fiber of f . Observe that $L_2 - \mathcal{O}_{X'}(6\sigma + n'\varphi) \in \text{Pic}^0(X')$. Since every element of the abelian variety $\text{Pic}^0(X')$ is divisible by 2, there exists $L_3 \in \text{Pic}^0(X')$ such that $L_2 - \mathcal{O}_{X'}(6\sigma + n'\varphi) = 2L_3$. The first statement holds by setting $L_1 := \mathcal{O}_{X'}(3\sigma + (n'/2)\varphi) + L_3$. For the second statement, the fact that $[K_{X'}] = -2[\sigma]$ implies $[K_{X'}] + [L_1] = [\sigma] + (n'/2)[\varphi]$. We conclude by using the following ampleness criterion (see [23, Chapter 5, Proposition 15]): the divisor $a\sigma + b\varphi$ in X' is ample if and only if $a > 0$ and $b > -a\chi'$. It remains to prove the third statement. We use Reider's criterion, see [57, Theorem 1]: if X is an algebraic complex surface and E is an ample line bundle, then $4E + K_X$ is very ample. Using again $[K_{X'}] = -2[\sigma]$ we observe that $[L_2] - [K_{X'}]$ is divisible by 4 in $H^2(X', \mathbb{Z})$. Hence there exists $E \in \text{Pic}(X')$ such that $L_2 = 4E + K_{X'}$. Similar arguments as above show that E is ample, hence L_2 is very ample by Reider's criterion. \square

We continue the proof of Theorem 2.13 using Proposition 2.15. Let \tilde{s} be a holomorphic section of L_2 such that $(\tilde{s}) = 6\sigma + \sum_1^{n'} \varphi_i$. Let $\{p_1, \dots, p_{n'}\} := \{f(\varphi_1), \dots, f(\varphi_{n'})\}$. Since σ and M' are disjoint, there exists a holomorphic section s of L_2 which is close to \tilde{s} , such that (s) is smooth and reduced (here we use Bertini's theorem, indeed L_2 is very ample) and such that $(s) \cap M'$ is a link isotopic to a family of n' distinct fibers $\varphi_i \cap M'$ of the circle fibration $f|_{M'} : M' \rightarrow \Sigma'$. We deduce that there exists a circle fibration $c' : M' \rightarrow \Sigma'$ which is C^1 -close to $f|_{M'}$ and such that $(s) \cap M' = c'^{-1}\{p_1, \dots, p_{n'}\}$. The fibers of c' are homologous to zero in X' , since c' and $f|_{M'}$ are C^1 -close and since the fibers of $f|_{M'}$ which are far from $(s) \cap M'$ bound discs in $X' \setminus (s)$.

Now we apply Proposition 2.14: let S be the surface of general type and let $\Theta : S \rightarrow X'$ be the ramified covering of degree 2 defined by L_2 and the section s . We set $M := \Theta^{-1}(M')$. The construction of Θ (see the proof of Proposition 2.14) shows that M is an oriented circle bundle $c : M \rightarrow \Sigma$ over a compact Riemann surface Σ , and there exists a continuous map $\theta : \Sigma \rightarrow \Sigma'$ such that $\theta \circ c = c' \circ \Theta|_M$. By Hurwitz formula, the Euler characteristic of Σ is $\chi := 2\chi' - n' = 10\chi' - 4$. The Euler class of M is equal to $e := 2e' = 2\chi'$, see the end of Subsection 2.3. Finally, the ratio e/χ is equal to $2\chi'/(10\chi' - 4)$

and tends to $1/5$ when χ' tends to $-\infty$. This completes the proof of Theorem 2.13.

Question 2.16. *Let $g \geq 2$ and $e \in \mathbb{Z}$ such that $|e| < 2g - 2$. Does there exist a Levi-flat hypersurface in a surface of general type which is diffeomorphic to an oriented circle bundle over a compact orientable surface of genus g with Euler class e ? Theorem 2.13 asserts that non-trivial values of e occur when g is large. Theorem 1.8 asserts that $|e| = 2g - 2$, corresponding to the geometry $\mathrm{PSL}(2, \mathbb{R})$ does not occur.*

2.5. Chern numbers. Let us compute the Chern numbers $c_1^2(S)$ and $c_2(S)$ of the surface of general type S we have constructed in the last paragraph of Subsection 2.4. We recall that $c_1^2(S) = [K_S] \cdot [K_S]$ and that $c_2(S)$ is the Euler characteristic of the compact manifold S . Using the notations of Proposition 2.14, we obtain $c_1^2(S) = 2([K_{X'}] + [L_1])^2$ and $c_2(S) = 2c_2(X') - \chi(C)$, where $\chi(C)$ is the Euler characteristic of $C := s^{-1}(0)$ (see [5, Section V.22]). Let us verify that $c_1^2(S) = 8 - 14\chi'$ and $c_2(S) = 40 - 52\chi'$. The first computation follows from $[L_1] = 3[\sigma] + 2(1 - 2\chi')[\varphi]$ and from

$$([K_{X'}] + [L_1])^2 = ([\sigma] + 2(1 - 2\chi')[\varphi])^2 = \chi' + 4(1 - 2\chi') = 4 - 7\chi'.$$

To compute $c_2(S)$ we first observe that $c_2(X') = 2\chi'$ since X' is diffeomorphic to $\Sigma' \times \mathbb{P}^1(\mathbb{C})$. Then, $\chi(C) = -[K_{X'}] \cdot [C] - [C] \cdot [C]$ and $[C] = 2[L_1]$ yield

$$\chi(C) = 2[\sigma] \cdot (6[\sigma] + 4(1 - 2\chi')[\varphi]) - (6[\sigma] + 4(1 - 2\chi')[\varphi])^2 = -40 + 56\chi'.$$

Let us make a comment. Every surface of general type S satisfies the Bogomolov-Miyaoka-Yau inequality $c_1^2(S) \leq 3c_2(S)$. A surface of general type which satisfies equality is a smooth quotient of the unit ball in \mathbb{C}^2 by a discrete group of automorphisms, see [5, Sections I.15 and VII.9]. With the values found above, the ratio $c_1^2(S)/c_2(S)$ tends to $7/26$ when χ' tends to $-\infty$, which is far from 3.

Question 2.17. *Does there exist a surface of general type S satisfying $c_1^2(S) = 3c_2(S)$ and containing a Levi-flat hypersurface?*

2.6. Minimals sets modelled on the limit set of Kleinian groups. Let $G \subset \mathrm{PSL}(2, \mathbb{C})$ be a *Kleinian group*, namely a non elementary finitely generated discrete subgroup of $\mathrm{PSL}(2, \mathbb{C})$, see [38, Section 4.6]. The *limit set* of G is the unique G -invariant closed subset $\Lambda \subset \mathbb{P}^1(\mathbb{C})$ such that the G -orbit of every point of Λ is dense in Λ . Let S be a complex surface. We say that a minimal set $M \subset S$ is *transversally modelled on G* if there exist an open covering $(U_j)_j$ of M and foliated charts $(\alpha_j, \beta_j) : U_j \rightarrow \mathbb{D} \times \mathbb{P}^1(\mathbb{C})$ such that $\beta_j(U_j) = \tau_j$ is an open subset of Λ and $\beta_{jj'}$ defined by Equation (1) in Section 1 is the restriction of some element $g \in G$.

Proposition 2.18. *Let G be a Kleinian group whose limit set Λ is different from $\mathbb{P}^1(\mathbb{C})$. There exists a surface of general type S , a non singular holomorphic foliation $\hat{\mathcal{F}}$ on an open set of S and a minimal set M of $\hat{\mathcal{F}}$ which is transversally modelled on G . Moreover $\hat{\mathcal{F}}$ extends to a singular holomorphic foliation of S .*

The remainder of this Section is devoted to the proof of Proposition 2.18. Let $g \geq 2$ be the cardinality of a generating set of G , let Σ be a compact Riemann surface of genus g and let $\rho : \pi_1(\Sigma) \rightarrow G$ be a surjective morphism. Such a morphism exists since $\pi_1(\Sigma)$ surjects to the free non abelian group on g generators. Let $S' := \Sigma \times_{\rho} \mathbb{P}^1(\mathbb{C})$ be the flat $\mathbb{P}^1(\mathbb{C})$ -bundle and $\mathcal{F}' := \mathcal{F}_{\rho}$ be the non singular holomorphic foliation on S' defined in Subsection 2.3. Let $M' := \Sigma \times_{\rho} \Lambda$ be the quotient of the ρ -invariant set $\mathbb{H} \times \Lambda$. It defines a minimal set for \mathcal{F}' since the G -orbit of every element of Λ is dense in Λ . This minimal set $M' \subset S'$ is transversally modelled on G .

Lemma 2.19. *There exist a surface of general type S and a double ramified covering $\Theta : S \rightarrow S'$ whose critical values intersect transversally every leaf of M' .*

Lemma 2.19 allows to finish the proof of Proposition 2.18. Indeed, let \mathcal{F} be the foliation on S defined as the pull-back of \mathcal{F}' by Θ . Let $M := \Theta^{-1}(M')$, this is a closed \mathcal{F} -saturated subset of S . The singular set of \mathcal{F} is the preimage of the set of tangency points between \mathcal{F}' and the critical values of Θ , hence M does not intersect the singular set of \mathcal{F} . Since every leaf of M' intersects the critical values of Θ , every leaf of M is the preimage of a leaf of M' , in particular M is minimal. Finally, M is modelled on G like M' . It remains to establish Lemma 2.19.

Proof of Lemma 2.19. Let $L \in \text{Pic}(S')$ such that $2L$ is very ample and $K_{S'} + L$ is ample (one can take for L a large multiple of a very ample line bundle on S'). Consider the embedding $S' \rightarrow \mathbb{P}^N(\mathbb{C})$ defined by the linear system $|2L|$ (see Equation (5)) and identify S' with its image in $\mathbb{P}^N(\mathbb{C})$. Let $\mathbb{P}^N(\mathbb{C})^*$ be the set of hyperplanes of $\mathbb{P}^N(\mathbb{C})$ endowed with the standard topology. By Bertini Theorem, there exists an open connected subset \mathcal{U} of $\mathbb{P}^N(\mathbb{C})^*$ such that every $H \in \mathcal{U}$ intersects S' along a smooth curve. Lemma 2.19 is a consequence of the following two claims and Proposition 2.14.

Claim 1. Every $H \in \mathcal{U}$ intersects every leaf of S' .

Let L be a leaf of \mathcal{F}' and let \mathcal{U}_L be the subset of $H \in \mathcal{U}$ intersecting L . It suffices to prove $\mathcal{U} = \mathcal{U}_L$. Observe that L is not compact since the Kleinian group G is not elementary. Hence L is not contained in H . We deduce that if H intersects L then any perturbation of H still intersects L , hence \mathcal{U}_L is an open subset of \mathcal{U} . Now let $(H_n)_n$ be a sequence in \mathcal{U}_L which converges to $H \in \mathcal{U}$. Then H contains a point $p \in \bar{L}$. As before, the leaf L_p is not contained in H . By working in a flow box around p (\mathcal{F}' has no singular point), we get that $H \in \mathcal{U}_L$. Hence \mathcal{U}_L is closed in \mathcal{U} and Claim 1 follows.

Claim 2. There exists $H \in \mathcal{U}$ which is transverse to every leaf of M' .

The leaves of \mathcal{F}' are not compact, hence they are not lines of $\mathbb{P}^N(\mathbb{C})$. We deduce that the set Z of points $p \in S'$ such that L_p has an order of tangency (with the line T_p in $\mathbb{P}^N(\mathbb{C})$ tangent to L_p at p) larger than or equal to 2 is a Zariski closed subset of S' . Let $\mathcal{U}' \subset \mathcal{U}$ be the set of hyperplanes which contain one of the line T_p for a point $p \in Z$; this is a Zariski closed subset

of \mathcal{U} . Let $H_0 \in \mathcal{U} \setminus \mathcal{U}'$, by construction the points of tangency between $S' \cap H_0$ and \mathcal{F}' are simple. Hence there exist a neighbourhood \mathcal{V} of H_0 and holomorphic functions $(p_i)_{i=1,\dots,r} : \mathcal{V} \rightarrow S'$ such that for every $H \in \mathcal{V}$ the points of tangency between $S \cap H$ and \mathcal{F} are $\{p_1(H), \dots, p_r(H)\}$. Since M' has empty interior and every p_i is open, $\mathcal{W} := \bigcap_{i=1}^r p_i^{-1}(S' \setminus M')$ is open and dense in \mathcal{V} , and every $H \in \mathcal{W}$ satisfies Claim 2.

This completes the proof of Lemma 2.19. \square

Corollary 2.20. *For every $\alpha \in]0, 2[$ there exist a surface of general type S_α and a singular holomorphic foliation on S_α having an exceptional minimal set whose transverse Hausdorff dimension is less than α .*

Proof. The proof consists in specifying for G in Proposition 2.18 suitable Schottky groups. We refer to [38, Section 4.6] and [43, Section 1.1] for the definition of Schottky groups and to [53, Section 13] for the result we need about Hausdorff dimension. \square

3. GEOMETRY AND ANALYSIS ON LAMINATIONS

In this section, M is either a minimal set or an immersed Levi-flat hypersurface in a complex surface S . Our reference for laminations and minimal sets is the article of Ghys [28]. We recall that there exists a covering of M by open sets $U_j \simeq \mathbb{D} \times \tau_j$ with $\tau_j \subset \mathbb{D}$ which overlap as (see Equation (1) in Section 1)

$$(z_{j'}, u_{j'}) = (\alpha_{jj'}(z_j, u_j), \beta_{jj'}(u_j)).$$

We denote by $p : M \rightarrow S$ the inclusion map if M is a minimal set or a Levi-flat hypersurface. If M is an immersed Levi-flat hypersurface then p is an immersion.

3.1. Calculus. Let $\mathcal{O}_{\mathcal{F}}$ be the sheaf of continuous functions on M which are leafwise holomorphic. By definition, the function $\alpha_{jj'}$ in Equation (1) belongs to $\mathcal{O}_{\mathcal{F}}$. The leafwise derivatives of every element of $\mathcal{O}_{\mathcal{F}}$ depend continuously on the transversal coordinate, by Cauchy's formula. Let $C_{\mathcal{F}}^\infty$ be the sheaf of continuous functions on M which are leafwise smooth and whose leafwise derivatives depend continuously on the transversal coordinate in the uniform topology (the elements of $\mathcal{O}_{\mathcal{F}}$ satisfy this property). We use the following topology on $C_{\mathcal{F}}^\infty$: a sequence c_n converges to c if c_n converges uniformly to c and if the leafwise derivatives of c_n converge uniformly to the leafwise derivatives of c .

Let $A_{\mathcal{F}}^p$ be the space of $C_{\mathcal{F}}^\infty$ -forms of degree p , let $A_{\mathcal{F}}^{p,q}$ be the space of $C_{\mathcal{F}}^\infty$ -forms of bidegree (p, q) . These spaces consist of forms along the leaves of \mathcal{F} . A form $\eta \in A_{\mathcal{F}}^{1,1}$ is *positive* (respectively *non negative*) if η is equal to $\eta_j(z_j, u_j) i dz_j \wedge d\bar{z}_j$ for some positive (respectively non negative) function $\eta_j \in C_{\mathcal{F}}^\infty$ in every chart U_j .

For every $\varphi \in A_{\mathcal{F}}^0$ we define $\partial\varphi \in A_{\mathcal{F}}^{1,0}$ and $\bar{\partial}\varphi \in A_{\mathcal{F}}^{0,1}$ as follows. Locally $\partial\varphi := \partial_{z_j}\varphi \cdot dz_j$ and $\bar{\partial}\varphi := \partial_{\bar{z}_j}\varphi \cdot d\bar{z}_j$, where $z_j = x_j + iy_j$, $\partial_{z_j} := \frac{1}{2}(\partial_{x_j} - i\partial_{y_j})$ and $\partial_{\bar{z}_j} := \frac{1}{2}(\partial_{x_j} + i\partial_{y_j})$. We set $d := \partial + \bar{\partial}$ so that d is a linear map from $A_{\mathcal{F}}^p$ to $A_{\mathcal{F}}^{p+1}$. We shall use the real operator $i\partial\bar{\partial} : A_{\mathcal{F}}^0 \rightarrow A_{\mathcal{F}}^{1,1}$. Observe that $i dz_j \wedge d\bar{z}_j = 2dx_j \wedge dy_j$.

3.2. Line bundles. A (holomorphic) *line bundle* on M is defined by a cocycle $(\gamma_{jj'})$ of non vanishing functions in $\mathcal{O}_{\mathcal{F}}$. We obtain it by identifying $(z_j, u_j, \xi_j) \in U_j \times \mathbb{C}$ with $(z_{j'}, u_{j'}, \xi_{j'}) \in U_{j'} \times \mathbb{C}$ using Equation (1) and $\xi_{j'} = \gamma_{jj'}(z_j, u_j) \cdot \xi_j$. We denote line bundles on M by $L_{\mathcal{F}}$. The group of isomorphism classes of line bundles on M is $H^1(M, \mathcal{O}_{\mathcal{F}}^*)$. A *holomorphic section* s of a line bundle $L_{\mathcal{F}}$ is defined by functions $s_j \in \mathcal{O}_{\mathcal{F}}$ satisfying $s_j = \gamma_{jj'} \cdot s_{j'}$ on $U_j \cap U_{j'}$. If $L \in \text{Pic}(S)$, then p^*L is a line bundle on M . We shall use the following intrinsic line bundles on M .

- **Tangent and Canonical line bundles $T_{\mathcal{F}}$ and $K_{\mathcal{F}}$:** the tangent line bundle $T_{\mathcal{F}}$ is the line bundle whose sections are holomorphic vector fields tangent to \mathcal{F} . It is defined by setting $\gamma_{jj'} := \partial_{z_j} \alpha_{jj'}$. The canonical line bundle $K_{\mathcal{F}}$ is defined by $\gamma_{jj'} := (\partial_{z_j} \alpha_{jj'})^{-1}$.
- **Normal line bundle $N_{\mathcal{F}}$:** for a (immersed or not) Levi-flat hypersurface $M \subset S$, the normal line bundle $N_{\mathcal{F}}$ is defined by setting $\gamma_{jj'} := d_{u_j} \beta_{jj'}$. This function is real valued and leafwise constant. For a minimal set $M \subset S$, the normal line bundle $N_{\mathcal{F}}$ is defined similarly by taking the derivative of $\hat{\beta}_{jj'}$, defined as the transition mapping of the ambient non-singular foliation $\hat{\mathcal{F}}$, see Definition 1.2. Here $\gamma_{jj'}$ is complex valued and leafwise constant.

If $\mathcal{L} \in \mathcal{F}$ is a compact leaf we respectively denote by $K_{\mathcal{L}}$ and $N_{\mathcal{L}}$ the canonical line bundle of the compact Riemann surface \mathcal{L} and the restriction of $N_{\mathcal{F}}$ to \mathcal{L} . These line bundles are linked with the canonical line bundle of S by the leafwise adjunction formula stated in Proposition 3.1 below. We refer to [7, Section 1] and [17, Section 3] for this formula.

Proposition 3.1. *Let S be a complex surface. Let M be a minimal set or an immersed Levi-flat hypersurface in S . Then $K_{\mathcal{F}} = p^*K_S + N_{\mathcal{F}}$. If \mathcal{L} is a compact leaf of \mathcal{F} then $K_{\mathcal{L}} = p^*K_S + N_{\mathcal{L}}$.*

3.3. Metric and curvature. Let $L_{\mathcal{F}}$ be a line bundle on M . A *continuous metric* m on $L_{\mathcal{F}}$ is a Riemannian metric on $L_{\mathcal{F}}$ which is locally defined by $m_j(z_j, u_j)|\xi_j|$, where $m_j \in C_{\mathcal{F}}^{\infty}$ is positive. By definition, if $L_{\mathcal{F}}$ is defined by the cocycle $(\gamma_{jj'})$ then we have $m_{j'}(z_{j'}, u_{j'}) = m_j(z_j, u_j)|\gamma_{jj'}(z_j, u_j)|$, where $(z_{j'}, u_{j'})$ depends on (z_j, u_j) using Equation (1). Continuous metrics always exist, they can be constructed by using partitions of unity. If m' is another metric on $L_{\mathcal{F}}$, then $m' = me^{\varphi}$ for some function $\varphi \in A_{\mathcal{F}}^0$. The *Chern curvature* of a continuous metric m is the form $\omega_m \in \mathcal{A}_{\mathcal{F}}^{1,1}$ locally defined by

$$(6) \quad \omega_{m_j} := \frac{1}{2i\pi} \partial \bar{\partial} \log m_j(z_j, u_j)^2.$$

In particular, if m_j is leafwise superharmonic, then ω_m is non negative. If $m' = me^{\varphi}$ is another metric, then $\omega_{m'} = \omega_m + \frac{1}{i\pi} \partial \bar{\partial} \varphi$. Given a continuous metric m on $T_{\mathcal{F}}$, we denote by vol_m its leafwise volume form and by Δ_m its leafwise Laplace-Beltrami operator. We note that $2i\partial \bar{\partial}$ is equal to $\Delta_m \cdot vol_m$ and to $\Delta \cdot dx_j \wedge dy_j$, where Δ stands for the usual euclidean laplacian.

3.4. Poincaré metric. We consider the conformal metric on \mathbb{D}

$$m_{\mathbb{D}} = 2 \frac{|dz|}{1 - |z|^2}.$$

It is complete of Gaussian curvature -1 (the Gaussian curvature of a conformal metric $\xi|dz|$ is equal to $-\xi^{-2}\Delta \log \xi$). We denote by $d_{\mathbb{D}}$ the induced metric on \mathbb{D} . Being invariant by every biholomorphism of the disc, $m_{\mathbb{D}}$ induces an infinitesimal metric on any hyperbolic Riemann surface. In particular, if \mathcal{F} is hyperbolic, one can endow the tangent bundle $T_{\mathcal{F}}$ with this metric. We call it the *leafwise Poincaré metric* and denote it by $m_{\mathcal{P}}$. It satisfies the following compactness properties, see [13] and [63]. See also [11, Section 7.1].

Theorem 3.2 (Candel, Verjovsky). *Let S be a complex surface and M be a minimal set or an immersed Levi-flat hypersurface in S . Assume that \mathcal{F} is hyperbolic. Then the leafwise Poincaré metric $m_{\mathcal{P}}$ is a continuous metric on $T_{\mathcal{F}}$ in the sense of Subsection 3.3. In particular, if $(x_n)_n$ converges toward x in M , then there exist holomorphic coverings of the leaves $\sigma_n : \mathbb{D} \rightarrow \mathcal{L}_{x_n}$ satisfying $\sigma_n(0) = x_n$ converging to a holomorphic covering $\sigma : \mathbb{D} \rightarrow \mathcal{L}_x$ satisfying $\sigma(0) = x$.*

3.5. Harmonic currents and foliation cycles. A harmonic current on M is a continuous linear form $T : A_{\mathcal{F}}^{1,1} \rightarrow \mathbb{R}$ such that

- (i) $T(\partial\bar{\partial}\varphi) = 0$ for every $\varphi \in A_{\mathcal{F}}^0$,
- (ii) $T(\eta) > 0$ for every positive form $\eta \in A_{\mathcal{F}}^{1,1}$.

The existence of harmonic currents can be proved using Hahn-Banach theorem, see [28, Proposition 3.5]. A harmonic current has the following local expression in $U_j \simeq \mathbb{D} \times \tau_j$. There exist a positive function H_j which is leafwise positive and harmonic, and a positive measure ν_j on τ_j such that for every $\eta = \eta_j dx_j \wedge dy_j \in A_{\mathcal{F}}^{1,1}$:

$$(7) \quad T(\eta) = \int_{\tau_j} \left[\int_{\mathbb{D} \times \{u_j\}} H_j \cdot \eta_j dx_j dy_j \right] d\nu_j(u_j).$$

A particular class of harmonic currents is given by foliation cycles. A *foliation cycle* is a continuous linear form $T : A_{\mathcal{F}}^{1,1} \rightarrow \mathbb{R}$ satisfying $T(d\varphi) = 0$ for every $\varphi \in A_{\mathcal{F}}^1$ and condition (ii). Sullivan established a correspondance between foliation cycles and transverse invariant measures, see [60] and [14, Chapter 10]. We note that a foliation cycle is a harmonic current such that H_j is leafwise constant and positive, the measures ν_j then define a transverse invariant measure. The existence of a foliation cycle is a very rare phenomenon. Two special cases imply their existence. The first case is when \mathcal{F} has a compact leaf \mathcal{L} , then the current of integration on this leaf is a foliation cycle. We shall also denote by $\{\mathcal{L}\}$ the current of integration on \mathcal{L} . The second case is when \mathcal{F} has a parabolic leaf, then the classical Ahlfors argument yields a foliation cycle, see [9, Section 1].

3.6. Intersections. Let T be a harmonic current and let $L_{\mathcal{F}}$ be a line bundle on M . We define the intersection

$$T \cdot L_{\mathcal{F}} := T(\omega_m),$$

where m is any continuous metric on $L_{\mathcal{F}}$ and $\omega_m \in A_{\mathcal{F}}^{1,1}$ is the curvature form of m . This definition does not depend on m . We shall use the following

Lemma 3.3. *Let S be a complex surface. Let M be a minimal set or an immersed Levi-flat hypersurface in S . Let T be a harmonic current on M and let $L \in \text{Pic}(S)$. Let \mathcal{C} be a finite set of curves of S . If the following conditions are satisfied*

- (1) L has a riemannian metric whose curvature is positive on $S \setminus \mathcal{C}$,
- (2) for every plaque U of \mathcal{F} , $p(U)$ is not included in a curve of \mathcal{C} ,

then $T \cdot p^*L > 0$.

Proof. Let ω be the curvature form of a riemannian metric on L which is positive outside \mathcal{C} . We want to show that $T(p^*\omega) > 0$. It suffices to work in a flow box U_j . If $p^*\omega = c_j dx_j \wedge dy_j$ in this flow box, then $c_j \in C_{\mathcal{F}}^{\infty}$ is non negative and vanishes at a finite number of points in every plaque. Since H_j is positive on every plaque, it follows from Equation (7) that

$$T(p^*\omega) = \int_{\tau_j} \left[\int_{\mathbb{D} \times \{u_j\}} H_j \cdot c_j dx_j dy_j \right] d\nu_j(u_j) > 0.$$

This completes the proof of the Lemma. □

Observe that if $\mathcal{L} \in \mathcal{F}$ is a compact leaf, then $-\{\mathcal{L}\} \cdot K_{\mathcal{L}}$ is equal to the Euler characteristic of the compact Riemann surface \mathcal{L} (Gauss-Bonnet formula). We extend this definition by defining the *Euler-Poincaré characteristic* of a harmonic current T as $-T \cdot K_{\mathcal{F}}$. We refer to [13, Section 4] for the following result.

Proposition 3.4 (Candel). *Let S be a complex surface and M be a minimal set or an immersed Levi-flat hypersurface in S . If $\mathcal{L} \in \mathcal{F}$ is a parabolic leaf, then there exists an Ahlfors current T associated to \mathcal{L} such that $T \cdot K_{\mathcal{F}} = 0$.*

The following Lemma is related to [12, Theorem 2].

Lemma 3.5. *Let S be a complex surface and M be a minimal set or an immersed Levi-flat hypersurface in S .*

- (1) $T \cdot N_{\mathcal{F}} = 0$ for every foliation cycle T .
- (2) $\{\mathcal{L}\} \cdot N_{\mathcal{L}} = 0$ for every compact leaf \mathcal{L} .

Proof. If T is a foliation cycle and $L_{\mathcal{F}}$ is any line bundle on M , then $T \cdot L_{\mathcal{F}} = T(\omega)$ where ω is the curvature of any smooth connexion on $L_{\mathcal{F}}$. Indeed, the curvatures of two connexions differ by an exact 2-form. The first item then follows by taking for ω the curvature form of the Bott connexion on $N_{\mathcal{F}}$, which is leafwise flat (see [15, Chapter 6]). The second item is proved similarly by considering the Bott connexion on $N_{\mathcal{L}}$. □

The following proposition is proved in [28, Lemma 4.5] when T is a foliation cycle, we extend the result to harmonic currents.

Proposition 3.6. *Let S be a complex surface and M be a minimal set or an immersed Levi-flat hypersurface in S . Let T be a harmonic current and $L_{\mathcal{F}}$ be a line bundle on M . Assume that $L_{\mathcal{F}}$ has a holomorphic section s which does not vanish identically on any leaf. Then $L_{\mathcal{F}}$ has a continuous metric with non negative curvature and consequently $T \cdot L_{\mathcal{F}} \geq 0$.*

Proof. We cover M by finitely many flow boxes $U_j \simeq \mathbb{D} \times \tau_j$ such that the line bundle $L_{\mathcal{F}}$ is trivial on every U_j . Let $D := \{s = 0\}$. We will construct a function $\phi : M \setminus D \rightarrow \mathbb{R}$ which satisfies the following conditions:

- (i) ϕ belongs to $C_{\mathcal{F}}^{\infty}$,
- (ii) ϕ is leafwise superharmonic,
- (iii) on every U_j , if we write $s = f_j \tilde{s}_j$ for some non vanishing holomorphic section \tilde{s}_j of $L_{\mathcal{F}}$, then ϕ differs from $\log |f_j|$ by some $\chi_j \in C_{\mathcal{F}}^{\infty}(U_j)$.

Then $|s|_* := e^{\phi}$ defines a metric on $L_{\mathcal{F}}$ restricted to $M \setminus D$. It is smooth by Property (i), it has leafwise non negative curvature by Property (ii). Property (iii) allows to extend it as a smooth metric $|\cdot|$ on $L_{\mathcal{F}}$, by setting $|\tilde{s}_j| = e^{\chi_j}$ on every U_j . Its leafwise curvature is still non negative since $\chi_j = \phi - \log |f_j|$ is leafwise smooth and superharmonic. Proposition 3.6 then follows.

The remainder is devoted to the construction of ϕ . Let $0 < \rho < 1$ and let $\{\psi_j\}_j$ be a finite partition of unity of D such that every ψ_j is supported by $U_j \simeq \mathbb{D} \times \tau_j$ and such that $D \cap U_j \subset \mathbb{D}_{\rho} \times \tau_j$ for every j . The functions $\psi_j : D \rightarrow [0, 1]$ are continuous and satisfies $\sum_j \psi_j = 1$ on D .

For every j we construct a function $\phi_j : M \setminus D \rightarrow \mathbb{R}$ which satisfies (i), (ii) and (iii'): it has a leafwise logarithmic pole at every $x \in D$ with residue equal to $\psi_j(x)$ times the leafwise order of annulation of s at x . For that purpose observe that every plaque $\mathbb{D} \times \{u_j\}$ of U_j contains the same number d of points of D (we repeat them according to the order of annulation of s along the plaques). For every $u_j \in \tau_j$, let $x_k(u_j) := (w_k(u_j), u_j)$ be these points for $k = 1, \dots, d$. We define ϕ_j on $U_j \setminus D$ by

$$\phi_j(z_j, u_j) := F \left(\sum_{k=1}^d \psi_j(x_k(u_j)) \log C |z_j - w_k(u_j)| \right),$$

where $C := 2/(1 - \rho)$ and $F : \mathbb{R} \rightarrow (-\infty, 0]$ is a smooth concave function which is equal to $F(a) = a + 1$ for $a \leq -2$ and $F(a) = 0$ for $a \geq 0$. The function ϕ_j is smooth and leafwise superharmonic on $U_j \setminus D$. It vanishes outside $\mathbb{D}_{(1+\rho)/2} \times \tau_j$, since $|w_k(u_j)| \leq \rho$ for every k and u_j . It also vanishes outside $\mathbb{D} \times p_j(\text{supp } \psi_j)$, where $p_j : U_j \rightarrow \tau_j$ denotes the projection. Hence ϕ_j extends to a smooth function on $M \setminus D$ by setting 0 outside U_j , this extension satisfies (i), (ii), (iii'). We finally set $\phi := \sum \phi_i : M \setminus D \rightarrow \mathbb{R}$. \square

4. HYPERBOLICITY

We prove that the immersed Levi-flat hypersurfaces and the minimal sets in surfaces of general type have a hyperbolic Cauchy-Riemann foliation. In particular we can endow them with the leafwise Poincaré metric introduced in Subsection 3.4. We deduce topological obstructions for Levi-flat hypersurfaces.

4.1. \mathcal{F} is hyperbolic: proof of Proposition 1.3. We recall classical facts about minimal surfaces and surfaces of general type, we refer to [5, Section VII] and [23, Chapter 10]. Let S be a complex surface. A $(-k)$ -curve in S is a curve biholomorphic to $\mathbb{P}^1(\mathbb{C})$ with self-intersection $-k$. A surface S is *minimal* if it does not contain any (-1) -curve. Every non minimal surface can be obtained from a minimal one by performing finitely many blowups.

A minimal surface of general type has no $(-k)$ -curves for $k \geq 3$ and has finitely many (-2) -curves. We denote by \mathcal{C}_S the union of these curves. The pluricanonical map $S \rightarrow \mathbb{P}^{N_n}$ associated to the line bundle $nK_S := K_S^{\otimes n}$ defined in Equation (5) contracts (for n large enough) every curve of \mathcal{C}_S to a singular point and is biholomorphic elsewhere. In particular, K_S supports a riemannian metric whose curvature form is non negative on S and positive on $S \setminus \mathcal{C}_S$ (this curvature form is the pullback of the Fubini-Study metric on \mathbb{P}^{N_n} by the pluricanonical map). The first assumption of Lemma 3.3 is thus satisfied for $K_S \in \text{Pic}(S)$ and the set \mathcal{C}_S of (-2) -curves we have introduced. The second assumption follows from:

Lemma 4.1. *Let S be a complex compact surface and let $C \subset S$ be a smooth compact analytic curve with non-zero self-intersection. Let (M, p) be a minimal set or an immersed Levi-flat hypersurface in S . Then for every plaque U of \mathcal{F} the image of U by the immersion p is not included in C .*

Proof. Assume that there exists a plaque U of \mathcal{F} such that $p(U) \subset C$. Let \mathcal{L}_U be the leaf which contains U . Then $p(\mathcal{L}_U) \subset C$ by analytic continuation. We actually have $p(\mathcal{L}_U) = C$. Indeed, $p(\mathcal{L}_U)$ is open and closed in C since M is covered by a finite union of flow boxes. Now the compactness of M implies that $p : \mathcal{L}_U \rightarrow C$ is a covering, hence \mathcal{L}_U is compact. Therefore \mathcal{L}_U has zero self-intersection by Lemma 3.5(2). This contradicts the covering property of $p : \mathcal{L}_U \rightarrow C$ and the fact that C has non-zero self-intersection. \square

This result allows to apply Lemma 3.3 with $L := K_S \in \text{Pic}(S)$ and $\mathcal{C} := \mathcal{C}_S$. Now let us prove Proposition 1.3 by assuming that the surface S is minimal. Assume that \mathcal{L} is a leaf of \mathcal{F} which is isomorphic to $\mathbb{P}^1(\mathbb{C})$. Adjunction formula (Proposition 3.1) then gives

$$-2 = \{\mathcal{L}\} \cdot K_{\mathcal{L}} = \{\mathcal{L}\} \cdot p^*K_S + \{\mathcal{L}\} \cdot N_{\mathcal{L}}.$$

The first term of the right hand side is > 0 by Lemma 3.3, and the second one vanishes by Lemma 3.5(2), this is a contradiction. Assume now that there exists a parabolic leaf \mathcal{L} and let T be an Ahlfors current such that $T \cdot K_{\mathcal{F}} = 0$ (Proposition 3.4). Using the leafwise adjunction formula (Proposition 3.1) we obtain

$$T \cdot K_{\mathcal{F}} = T \cdot p^*K_S + T \cdot N_{\mathcal{F}}.$$

The first term of the right hand side is > 0 by Lemma 3.3, and the second term vanishes by Lemma 3.5(1). This yields a contradiction.

Now we focus on the non minimal case. We consider a blow-up $\zeta : S' \rightarrow S$ at a point P of a minimal surface of general type, we have $K_{S'} = \zeta^*K_S + \mathcal{O}_{S'}(E)$ where E is the exceptional divisor of the blow-up (see [23, Section 3]). As before assume that there exists a parabolic leaf \mathcal{L} in M and let T be an Ahlfors current such that $T \cdot K_{\mathcal{F}} = 0$. By taking the pull-back of the preceding equation by p and then intersecting with T , we obtain

$$(8) \quad T \cdot p^*K_{S'} = T \cdot p^*(\zeta^*K_S) + T \cdot p^*\mathcal{O}_{S'}(E).$$

Let \mathcal{C}^* be the union of the proper transforms of the elements of \mathcal{C}_S . We want to apply Lemma 3.3. The line bundle ζ^*K_S has a riemannian metric whose curvature is non negative on S' and positive outside $E \cup \mathcal{C}^*$. Moreover, $p(U)$ is not included in $E \cup \mathcal{C}^*$ for every plaque U of \mathcal{F} . This follows from Lemma

4.1, since $[E]^2 \leq -1$ and since the blow-up ζ does not increase the self-intersection of the elements of \mathcal{C}^* . Thus we can apply Lemma 3.3 to prove that the first term of the right side of Equation (8) is > 0 . The second term is ≥ 0 by Proposition 3.6. This implies that $T \cdot p^*K_{S'} > 0$. We conclude as before by using the leafwise adjunction formula:

$$T \cdot K_{\mathcal{F}} = T \cdot p^*K_S + T \cdot N_{\mathcal{F}}.$$

The left hand side vanishes and the right hand side is > 0 , contradiction.

4.2. \mathbb{S}^3 , $\mathbb{S}^2 \times \mathbb{R}$, \mathbb{R}^3 and Nil do not appear in surfaces of general type. The arguments of the following proposition are due to Plante, see [55, Lemma 7.2]. A similar result was obtained for (hypothetical) Levi-flat hypersurfaces in $\mathbb{P}^2(\mathbb{C})$ in [34]. The proof relies on the volume growth of the leaves. Since M is a compact manifold, every leaf has a well defined bi-Lipschitz class of metrics (as well as their universal cover) defined by the restriction of continuous Riemannian metrics on M . When the foliation is hyperbolic, this class can be represented by the leafwise Poincaré metric, see Theorem 3.2. We refer to [14, Chapter 12] for an introduction concerning the growth of leaves.

Proposition 4.2. *Let S be a surface of general type and let M be an immersed Levi-flat hypersurface of class C^2 in S . Then the fundamental group of M has exponential growth. In particular M does not carry the geometries \mathbb{S}^3 , $\mathbb{S}^2 \times \mathbb{R}$, \mathbb{R}^3 nor Nil.*

Proof. We can assume that M is orientable and that \mathcal{F} is transversally orientable by using Lemma 2.2. By Proposition 1.3, \mathcal{F} is hyperbolic. By Theorem 2.1 (Novikov) and Lemma 2.4, the natural map $\pi_1(\mathcal{L}) \rightarrow \pi_1(M)$ is injective for every leaf $\mathcal{L} \in \mathcal{F}$. Hence the pull-back foliation $\tilde{\mathcal{F}}$ on the universal cover \tilde{M} has simply connected leaves. We obtain that the leaves of $\tilde{\mathcal{F}}$ have exponential growth by using the leafwise Poincaré metric. Assume that the fundamental group of M has subexponential growth. Let $(U_j)_j$ be a finite covering of M by foliated charts. Let \tilde{U}_j be a lift of U_j in \tilde{M} . Observe that $(g\tilde{U}_j)_{j,g \in \pi_1(M)}$ is a covering of \tilde{M} by foliated charts of $\tilde{\mathcal{F}}$. Moreover, because $\pi_1(M)$ has subexponential growth, the number of charts of this covering in $B_{\tilde{M}}(x, R)$ is subexponential. Let us fix $\mathcal{L} \in \mathcal{F}$. By the pigeon hole principle, there exists $g\tilde{U}_j$ whose intersection with \mathcal{L} contains at least two different plaques. Hence there exists a simple closed loop in \tilde{M} which is transverse to $\tilde{\mathcal{F}}$. Its projection in M is a closed and homotopically trivial loop transverse to \mathcal{F} . This contradicts Theorem 2.1 combined with Lemma 2.4. \square

4.3. **Green-Griffiths conjecture.** The hyperbolicity of the foliation is related to the following question.

Conjecture 4.3. *(Green-Griffiths) Let S be a surface of general type. There exists a proper subvariety $Y \subset S$ such that every entire curve $\sigma : \mathbb{C} \rightarrow S$ satisfies $\sigma(\mathbb{C}) \subset Y$.*

This problem was solved by McQuillan [48] for surfaces of general type satisfying $c_1^2(S) > c_2(S)$. He proved that every non-degenerate entire curve

$\sigma : \mathbb{C} \rightarrow S$ is tangent to a singular holomorphic foliation on (a finite cover of) S . A contradiction is deduced from positivity properties of the tangent bundle of the foliation. Brunella provided an alternative proof in [9] by using the normal bundle of the foliation. An important difficulty in these works is that $\sigma(\mathbb{C})$ can contain a singular point of the foliation. In our non-singular context the proof is simpler because we directly use adjunction formula. We refer to the survey [20] for recent results concerning Green-Griffiths conjecture.

5. HARMONIC MEASURES AND LYAPUNOV EXPONENTS

5.1. Heat kernel and Brownian motion. Let M be a minimal set or an immersed Levi-flat hypersurface in a complex surface S . Let m be continuous metric on $T_{\mathcal{F}}$. Since M is compact, every leaf $\mathcal{L} \in \mathcal{F}$ has bounded geometry. The leafwise heat kernel $p_m(t, x, y)$ is therefore well-defined, this is the fundamental solution of the heat equation

$$\frac{\partial}{\partial t} = \Delta_m \quad , \quad \lim_{t \rightarrow 0} p_m(t, x, y) = \delta_x(y)$$

where δ_x is the Dirac mass at x , the limit being in the sense of distributions, see [15, appendix B] and [16, Chapter VI]. We consider the Brownian motion on \mathcal{L}_x with transition probability $p_m(t, x, y)$. This is a diffusion process with continuous sample paths, it is defined for every $t \in \mathbb{R}^+$. Let $\Gamma_x := \{\gamma : \mathbb{R}^+ \rightarrow \mathcal{L}_x, \gamma \text{ continuous}, \gamma(0) = x\}$ and let W_x be the Wiener probability measure on Γ_x given by Brownian motion.

5.2. Harmonic measure and Garnett's theory. We refer to [25]. A *harmonic measure* is a probability measure μ on M satisfying

$$\forall \varphi \in A_{\mathcal{F}}^0, \quad \int_M \Delta_m \varphi d\mu = 0.$$

The set of harmonic measures is convex, its extremal points are *ergodic*.

We recall that the continuous metric m on $T_{\mathcal{F}}$ is a conformal metric on the leaves locally defined by $m_j(z_j, u_j)|\xi_j|$, see Subsection 3.3. Let T be a harmonic current on M locally defined by the leafwise harmonic function H_j and by the transverse measure ν_j , see Subsection 3.5. Let $T \wedge \text{vol}_m$ denote the positive measure on M locally defined by

$$T \wedge \text{vol}_m(\varphi) = \int_{\tau_j} \left[\int_{\mathbb{D} \times \{u_j\}} H_j \cdot m_j \cdot \varphi dx_j dy_j \right] d\nu_j(u_j)$$

for every continuous function $\varphi : M \rightarrow \mathbb{R}$. This is a harmonic measure on M up to a multiplicative constant. We shall use the following remark.

Remark 5.1. *Let m be a continuous metric on $T_{\mathcal{F}}$. Let \mathcal{M} be the convex set of harmonic measures on M and \mathcal{T} be the convex set of harmonic currents on M such that $T \wedge \text{vol}_m(M) = 1$. The relation $\mu = T \wedge \text{vol}_m$ defines a bijection between \mathcal{M} and \mathcal{T} . We refer to [28, Section 3] for more details.*

Harmonic measures allow to introduce ergodic theory and Lyapunov exponents. Let us specify the associated dynamical system $(\Gamma, (a_t)_{t \geq 0}, \bar{\mu})$. We set $\Gamma := \cup_{x \in M} \Gamma_x$ and $(a_t)_{t \geq 0}$ as the shift semi-group acting on Γ by

$$a_t(\gamma)(t') := \gamma(t + t').$$

The invariant probability measure $\bar{\mu}$ on Γ is defined by

$$\bar{\mu} := \int_M W_x d\mu(x).$$

If μ is an ergodic harmonic measure, then $(\Gamma, (a_t)_{t \geq 0}, \bar{\mu})$ is also ergodic by Garnett's random ergodic theorem, see [25, Theorem 2].

5.3. Lyapunov exponent. If M is a minimal set (resp. an immersed Levi-flat hypersurface), we denote by $(\tau_x)_{x \in M}$ a family of germs of topological discs (resp. intervals) which are transverse to $\hat{\mathcal{F}}$ (resp. \mathcal{F}) and centered at x . For any $x \in M$ and $\gamma \in \Gamma_x$, let

$$h_{\gamma,t} : \tau_{\gamma(0)=x} \rightarrow \tau_{\gamma(t)}$$

be the germ of holonomy map over the continuous path $\gamma : [0, t] \rightarrow \mathcal{L}_x$. Let $|\cdot|$ be a continuous metric on $N_{\mathcal{F}}$. Let μ an ergodic harmonic measure on M . The *Lyapunov exponent* of μ is defined by

$$(9) \quad \lambda(\mu) := \int_M \int_{\Gamma_x} \log |h'_{\gamma,1}(x)| dW_x(\gamma) d\mu(x).$$

The compactness of M ensures that $\lambda(\mu)$ is finite and is independent of the metric $|\cdot|$. Applying Birkhoff's ergodic theorem to the system $(\Gamma, (a_t)_{t \geq 0}, \bar{\mu})$ and to the function $H(\gamma) := \log |h'_{\gamma,1}(\gamma(0))|$ one obtains

$$\lambda(\mu) = \lim_{t \rightarrow +\infty} \frac{1}{t} \log |h'_{\gamma,t}(x)|$$

for μ -almost every $x \in M$ and W_x -almost every $\gamma \in \Gamma_x$. We shall use the following theorem, see [19, Theorem 1.1] and its proof.

Theorem 5.2 (Deroin-Kleptsyn). *Let M be a minimal set or an immersed Levi-flat hypersurface in a complex surface S . Let m be a continuous metric on $T_{\mathcal{F}}$. Assume that \mathcal{F} does not have any transverse invariant measure. Then M supports an ergodic harmonic measure μ whose Lyapunov exponent satisfies $\lambda(\mu) < 0$. Moreover, there exist a non-empty open subset $V_{\mu} \subset M$ and Borel subsets $\Lambda_x \subset \Gamma_x$ of positive W_x -measure satisfying the following properties for every $x \in V_{\mu}$. Given $\epsilon > 0$ and $\gamma \in \Lambda_x$, there exist $r_{\gamma}(\epsilon) > 0$ and $t_{\gamma}(\epsilon) \geq 0$ such that for every $r \leq r_{\gamma}(\epsilon)$ and $t \geq t_{\gamma}(\epsilon)$:*

- (1) $h_{\gamma,t}$ is well defined from $\mathbb{D}_x(r) \subset \tau_x$ to $\tau_{\gamma(t)}$,
- (2) $e^{t(\lambda(\mu)-\epsilon)} \leq |h'_{\gamma,t}(x)| \leq e^{t(\lambda(\mu)+\epsilon)}$.

5.4. Cohomological expression of the Lyapunov exponent. We refer to [17, appendix] for the following formula.

Proposition 5.3. *Let S be a complex surface. Let M be a minimal set or an immersed Levi-flat hypersurface in S . Let m be a continuous metric on $T_{\mathcal{F}}$, μ be an ergodic harmonic measure and T be the normalized harmonic current satisfying $\mu = T \wedge \text{vol}_m$. Then the Lyapunov exponent of μ is*

$$\lambda(\mu) = -2\pi T \cdot N_{\mathcal{F}}.$$

Sketch of proof. Let $|\cdot|$ be a metric on the normal bundle $N_{\mathcal{F}}$. Using the Markov property of the heat kernel, and the following cocycle relation for every $\gamma \in \Gamma_x$ and $(t_1, t_2) \in \mathbb{R}^+ \times \mathbb{R}^+$

$$\log |h'_{\gamma,t_1+t_2}(x)| = \log |h'_{a_{t_1}(\gamma),t_2}(\gamma(t_1))| + \log |h'_{\gamma,t_1}(x)|,$$

we deduce:

$$\forall t > 0, \quad t\lambda(\mu) = \int_M \int_{\Gamma_x} \log |h'_{\gamma,t}(x)| dW_x(\gamma) d\mu(x).$$

Now we work on the universal covers $\tilde{\mathcal{L}}$ of the leaves, we keep the same notations for the pull-back of x, y, m and p_m to $\tilde{\mathcal{L}}$. Let $h_{x,y}$ be the holonomy map between x and y , it is well-defined without specifying γ since $\tilde{\mathcal{L}}$ is simply connected. We obtain

$$\forall t > 0, \quad t\lambda(\mu) = \int_M \int_{\tilde{\mathcal{L}}_x} \log |h'_{x,y}(x)| p_m(t, x, y) d\text{vol}_m(y) d\mu(x).$$

Taking the derivatives with respect to t and the limit when t tends to zero, we get (recall that $|\cdot|$ stands for a metric we have fixed on $N_{\mathcal{F}}$)

$$\lambda(\mu) = \int_M \Delta_m \log |\cdot| d\mu.$$

The conclusion follows from $\mu = T \wedge \text{vol}_m$, $\Delta_m \cdot \text{vol}_m = 2i\partial\bar{\partial}$, and the definition of the curvature $\frac{1}{2i\pi}\partial\bar{\partial}\log|\cdot|^2$ (see Subsection 3.3). \square

5.5. Pinching: proof of Proposition 1.4. Let T be the harmonic current such that $\mu = T \wedge \text{vol}_m$. Since the Gaussian curvature of the Poincaré metric is equal to -1 , we have $T \cdot T_{\mathcal{F}} = -\frac{1}{2\pi}$. Propositions 3.1 (adjunction formula) and 5.3 yield

$$\lambda(\mu) = 2\pi T \cdot T_{\mathcal{F}} + 2\pi T \cdot p^*K_S = -1 + 2\pi T \cdot p^*K_S.$$

It remains to prove that $T \cdot p^*K_S$ is positive. We use similar arguments as for the proof of the hyperbolicity of \mathcal{F} (see Subsection 4.1). If S is a minimal surface of general type, then K_S has a metric of positive curvature outside the curves of \mathcal{C}_S , and these curves are not leaves of \mathcal{F} , see Lemma 4.1. Lemma 3.3 then implies $T \cdot p^*K_S > 0$. If S is not a minimal surface, we follow the arguments of Subsection 4.1 by replacing foliated cycles by harmonic currents.

6. ANOSOV LEVI-FLAT HYPERSURFACES

In this section we prove Theorem 1.5 and Corollary 1.6.

6.1. Stretching.

Definition 6.1. *Let M be an immersed Levi-flat hypersurface in a complex surface S . We say that M has a point at infinity if \mathcal{F} is hyperbolic and if it supports a continuous flow, that is a continuous mapping $\psi : M \times \mathbb{R} \rightarrow M$ satisfying the following properties.*

- (1) $\psi(x, 0) = x$ and $\psi(x, t) \in \mathcal{L}_x$ for every $x \in M$ and $t \in \mathbb{R}$.
- (2) For every leaf \mathcal{L} and every holomorphic covering $\sigma : \mathbb{D} \rightarrow \mathcal{L}$ there exists $\delta \in \partial\mathbb{D}$ such that: for every $x \in \mathcal{L}$, every σ -lift of $t \mapsto \psi(x, t)$ to \mathbb{D} is a geodesic for the Poincaré metric parametrized by arc length which tends to $\delta \in \partial\mathbb{D}$ when t tends to $-\infty$.

We recall (see Subsection 1.2) that a Levi-flat hypersurface M is Anosov if there exists a compact 3-manifold N endowed with an Anosov flow $A = \{A^t\}_{t \in \mathbb{R}}$ such that \mathcal{F} is topologically conjugate to the weak unstable foliation of the flow A .

Lemma 6.2 (Stretching). *Let M be an immersed Levi-flat hypersurface in a complex surface S . If M is Anosov then M has a point at infinity.*

In the sequel we denote by \mathcal{G} the weak unstable foliation of the Anosov flow on N and by $\Phi : M \rightarrow N$ a homeomorphism sending the leaves of \mathcal{F} to the leaves of \mathcal{G} . Let us prove Lemma 6.2. We have to show that \mathcal{F} is hyperbolic and that \mathcal{F} supports a continuous flow in the sense of Definition 6.1. To show that \mathcal{F} is hyperbolic, it suffices to prove that \mathcal{G} has no transverse invariant measure, by Ahlfors argument. For that purpose we first observe that a leafwise volume form on \mathcal{G} is uniformly and exponentially dilated by the Anosov flow. Assume now that there exists a transverse invariant measure ρ on \mathcal{G} . It is invariant by the Anosov flow, since this flow is along the leaves. Hence by taking the product of ρ with a leafwise volume form we produce a finite measure on N which is uniformly dilated by A . This contradicts the preservation of the volume of N . Now let us prove the existence of a continuous flow ψ on M , namely the existence of a continuous mapping $\psi : M \times \mathbb{R} \rightarrow M$ satisfying (1) and (2) of Definition 6.1. Using the leafwise homeomorphism $\Phi : M \rightarrow N$ we set

$$\forall x \in M, \forall t \in \mathbb{R}, B^t(x) := \Phi^{-1} A^t \Phi(x).$$

The following proposition is proved in Subsection 9.1.

Proposition 6.3. *Let $\mathcal{L} \in \mathcal{F}$ and let $\sigma : \mathbb{D} \rightarrow \mathcal{L}$ be a holomorphic covering. For every $x \in \mathcal{L}$ and $\tilde{x} \in \sigma^{-1}(x)$, we denote by $B_{\sigma, \tilde{x}} : \mathbb{R} \rightarrow \mathbb{D}$ the σ -lift of $t \mapsto B^t(x)$ satisfying $B_{\sigma, \tilde{x}}(0) = \tilde{x}$.*

- (1) *There exist $a > 1$ and $b > 0$ independant of $\mathcal{L}, \sigma, x, \tilde{x}$ such that for every $(t, t') \in \mathbb{R}^2$:*

$$a^{-1}|t - t'| - b \leq d_{\mathbb{D}}(B_{\sigma, \tilde{x}}(t), B_{\sigma, \tilde{x}}(t')) \leq a|t - t'| + b.$$

- (2) *For every $(x, y) \in \mathcal{L} \times \mathcal{L}$, the function $t \mapsto d_{\mathbb{D}}(B_{\sigma, \tilde{x}}(t), B_{\sigma, \tilde{y}}(t))$ is bounded on $] -\infty, 0]$.*

The first item asserts that $B_{\sigma, \tilde{x}}$ is a (a, b) -quasigeodesic for the metric $d_{\mathbb{D}}$. The crucial consequence is that there exists a limit $\delta := \lim_{t \rightarrow -\infty} B_{\sigma, \tilde{x}}(t) \in \partial\mathbb{D}$, see [29, Chapitre 7]. The second part of Proposition 6.3 asserts that $\lim_{t \rightarrow -\infty} B_{\sigma, \tilde{y}}(t) = \delta$ for every $y \in \mathcal{L}$. Let $G_{\sigma, \tilde{x}} : \mathbb{R} \rightarrow \mathbb{D}$ be the unique geodesic such that $G_{\sigma, \tilde{x}}(0) = \tilde{x}$ and $\lim_{t \rightarrow -\infty} G_{\sigma, \tilde{x}}(t) = \delta$. We define our continuous flow $\psi : M \times \mathbb{R} \rightarrow M$ by setting

$$\psi(x, t) := \sigma(G_{\sigma, \tilde{x}}(t)).$$

Let us verify that this definition does not depend on $\tilde{x} \in \sigma^{-1}(x)$. Set $\tilde{x}' := g\tilde{x}$ where g is an element of $\text{Aut}(\mathbb{D})$ which leaves invariant σ . Then $B_{\sigma, \tilde{x}'} = gB_{\sigma, \tilde{x}}$ by unicity of the σ -lift starting at a point, and this quasigeodesic converges to $g(\delta)$ in the past. The geodesics $G_{\sigma, \tilde{x}'}$ and $gG_{\sigma, \tilde{x}}$ are then equal since they coincide at $t = 0$ and both converge to $g(\delta)$ in the past. Composing by σ we see that the definition of ψ does not depend on $\tilde{x} \in \sigma^{-1}(x)$. The fact that it does not depend on σ is easy and left to the reader. This proves items (1) and (2) of Definition 6.1. It remains to check the continuity of ψ . This follows from Theorem 3.2 and from the following classical fact.

Proposition 6.4. *Let $\alpha_n : \mathbb{R}^+ \rightarrow \mathbb{D}$ be a sequence of (a, b) -quasigeodesics starting at 0 and converging uniformly on compact subsets to a (a, b) -quasigeodesic α . If $\delta_n := \lim_{t \rightarrow +\infty} \alpha_n(t)$ and $\delta := \lim_{t \rightarrow +\infty} \alpha(t)$, then $\lim_n \delta_n = \delta$.*

Proof. Let α be a (a, b) -quasigeodesic and let $\delta := \lim_{t \rightarrow +\infty} \alpha(t)$. It suffices to prove that there exist A, B depending on a, b (not on γ) such that

$$\forall t \geq 1, \angle(\alpha(t), \delta) \leq A e^{-Bt}.$$

Let $s \geq t \geq 1$ and consider the geodesic triangle with vertices $0, \alpha(s), \alpha(s+1)$. Let a_s and b_s be the lengths of the sides containing 0 and let c_s be the length of the third side. Since α is a (a, b) -quasigeodesic, we have $a_s, b_s \geq a^{-1}s - b$ and $c_s \leq a + b$. Hyperbolic trigonometry then yields $|\sin(\angle(\alpha(s), \alpha(s+1)))| \leq A(a, b)e^{-(a^{-1}/2)s}$ for s large enough. \square

6.2. Volume estimates.

Lemma 6.5 (Volume estimates). *Let M be an immersed Levi-flat hypersurface in a complex surface S . Assume that M has a point at infinity and let us endow $T_{\mathcal{F}}$ with the leafwise Poincaré metric. Then the Lyapunov exponent of any ergodic harmonic measure μ satisfies $\lambda(\mu) \leq -1$.*

Let us begin with the following proposition.

Proposition 6.6. *Let M be an immersed Levi-flat hypersurface in a complex surface S . Assume that M has a point at infinity $\psi : M \times \mathbb{R} \rightarrow M$ and let us set $\psi^t(x) := \psi(x, t)$. Let us endow $T_{\mathcal{F}}$ with the leafwise Poincaré metric and $N_{\mathcal{F}}$ with an arbitrary metric. Let vol_P and θ be their respective volume form and let $\nu := \text{vol}_P \otimes \theta$, this defines a finite smooth volume form on M . Let $B_x(r) \subset M$ be the ball centered at x of radius r . Then*

$$\forall x \in M, \forall t \geq 0, \lim_{r \rightarrow 0} \frac{\nu(\psi^t B_x(r))}{\nu(B_x(r))} = e^t \cdot |h'_{(\psi^s(x))_{0 \leq s \leq t}}(x)|,$$

where $h_{(\psi^s(x))_{0 \leq s \leq t}}$ is the holonomy map over $s \in [0, t] \mapsto \psi^s(x) \in \mathcal{L}_x$.

Proof. Let (z, u) be coordinates of a flow box $U = \mathbb{D} \times \tau$ around $x \in M$. Let $U' \Subset U$ and $t > 0$ small enough such that $\psi^t(U') \subset U$. Let us write $\psi^t(z, u) = (\psi_z^t(z, u), u)$. By Definition 6.1, the map $(z, u) \mapsto \psi_z^t(z, u)$ is continuous and leafwise smooth. Let us write $\theta = \tilde{\theta}(z, u)du$ and set $\tilde{\theta}^t := \tilde{\theta} \circ \psi^t$. We also denote by $\{\cdot\}_u$ the u -horizontal slice in U . Then

$$\nu(\psi^t B_x(r)) = \int_I \left[\int_{\{\psi^t B_x(r)\}_u} \tilde{\theta}(z, u) \cdot \text{vol}_P \right] du.$$

Using the fact that $z \mapsto \psi_z^t(z, u)$ is smooth, we can perform a change of coordinates in the leaves:

$$\nu(\psi^t B_x(r)) = \int_I \left[\int_{\{B_x(r)\}_u} \tilde{\theta}^t(z, u) \cdot (\psi_z^t)^*(\text{vol}_P) \right] du.$$

Let $u \mapsto h(u)$ be the holonomy map over $(\psi^s(x))_{0 \leq s \leq t}$. Then $\tilde{\theta}^t(z, u) = |h'(u)| \cdot \tilde{\theta}(z, u)$ for every $(z, u) \in U'$. Moreover we can uniformize every leaf by the upper half plane $\{(a, b) \in \mathbb{R}^2, b > 0\}$ in such a way that $z \mapsto$

$\psi_z^t(u) := \psi^t(z, u)$ acts as $a + ib \mapsto a + e^{-t}ib$. Since $\text{vol}_P = da \wedge db/b^2$ in these coordinates, we have $(\psi_z^t)^*\text{vol}_P = e^t\text{vol}_P$. We deduce

$$\nu(\psi^t B_x(r)) = e^t \int_I \left[\int_{[B_x(r)]_u} |h'(u)| \cdot \tilde{\theta}(z, u) \cdot \text{vol}_P \right] du.$$

The Lemma follows by dividing by $\nu(B_x(r))$ and letting r to zero. \square

Proposition 6.7. *We consider the hypothesis of Proposition 6.6. Let μ be an ergodic harmonic measure and $\lambda(\mu)$ be its Lyapunov exponent. There exists a Borel set $B_\mu \subset M$ satisfying $\nu(B_\mu) > 0$ and*

$$\forall x \in B_\mu, \quad \lim_{t \rightarrow +\infty} \frac{1}{t} \log |h'_{(\psi^s(x))_{0 \leq s \leq t}}(x)| = \lambda(\mu).$$

To prove Proposition 6.7 we shall need the two classical results below.

Proposition 6.8. *Let us endow \mathbb{D} with the Poincaré metric (see Subsection 3.4). Let Γ be the set of continuous paths in \mathbb{D} starting at the origin and W be the Wiener measure on Γ . For W -almost every $\gamma \in \Gamma$,*

$$\lim_{t \rightarrow +\infty} \frac{1}{t} d_{\mathbb{D}}(0, \gamma(t)) = 1.$$

We refer to [54, Section 9.6, Theorem 6.1] for a standard proof, we provide another one in Subsection 9.2. For the second result we recall that the limit

$$s(\gamma) = \lim_{t \rightarrow +\infty} \gamma(t) \in \partial\mathbb{D}$$

exists for W -almost every $\gamma \in \Gamma$. We have the following shadowing property of geodesics by Brownian paths, see [1, théorème 7.3, page 103]. For $s \in \partial\mathbb{D}$ we define the radius $[0, s[:= \{ts \in \mathbb{D}, 0 \leq t < 1\}$.

Theorem 6.9 (Ancona). *There exists a subset $A \subset \Gamma$ of full W -measure satisfying the following properties. For every $\gamma \in A$, there exists $c(\gamma) > 0$ such that $d_{\mathbb{D}}(\gamma(t), [0, s(\gamma)[) \leq c(\gamma) \log t$ for t large enough.*

The following Lemma is a step towards Proposition 6.7. The open set $V_\mu \subset M$ was introduced in Theorem 5.2.

Lemma 6.10. *We consider the hypothesis of Proposition 6.6. Let μ be an ergodic harmonic measure and $\lambda(\mu)$ be its Lyapunov exponent. Let $x \in V_\mu$ and $\sigma : \mathbb{D} \rightarrow \mathcal{L}_x$ be a uniformization such that $\sigma(0) = x$. There exists $\mathcal{S} \subset \partial\mathbb{D}$ of positive Lebesgue measure such that for every hyperbolic geodesic $\beta : \mathbb{R} \rightarrow \mathbb{D}$ tending to s at $+\infty$:*

$$\lim_{t \rightarrow +\infty} \frac{1}{t} \log |h'_{\beta, t}(\beta(0))| = \lambda(\mu).$$

Proof. We use the subsets $\Lambda_x \subset \Gamma_x$ and $A \subset \Gamma$ defined in Theorems 5.2 and 6.9. We identify $\Gamma \simeq \Gamma_x$. Let $\mathcal{S} := s(A \cap \Lambda_x)$. It has positive Lebesgue measure since $W_x(\Lambda_x) > 0$ by Theorem 5.2. Let $s \in \mathcal{S}$ and $\gamma \in A \cap \Lambda_x$ such that $\lim_{t \rightarrow +\infty} \gamma(t) = s$. Let $\beta : \mathbb{R} \rightarrow \mathbb{D}$ be a geodesic such that $\lim_{t \rightarrow +\infty} \beta(t) = s$. Since the geodesics $[0, s[$ and β are exponentially asymptotic, we can replace $[0, s[$ by β in Ancona's theorem. Hence, there exist $c' = c'(\gamma, \beta) \geq c(\gamma)$ and a time $\tilde{t} \geq 0$ depending on t satisfying

$$(10) \quad d_{\mathbb{D}}(\gamma(t), \beta(\tilde{t})) \leq c' \log t.$$

Now let $\epsilon > 0$. Using Proposition 6.8 we deduce that

$$(11) \quad |\tilde{t} - t| = d_{\mathbb{D}}(\beta(t), \beta(\tilde{t})) \leq \epsilon t + c' \log t.$$

In the remainder γ stands for the restriction of γ to $[0, t]$. Let $\tilde{\gamma}$ be the geodesic joining $\gamma(t), \beta(\tilde{t})$ and let $\hat{\beta}$ be the geodesic joining $\beta(\tilde{t})$ to $\beta(t)$. The endpoints of the path $\gamma * \tilde{\gamma} * \hat{\beta}$ are $\gamma(0)$ and $\beta(t)$. If $\hat{\gamma}$ is the geodesic joining $\gamma(0)$ to $\beta(0)$, then $\gamma * \tilde{\gamma} * \hat{\beta}$ and $\beta * \hat{\gamma}$ have identical endpoints and are homotopic in \mathbb{D} . The chain rule then implies (the derivatives are implicitly evaluated at the starting point of the path)

$$(12) \quad h'_{\beta,t} \cdot h'_{\tilde{\gamma},\rho} = h'_{\hat{\beta},|\tilde{t}-t|} \cdot h'_{\tilde{\gamma},l_t} \cdot h'_{\gamma,t},$$

where l_t , $|\tilde{t} - t|$ and ρ are the respective lengths of $\tilde{\gamma}$, $\hat{\beta}$ and $\hat{\gamma}$. Now we look at the exponential growths. Using $\gamma \in \Lambda_x$, we obtain

$$\lim_{t \rightarrow +\infty} \frac{1}{t} \log |h'_{\gamma,t}| = \lambda(\mu).$$

Now the compactness of M yields $C > 0$ such that (use Equations (10) and (11)):

$$\begin{aligned} |\log |h'_{\tilde{\gamma},l_t}| | &\leq C \cdot l_t \leq C \cdot c' \log t, \\ |\log |h'_{\hat{\beta},|\tilde{t}-t|}| | &\leq C \cdot |\tilde{t} - t| \leq C \cdot (\epsilon t + c' \log t). \end{aligned}$$

Taking logarithm and dividing by t the Equation (12), we deduce

$$\lambda(\mu) - C\epsilon \leq \liminf_{t \rightarrow +\infty} \frac{1}{t} \log |h'_{\beta,t}| \leq \limsup_{t \rightarrow +\infty} \frac{1}{t} \log |h'_{\beta,t}| \leq \lambda(\mu) + C\epsilon.$$

Then Lemma 6.10 follows by letting ϵ to zero. \square

Let us prove Proposition 6.7. We cover M by charts $U_j = \cup_{u \in]0,1[} \mathbb{D} \times \{u\}$. Let $V_j = \cup_{u \in I_j} \mathbb{D} \times \{u\}$, where I_j consists of $u \in]0,1[$ satisfying

$$\text{vol}_P \{ y \in \mathbb{D} \times \{u\} \mid \lim_{t \rightarrow +\infty} \frac{1}{t} \log |h'_{(\psi^s(y))_{0 \leq s \leq t}}(y)| = \lambda(\mu) \} > 0.$$

Recalling that $s \mapsto \psi^s(y)$ are hyperbolic geodesics (Definition 6.1), Lemma 6.10 ensures that $V_\mu \subset \cup_j V_j$. Since $V_\mu \subset M$ is an open subset and ν is a volume form on M , there exists j' such that $\nu(V_{j'}) > 0$. We set $B_\mu := V_{j'}$, it satisfies the requirements of Proposition 6.7.

Let us explain how Propositions 6.6 and 6.7 imply Lemma 6.5. Let μ be an ergodic harmonic measure and $\lambda(\mu)$ be its Lyapunov exponent. We want to prove $\lambda(\mu) \leq -1$. Suppose to the contrary that $\lambda(\mu) + 1 > \epsilon$ for some $\epsilon > 0$. By Proposition 6.7 there exist $B_\mu(\epsilon) \subset B_\mu$ of positive ν -measure and $t_\mu(\epsilon) \geq 0$ such that

$$\forall x \in B_\mu(\epsilon), \forall t \geq t_\mu(\epsilon), |h'_{(\psi^s(x))_{0 \leq s \leq t}}(x)| \geq e^{t(\lambda(\mu) - \epsilon)}.$$

Taking into account Proposition 6.6, we obtain

$$\forall x \in B_\mu(\epsilon), \forall t \geq t_\mu(\epsilon), \lim_{r \rightarrow 0} \frac{\nu(\psi^t B_x(r))}{\nu(B_x(r))} \geq e^{t(\lambda(\mu) + 1 - \epsilon)}.$$

This implies $\nu(\psi^t(B_\mu(\epsilon))) \geq e^{t(\lambda(\mu) + 1 - \epsilon)} \cdot \nu(B_\mu(\epsilon))$ for every $t \geq t_\mu(\epsilon)$ (see [47, Theorem 2.12]). This contradicts the finiteness of the measure ν .

6.3. Proof of Theorem 1.5 and Corollary 1.6. Theorem 1.5 is a consequence of Lemmas 6.2 and 6.5. Corollary 1.6 follows from the combination of Proposition 1.4 and Theorem 1.5.

7. IMPOSSIBILITY OF Sol AND $\mathrm{PSL}(2, \mathbb{R})$

This section is devoted to the following result.

Theorem 7.1. *Let M be an immersed Levi-flat hypersurface of class C^2 in a surface of general type S . Then M neither carries the geometry Sol nor $\mathrm{PSL}(2, \mathbb{R})$.*

The geometry Sol is carried (up to a finite cover) by hyperbolic torus bundles, see [58, Theorem 5.5] (and Subsection 2.2). The geometry $\mathrm{PSL}(2, \mathbb{R})$ is carried (up to a finite cover) by the unitary tangent bundles of compact orientable surfaces Σ of genus ≥ 2 , see [58, page 466] (and Subsection 2.3). Hence Theorem 7.1 is an immediate consequence of the following proposition and Corollary 1.6.

Proposition 7.2. *Let M be an immersed Levi-flat hypersurface of class C^2 in a surface of general type S . Assume that M is diffeomorphic to*

- (1) *a hyperbolic torus bundle $\mathbb{S}^1 \times_{\Phi} \Sigma$, where Σ is a compact orientable surface of genus 1, or to*
- (2) *the unitary tangent bundle $\Sigma \times_{\rho} \mathbb{S}^1$ of a compact orientable surface Σ of genus ≥ 2 .*

Then M is an Anosov immersed Levi-flat hypersurface of class C^2 .

The proof occupies the remainder of this section. It relies on deep rigidity theorems by Ghys-Sergiescu [30, Section B, Theorem 1] and Matsumoto [46, Theorem 1.4], we also use Theorem 2.11 due to Thurston. These theorems allow to prove that the Cauchy-Riemann foliation on M is isotopic or conjugate to a standard foliation which is the weak unstable foliation of an Anosov flow.

7.1. Proof of case (1): hyperbolic torus bundle. Let M be an immersed Levi-flat hypersurface of class C^2 in a surface of general type S . Assume that M is diffeomorphic to a hyperbolic torus bundle $\mathbb{S}^1 \times_{\Phi} \Sigma$ (see Subsection 2.2). Its Cauchy-Riemann foliation \mathcal{F} is of class C^2 by [4]. Taking a double cover, we can assume that M is an orientable hyperbolic torus bundle and that \mathcal{F} is transversally orientable, see Lemma 2.2. Let us prove that \mathcal{F} has no compact leaf. Suppose on the contrary that \mathcal{F} has a compact leaf, denoted by \mathcal{L} . It has genus $g \geq 2$ since S is a surface of general type, see Proposition 1.3. The natural map $\pi_1(\mathcal{L}) \rightarrow \pi_1(M)$ can not be injective, because $g \geq 2$ and the topology of a hyperbolic torus bundle is not rich enough (the Lie group Sol is solvable). By Theorem 2.1 due to Novikov this implies that \mathcal{F} has a compact leaf of genus 1, which is excluded by Proposition 1.3. This proves that \mathcal{F} does not contain any compact leaf. Now we can use the following theorem.

Theorem 7.3 (Ghys-Sergiescu). *Let M be an orientable hyperbolic torus bundle $\mathbb{S}^1 \times_{\Phi} \Sigma$ endowed with a transversally orientable C^{k+2} foliation \mathcal{F} ($0 \leq k \leq \infty$). If \mathcal{F} does not have any compact leaf, then \mathcal{F} is C^k -conjugate to the suspension of the unstable foliation on $\mathbb{S}^1 \times_{\Phi} \Sigma$.*

This theorem applied with $k = 0$ shows that M is an Anosov Levi-flat hypersurface of class C^2 in S . The proof of Proposition 7.2(1) is complete.

7.2. Proof of case (2): unitary tangent bundles. Let M be an immersed Levi-flat hypersurface of class C^2 in a surface of general type S . Assume that M is diffeomorphic to the unitary tangent bundle $\Sigma \times_{\rho} \mathbb{S}^1$ of a compact orientable surface Σ of genus $g \geq 2$. The Cauchy-Riemann foliation \mathcal{F} of M is transversally orientable since M is orientable and the leaves of \mathcal{F} are orientable, see Lemma 2.2. The arguments of the proof of Proposition 2.10 show that \mathcal{F} does not contain any compact leaf. Theorem 2.11 due to Thurston then ensures the existence of an isotopy between the foliation \mathcal{F} and a foliation \mathcal{G} transverse to the circle fibration of $\Sigma \times_{\rho} \mathbb{S}^1$. Now we can use the following theorem.

Theorem 7.4 (Matsumoto). *Let $\Sigma \times_{\rho} \mathbb{S}^1$ be the unitary tangent bundle of a compact orientable surface Σ of genus $g \geq 2$. Let \mathcal{G} be a transversally orientable 2-dimensional foliation of class C^2 on $\Sigma \times_{\rho} \mathbb{S}^1$ without any compact leaf. Then \mathcal{G} is topologically conjugate to the weak unstable foliation of the geodesic flow for the Poincaré metric.*

This theorem shows that M is an immersed Anosov Levi-flat hypersurface in S , which completes the proof of Proposition 7.2(2). We note that Theorem 7.4 was improved by Ghys [26, Théorème principal]: the conjugation is of class C^r if the transverse regularity is of class C^r , with $r \geq 3$.

8. HAUSDORFF DIMENSION OF MINIMAL SETS

In this section we prove Theorems 1.9 and 1.10. Let S be a complex surface, let $\hat{\mathcal{F}}$ be a non singular holomorphic foliation on an open set of S and let M be a minimal set of $\hat{\mathcal{F}}$. We recall that \mathcal{F} is the restriction of $\hat{\mathcal{F}}$ to M . Let m be a continuous metric on $T_{\mathcal{F}}$, let $p_m(t, x, y)$ be its leafwise heat kernel and vol_m its volume form. Assume that \mathcal{F} does not support any transverse invariant measure and let μ be an ergodic harmonic measure with negative Lyapunov exponent, see [19, Theorem B]. The Kaimanovich entropy $h(\mu)$ is defined in Subsection 1.4. We shall use the following geometrical interpretation of $h(\mu)$: it is the exponential growth of "separated positions" for leafwise Brownian motions starting at μ -generic points. Let us be more precise. Let $x \in M$ and d_m be the distance on \mathcal{L}_x induced by the metric m . A subset $Z \subset \mathcal{L}_x$ is a (C, D) -lattice if its points are C -separated and if any point of \mathcal{L}_x lies at a distance $\leq D$ from Z for the distance d_m . For every $z \in \mathcal{L}_x$, let

$$\cap_{z' \in Z} \{y \in \mathcal{L}_x, d_m(y, z) \leq d_m(y, z')\}$$

be the tile of the Voronoi tessellation associated to Z containing z . For $n \geq 1$ let B_n be the distribution of leafwise Brownian motion starting at x at time n (with density $p_m(n, x, y) vol_m(dy)$). Let v be the projection $\mathcal{L}_x \rightarrow Z$ which contracts every tile to its center, it is defined B_n -almost everywhere. Let $\beta_n := v_* B_n$, this is a probability measure on Z . We shall use the following result (see [36, Theorem 3] and [41, page 195]).

Proposition 8.1. *Let μ be an ergodic harmonic measure on M . Let x be a μ -generic point, Z be a lattice in \mathcal{L}_x and $(Z_n)_n$ be subsets of Z such that $\beta_n(Z_n) \geq 1/2$. Then*

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \log |Z_n| \geq h(\mu).$$

8.1. Proof of Theorem 1.9. As before let M be a minimal set in a complex surface S and let μ be an ergodic harmonic measure with negative Lyapunov exponent $\lambda(\mu)$. Let $x \in M$ be a μ -generic point and let τ_x be a transversal to the ambient foliation $\hat{\mathcal{F}}$ at x . For simplicity we set $h := h(\mu)$ and $\lambda := \lambda(\mu)$. Given $\epsilon > 0$ we shall prove

$$\dim_H M \cap \tau_x \geq \frac{h - 3\epsilon}{|\lambda| + 3\epsilon}.$$

Let $\mathbb{D}(r) \subset \tau_x$ be the disc centered at zero of radius r , we put a subscript to indicate the center of the disc. Let us recall Koebe distortion theorem: there exists $\kappa > 0$ such that for every holomorphic injective function defined on $\mathbb{D}(2\rho)$,

$$(13) \quad \forall r \leq \rho, \quad \mathbb{D}_{h(0)}(e^{-\kappa}|h'(0)|r) \subset h(\mathbb{D}(r)) \subset \mathbb{D}_{h(0)}(e^{\kappa}|h'(0)|r).$$

Now we use the functions $r_\gamma(\epsilon)$ and $t_\gamma(\epsilon)$ of Theorem 5.2. Let r_0 small and t_0 large enough such that

$$\Gamma_{x,\epsilon} := \{\gamma \in \Gamma_x, r_\gamma(\epsilon) \geq 2r_0, t_\gamma(\epsilon) \leq t_0\}$$

satisfies $W_x(\Gamma_{x,\epsilon}) \geq 1/2$. Let $r_1 := r_0 e^{-2\kappa}/8$, where κ is the distortion constant in Equation (13). The next proposition will be proved in Subsection 8.2.

Proposition 8.2. *Under the assumption of Theorem 1.9, there exist for n large enough:*

- (i) a set $I'_n \subset \mathcal{L}_x \cap \tau_x$ of cardinality larger than $e^{n(h-3\epsilon)}$,
- (ii) a negative number $\lambda_n \in [n(\lambda - 2\epsilon), n(\lambda + 2\epsilon)]$,
- (iii) continuous paths $\gamma_z : [0, n+1] \rightarrow \mathcal{L}_x$ with endpoints x and $z \in I'_n$,

such that every holonomy map $h_{x,z} := h_{\gamma_z, n+1}$ satisfies:

- (1) $\mathbb{D}_z(r_1 e^{\lambda_n} e^{-\kappa}) \subset h_{x,z}(\mathbb{D}(r_1)) \subset \mathbb{D}_z(r_1 e^{\lambda_n + \log 2} e^{\kappa})$,
- (2) $h_{x,z}(\mathbb{D}(r_1)) \subset \mathbb{D}(r_1)$,
- (3) The sets $h_{x,z}(\mathbb{D}(r_1))$ for $z \in I'_n$ are pairwise disjoint.

Let us see how Theorem 1.9 follows. Let $m \geq 1$ and

$$\mathcal{H}_m := \{h_{x,z_1} \circ \dots \circ h_{x,z_m}, z_i \in I'_n\}.$$

The elements of \mathcal{H}_m are well defined on $\mathbb{D}(r_1)$ by item (2). They are pairwise distinct by item (3), in particular $|\mathcal{H}_m| = |I'_n|^m$. Let us introduce

$$K_m := \bigcup_{z \in \{h(0), h \in \mathcal{H}_m\}} \mathbb{D}_z \left(r_1 (e^{\lambda_n} e^{-2\kappa})^m \right),$$

these discs are pairwise disjoint by item (3) and $K_m \subset \bigcup_{h \in \mathcal{H}_m} h(\mathbb{D}(r_1))$ by item (1). The limit set $K := \bigcap_m K_m$ then satisfies (see [53, Section 13])

$$\dim_H K \geq \frac{n(h - 3\epsilon)}{-\lambda_n + 2\kappa} \geq \frac{h - 3\epsilon}{-\lambda + 3\epsilon},$$

where the second inequality relies on $\lambda_n \geq n(\lambda - 2\epsilon)$ and $n\epsilon \geq 2\kappa$ for n large enough. The Hausdorff dimension of $M \cap \tau_x$ satisfies the same lower bound, since K is contained in this set.

8.2. Proof of Proposition 8.2. Let $Z := \mathcal{L}_x \cap \mathbb{D}(r_1/2)$. By compactness and minimality of M , the subset Z is a (C, D) -lattice. Recall that v is the projection $\mathcal{L}_x \rightarrow Z$ and $\beta_n = v_* B_n$, where B_n is the distribution of Brownian motion at time n . We set $\Gamma_{x,\epsilon}(n) := \{\gamma(n), \gamma \in \Gamma_{x,\epsilon}\}$ and $J_n := v(\Gamma_{x,\epsilon}(n))$. Since $\beta_n(J_n) \geq 1/2$ by definition of $\Gamma_{x,\epsilon}$, Proposition 8.1 yields $|J_n| \geq e^{n(h-\epsilon)}$.

Using again the compactness of M , there exist $E > 0$ and $r_2 > 0$ such that if $\hat{\gamma} : [0, 1] \rightarrow \mathcal{L}_x$ is a smooth path of length $\leq D$, then the holonomy map $h_{\hat{\gamma},1}$ is defined on $\mathbb{D}(r_2)$ and satisfies $-E \leq \log|h'_{\hat{\gamma},1}| \leq E$. Let us prove the following claim. There exist

- (i') a set $I_n \subset J_n$ of cardinality larger than $e^{n(h-2\epsilon)}$,
- (ii') a negative number $\lambda_n \in [n(\lambda - 2\epsilon), n(\lambda + 2\epsilon)]$,
- (iii') continuous paths $\gamma_z : [0, n+1] \rightarrow \mathcal{L}_x$ with endpoints x and $z \in I_n$,

such that $h_{x,z} := h_{\gamma_z, n+1}$ satisfies for every $z \in I_n$:

- (a) $h_{x,z}$ is defined from $\mathbb{D}(2r_0) \subset \tau_x$ to τ_x ,
- (b) $\lambda_n \leq \log|h'_{x,z}(0)| \leq \lambda_n + \log 2$,
- (c) $\mathbb{D}_z(r_1 e^{\lambda_n} e^{-\kappa}) \subset h_{x,z}(\mathbb{D}(r_1)) \subset \mathbb{D}_z(r_1 e^{\lambda_n + \log 2} e^{\kappa})$,
- (d) $h_{x,z}(\mathbb{D}(r_1)) \subset \mathbb{D}(r_1)$.

By construction, every point $z \in J_n$ is D -distant from a point $y \in \Gamma_{x,\epsilon}(n)$. Let $\hat{\gamma} : [0, 1] \rightarrow \mathcal{L}_x$ be a smooth path, of length $\leq D$, with endpoints y, z . Let $\gamma \in \Gamma_{x,\epsilon}$ such that $y = \gamma(n)$. We set $\gamma_z := \hat{\gamma} * \gamma|_{[0,n]}$ and also $h_{x,y} := h_{\gamma,n}$, $h_{y,z} := h_{\hat{\gamma},1}$ so that $h_{x,z} := h_{\gamma_z, n+1} = h_{y,z} \circ h_{x,y}$. By definition of $\Gamma_{x,\epsilon}$, the map $h_{x,y}$ is well-defined on $\mathbb{D}(2r_0)$. By Koebe distortion theorem described in Equation (13), the image of $\mathbb{D}(2r_0)$ by $h_{x,y}$ is contained in a disc of radius $2r_0 e^{n(\lambda+\epsilon)} e^{\kappa}$. Since that radius is smaller than r_2 for n large enough, $h_{x,z}$ is well-defined on $\mathbb{D}(2r_0)$ and satisfies

$$-E + n(\lambda - \epsilon) \leq \log|h'_{x,z}(0)| \leq E + n(\lambda + \epsilon).$$

This implies (a). Moreover this ensures the existence of λ_n satisfying

$$-E + n(\lambda - \epsilon) \leq \lambda_n \leq E + n(\lambda + \epsilon)$$

and a subset $I_n \subset J_n$ of cardinality

$$|I_n| \geq \frac{|J_n| \log 2}{2(E + \epsilon n)}$$

such that (b) is satisfied for every $z \in I_n$. Items (i') and (ii') then follow for n large. Koebe distortion theorem recalled in Equation (13) ensures (c). Finally we get (d) from $h_{x,z}(0) \in Z \subset \mathbb{D}(r_1/2)$ and $\text{diam } h_{x,z}(\mathbb{D}(r_1)) \leq 2r_1 e^{\lambda_n + \log 2} e^{\kappa}$. That completes the proof of the claim.

At this stage, items (ii), (iii), (1), (2) of Proposition 8.2 are established. It remains to build $I'_n \subset I_n$ satisfying (i) and (3). Let $\kappa' := 2\kappa + \log 2$ and

$$\mathcal{H} := \{h : \mathbb{D}(r_1) \rightarrow \tau_x \text{ holonomy map}, -\kappa' \leq \log|h'(0)| \leq \kappa'\}.$$

This set is finite because the holonomy pseudogroup of \mathcal{F} is assumed to be discrete. Let $\rho_n := 2r_1 e^{\lambda_n + \log 2} e^\kappa$ and $(z, z') \in I_n \times I_n$ satisfying $|z - z'| \leq 2\rho_n$. Item (c) established before yields

$$h_{x,z}(\mathbb{D}(2r_1)) \subset \mathbb{D}_z(\rho_n),$$

$$h_{x,z'}(\mathbb{D}(r_0)) \supset \mathbb{D}_{z'}(r_0 e^{\lambda_n} e^{-\kappa}) = \mathbb{D}_{z'}(2\rho_n) \supset \mathbb{D}_z(\rho_n).$$

We deduce that $h := h_{x,z'}^{-1} \circ h_{x,z} : \mathbb{D}(2r_1) \rightarrow \mathbb{D}(r_0)$ is well defined. One easily verifies that $-\kappa' \leq \log |h'(0)| \leq \kappa'$, hence the restriction $h : \mathbb{D}(r_1) \rightarrow \mathbb{D}(r_0)$ belongs to \mathcal{H} . Let $I'_n \subset I_n$ such that $\{h_{x,z}\}_{z \in I'_n}$ represents the classes of I_n modulo \mathcal{H} by right composition. This set has cardinality

$$|I'_n| \geq |I_n|/|\mathcal{H}| \geq e^{n(h-2\epsilon)}/|\mathcal{H}| \geq e^{n(h-3\epsilon)},$$

this proves (i). Item (3), that is $\{h_{x,z}(\mathbb{D}(r_1))\}_{z \in I'_n}$ are pairwise disjoint, follows from $\text{diam } h_{x,z}(\mathbb{D}(r_1)) \leq \rho_n$ and $|z - z'| > 2\rho_n$ for every $(z, z') \in I'_n \times I'_n$. This completes the proof of Proposition 8.2.

8.3. Proof of Theorem 1.10. We show the more precise statement:

Theorem 8.3. *Let S be a complex surface, let $\hat{\mathcal{F}}$ be a non singular holomorphic foliation on an open set of S and let M be a minimal set of $\hat{\mathcal{F}}$. Assume that \mathcal{F} has no transverse invariant measure.*

- (1) *If the holonomy pseudogroup of \mathcal{F} is not discrete, then $\dim_H M^{\text{th}} \geq 1$. Moreover, M is an analytic Levi-flat hypersurface in S if and only if $\dim_H M^{\text{th}} = 1$.*
- (2) *If S is a surface of general type, the holonomy pseudogroup of \mathcal{F} is discrete and \mathcal{F} has a simply connected leaf, then $\dim_H M^{\text{th}} > 1$.*

8.3.1. Proof of case (1): the holonomy pseudo-group of \mathcal{F} is not discrete. Since \mathcal{F} has no transverse invariant measure, there exists a holonomy mapping with a hyperbolic fixed point in M (see [19, Corollary 1.3]). The holonomy pseudogroup of \mathcal{F} being not discrete, this implies that there are local holomorphic flows Ψ in $\bar{\mathcal{G}}$, the closure of the pseudogroup (see [44, Section 4]). This implies that $\dim_H M^{\text{th}} \geq 1$.

Now let us assume that $\dim_H M^{\text{th}} = 1$ and let us show that M is an analytic Levi-flat hypersurface in S . If the closure $\bar{\mathcal{G}}$ contains two transverse local holomorphic flows then M has non empty interior and $\dim_H M^{\text{th}} = 2$, which is excluded. Therefore, in a transversal τ to M , every local holomorphic flow in $\bar{\mathcal{G}}$ acts, say, horizontally. The orbit of $\bar{\mathcal{G}}$ in τ is thus a union of horizontal lines parametrized by a compact set in the vertical direction, denoted F . There are two cases. In the first case the compact set F is finite and M is an analytic Levi-flat hypersurface as desired. In the second case F is not finite and it contains a Cantor set of positive Hausdorff dimension (similar ideas than in Section 8.1): this implies that $\dim_H M^{\text{th}} > 1$, which is excluded.

8.3.2. Proof of case (2): S is a surface of general type, the holonomy pseudo-group of \mathcal{F} is discrete and there is a simply connected leaf. We want to prove $\dim_H M^{\text{th}} > 1$. The leaves are hyperbolic since \mathcal{F} has no transverse invariant measure. We endow \mathcal{F} with the Poincaré metric m_P , it is continuous of curvature -1 (see Subsection 3.4). Let μ be a harmonic measure associated

to m_P (see Theorem 5.2). The discreteness of \mathcal{F} allows to apply Theorem 1.9: we get $\dim_H M^{\text{th}} \geq h(\mu)/|\lambda(\mu)|$. The fact that S is a surface of general type implies $|\lambda(\mu)| < 1$ (see Proposition 1.4). It remains to prove $h(\mu) = 1$.

Proposition 8.4. *Let S be a Kähler surface and let M be a minimal set with no transverse invariant measure. Let μ be a harmonic measure associated with the Poincaré metric. If \mathcal{F} has a simply connected leaf then $h(\mu) = 1$.*

The remainder of this subsection is devoted to the proof.

Lemma 8.5. *Let S be a Kähler surface and M a minimal set. If M has a simply connected leaf then every leaf, except for a countable number, is simply connected. In particular, if M has no transverse invariant measure and if μ is a harmonic measure associated with the Poincaré metric, then \mathcal{L}_x is simply connected for μ -almost every $x \in M$.*

Proof. First observe that the Kähler property implies that there is no vanishing cycle, see [35, Corollary 1] (this article contains the definition of vanishing cycles). Then, by minimality and simply connected assumption, the holonomy of a leafwise (homotopically) non trivial loop is not the identity. Indeed, any deformation of that loop in a close leaf remains non trivial, and by minimality the simply connected leaf must contain such a deformation.

Now suppose that there are uncountably many leaves which are not simply connected. Let $(U_j)_{1, \dots, N}$ be a covering of M by flow boxes. Let us fix in each non simply connected leaf a non trivial loop γ . To each of them we associate a cyclic sequence $(c_n(\gamma))_n \in \{1, \dots, N\}^{\mathbb{N}}$ given by the indices of the flow boxes which are successively crossed. By a cardinality argument, there must exist such a cyclic sequence attained by uncountably many loops. Since the domain of definition of the corresponding holonomy map is open, one of its connected component has to intersect uncountably many loops. The holonomy map being analytic, it is therefore equal to identity on this component, which has been excluded. This proves the first part of the lemma. The second part follows from the fact that a harmonic measure which is not invariant has no atom. \square

Proposition 8.4 is then a consequence of the following classical result (see [37, Theorem 10]) and of the definition of $h(\mu)$ in Equation (3).

Lemma 8.6. *Let us endow \mathbb{D} with the Poincaré metric of curvature -1 . Let $p_{m_{\mathbb{D}}}$ and $\text{vol}_{m_{\mathbb{D}}}$ be the heat kernel and the volume of this metric. Then*

$$\lim_{t \rightarrow +\infty} -\frac{1}{t} \int_{\mathbb{D}} p_{m_{\mathbb{D}}}(x, y, t) \log p_{m_{\mathbb{D}}}(x, y, t) \text{vol}_{m_{\mathbb{D}}}(dy) = 1.$$

This completes the proof of the case (2) of Theorem 8.3.

8.4. Other applications of Theorem 1.9. We begin with Margulis-Ruelle type inequalities. From $\dim_H M^{\text{th}} \leq 2$ for minimal sets and $\dim_H M^{\text{th}} = 1$ for Levi-flat hypersurfaces, we obtain:

Corollary 8.7. *Let S be a complex surface, let $\hat{\mathcal{F}}$ be a non singular holomorphic foliation on an open set of S and let M be a minimal set of $\hat{\mathcal{F}}$. Assume that \mathcal{F} has no transverse invariant measure and that the holonomy pseudogroup of \mathcal{F} is discrete. We endow $T_{\mathcal{F}}$ with the Poincaré metric. If*

\mathcal{F} has a simply connected leaf, then the Lyapunov exponent of its harmonic measure μ satisfies $\lambda(\mu) \leq -1/2$.

Corollary 8.8. *Let S be a complex surface and let M be an analytic Levi-flat hypersurface in S . Assume that \mathcal{F} has no transverse invariant measure and that the holonomy pseudogroup of \mathcal{F} is discrete. Then the Lyapunov exponent of its harmonic measure satisfies $\lambda(\mu) \leq -1$.*

This result provides an alternative proof of Theorem 1.5 when M is the unitary tangent bundle of a compact orientable surface of genus $g \geq 2$. However, such a method does not work for a hyperbolic torus bundle, because its holonomy pseudogroup is not discrete: it contains an irrational rotation.

8.5. The case of $\mathbb{P}^2(\mathbb{C})$. Let $\mathcal{F}_{\mathbb{P}^2(\mathbb{C})}$ be a singular holomorphic foliation on $\mathbb{P}^2(\mathbb{C})$. By definition, its degree is the number of tangency points between its leaves and a generic complex line in $\mathbb{P}^2(\mathbb{C})$. Let $\hat{\mathcal{F}}$ be the non singular part of $\mathcal{F}_{\mathbb{P}^2(\mathbb{C})}$. Let M be a (hypothetical) minimal set of $\hat{\mathcal{F}}$. The restriction \mathcal{F} of the foliation $\hat{\mathcal{F}}$ to M has no transverse invariant measure, hence \mathcal{F} is hyperbolic, see [12, Theorems 2 and 3]. We recall a formula for the Lyapunov exponent, see [19, Proposition 3.12].

Proposition 8.9. *(Derooin-Kleptsyn) Let $\hat{\mathcal{F}}$ be the non singular part of a holomorphic foliation $\mathcal{F}_{\mathbb{P}^2(\mathbb{C})}$ of degree $d \geq 2$. Let M be a minimal set of $\hat{\mathcal{F}}$. We endow $T_{\mathcal{F}}$ with the Poincaré metric m_P . Then the Lyapunov exponent of its harmonic measure μ is equal to*

$$\lambda(\mu) = -\frac{d+2}{d-1}.$$

Let us sketch the proof. First the normal bundle and the cotangent bundle to the foliation are $N_{\mathcal{F}_{\mathbb{P}^2(\mathbb{C})}} = \mathcal{O}_{\mathbb{P}^2(\mathbb{C})}(d+2)$ and $K_{\mathcal{F}_{\mathbb{P}^2(\mathbb{C})}} = \mathcal{O}_{\mathbb{P}^2(\mathbb{C})}(d-1)$, see [8, Chapter 2, Section 3]. Let T be the harmonic current satisfying $\mu = T \wedge \text{vol}_{m_P}$ (see Remark 5.1). Then $\lambda(\mu) = -2\pi T \cdot N_{\mathcal{F}} = 2\pi(d+2)(d-1)^{-1} T \cdot T_{\mathcal{F}}$, and the expected value follows from $T \cdot T_{\mathcal{F}} = -1/2\pi$ (see Subsection 5.5). Theorem 1.9 implies the following result.

Proposition 8.10. *Let $\mathcal{F}_{\mathbb{P}^2(\mathbb{C})}$ be a holomorphic foliation on $\mathbb{P}^2(\mathbb{C})$ of degree $d \geq 1$. Let M be a minimal set of $\hat{\mathcal{F}}$. If \mathcal{F} has a simply connected leaf then*

$$\dim_H M^{\text{th}} \geq \frac{d-1}{d+2}.$$

Proof. This lower bound is trivial when $d = 1$, so let $d \geq 2$. If the holonomy pseudogroup of \mathcal{F} is not discrete, then $\dim_H M^{\text{th}} \geq 1$ by Assertion 1 of Theorem 8.3, in particular $\dim_H M^{\text{th}} \geq (d-1)/(d+2)$. If the holonomy pseudogroup is discrete, then the Hausdorff dimension is $\geq 1/|\lambda|$ by using Theorems 1.9 and 8.4. Proposition 8.9 yields the desired lower bound. \square

Note that the conformal dimension of any conformal harmonic current is bounded above by $(d-1)/(d+2)$, see [10, Theorem 1.1]. Note also that these sets are not too small in the sense of potential theory since their harmonic current has finite energy, see [21].

9. APPENDIX

9.1. Proof of Proposition 6.3. The proof follows from Lemmas 9.1 and 9.2 below. Let us recall the context. Let M be an immersed Levi-flat hypersurface in a complex surface S . We assume that \mathcal{F} is hyperbolic and we endow $T_{\mathcal{F}}$ with the leafwise Poincaré metric. Let N be a compact 3-manifold supporting an Anosov flow $A = \{A^t\}_{t \in \mathbb{R}}$. Let \mathcal{G} be the weak unstable foliation of the Anosov flow and let us endow $T_{\mathcal{G}}$ with the restriction of a smooth riemannian metric g_N on the tangent bundle TN . For every $\mathcal{L}' \in \mathcal{G}$ we denote by $\tilde{\mathcal{L}}'$ the universal cover of \mathcal{L}' and $d_{\tilde{\mathcal{L}}'}$ the distance induced on $\tilde{\mathcal{L}}'$ by g_N . We assume that \mathcal{F} and \mathcal{G} are topologically conjugate by $\Phi : M \rightarrow N$.

Lemma 9.1. *There exist $\alpha > 1$ and $\beta > 0$ satisfying the following property. For every $\mathcal{L} \in \mathcal{F}$ and every lift $\tilde{\Phi} : \mathbb{D} \rightarrow \tilde{\mathcal{L}}'$ of $\Phi : \mathcal{L} \rightarrow \mathcal{L}'$,*

$$\forall (x, y) \in \mathbb{D} \times \mathbb{D}, \quad \alpha^{-1}d_{\mathbb{D}}(x, y) - \beta \leq d_{\tilde{\mathcal{L}}'}(\tilde{\Phi}(x), \tilde{\Phi}(y)) \leq \alpha d_{\mathbb{D}}(x, y) + \beta.$$

Proof. Let $(U_j)_{j \in J}$ and $(V_j)_{j \in J}$ be coverings of M by flow boxes satisfying $\bar{V}_j \subset U_j$. For every leaf $\mathcal{L} \in \mathcal{F}$, the lifts of the plaques to the universal cover induce coverings $(\tilde{U}_j)_{j \in J}$ and $(\tilde{V}_j)_{j \in J}$ of \mathbb{D} by discs. We define a distance $\delta : J \times J \rightarrow \mathbb{N}$ by setting $\delta(j, j')$ as the minimal $k \geq 0$ such that there exist $j_0 = j, j_1, \dots, j_k = j'$ satisfying $U_{j_l} \cap U_{j_{l+1}} \neq \emptyset$ for $l = 0, \dots, k-1$. We define $v : J \rightarrow \mathbb{D}$ by fixing some $v(j) \in \tilde{V}_j$ for every $j \in J$. We claim that there exist $\alpha_1 > 1$ and $\beta_1 > 0$ depending on $(U_j)_{j \in J}$, $(V_j)_{j \in J}$ and v satisfying

$$(14) \quad \forall (j, j') \in J^2, \quad \alpha_1^{-1}\delta(j, j') - \beta_1 \leq d_{\mathbb{D}}(v(j), v(j')) \leq \alpha_1\delta(j, j') + \beta_1.$$

The right hand side of (14) easily follows from $d_{\mathbb{D}}(v(j), v(j')) \leq 2kc\delta(j, j')$ where c is the sum of $\max_j d_{\mathbb{D}}(0, v(j))$ and of the supremum of the diameter of the plaques of $(U_j)_{j \in J}$. For the left hand side, let $\epsilon > 0$ be the minimal distance in a plaque between points of ∂V_j and points of ∂U_j . The following property holds: for every geodesic arc γ in \mathbb{D} , every $x \in \gamma$, and every \tilde{V}_j containing x , \tilde{U}_j contains the ϵ -neighborhood of x in γ . We deduce that for every geodesic γ between $v(j)$ and $v(j')$, there exist $k \leq \text{length}(\gamma)/\epsilon + 1$ and a sequence $j_0 = j, j_1, \dots, j_k = j'$ such that $U_{j_l} \cap U_{j_{l+1}} \neq \emptyset$. Hence $\delta(j, j') \leq \epsilon^{-1}d_{\mathbb{D}}(v(j), v(j')) + 1$ as desired. To conclude, we introduce the open coverings $(\Phi(U_j))_{j \in J}$ and $(\Phi(V_j))_{j \in J}$ of N . Defining a distance $\delta' : J \times J \rightarrow \mathbb{N}$ and a map w as before, we obtain $\alpha_2 > 1$ and $\beta_2 > 0$ such that

$$(15) \quad \forall (j, j') \in J^2, \quad \alpha_2^{-1}\delta'(j, j') - \beta_2 \leq d_{\tilde{\mathcal{L}}'}(w(j), w(j')) \leq \alpha_2\delta'(j, j') + \beta_2.$$

The combination of Equations (14), (15) and $\delta = \delta'$ completes the proof. \square

Proposition 9.2. *There exists $\rho > 0$ satisfying the following property. For every $\mathcal{L}' \in \mathcal{G}$, every lift $\tilde{A}^t : \mathbb{R} \rightarrow \tilde{\mathcal{L}}'$ of $A^t : \mathbb{R} \rightarrow \mathcal{L}'$ and every $p \in \tilde{\mathcal{L}}'$:*

$$\forall (t, t') \in \mathbb{R}^2, \quad \rho^{-1}|t - t'| \leq d_{\tilde{\mathcal{L}}'}(\tilde{A}^t(p), \tilde{A}^{t'}(p)) \leq \rho|t - t'|.$$

Given $(p, q) \in \tilde{\mathcal{L}}' \times \tilde{\mathcal{L}}'$, the function $t \mapsto d_{\tilde{\mathcal{L}}'}(\tilde{A}^t(p), \tilde{A}^t(q))$ is bounded on $]-\infty, 0]$.

Proof. Let Ξ be the infinitesimal generator of the Anosov flow A , this is a vector field on N . Then $\rho = \max_N g_N(\Xi, \Xi)$ is convenient for the upper

bound. Let us verify the lower bound. We identify the normal bundle of \mathcal{G} with the strong stable bundle E^{ss} . The Anosov property yields a continuous (adapted) metric on this bundle and $\xi > 0$ such that

$$\forall t \leq 0, \forall v \in E^{ss}, |dA^t(v)| \geq e^{\xi t}|v|.$$

In particular, the holonomy of \mathcal{G} satisfies

$$\forall p \in N, \forall t \leq t', |h'_{(A^{-s}(p))_{t \leq s \leq t'}}(p)| \geq e^{\xi(t'-t)}.$$

On the other hand, there exists $\xi' > 0$ such that the derivative of a holonomy map along a path γ is always $\leq e^{\xi' \text{length}(\gamma)}$ (use local coordinates). Hence $d_{\tilde{\mathcal{G}}}(\tilde{A}^t(p), \tilde{A}^{t'}(p)) \geq (\xi - \xi')(t' - t)$ as desired. The second statement follows from the fact that every Anosov flow contracts exponentially the distance in the past on the unstable foliation. \square

9.2. Proof of Proposition 6.8. One can find a proof in [54, Section 9.6], our purpose is to employ arguments adapted to the unit disc without using the law of iterated logarithm. Recall that \mathbb{D} is endowed with the Poincaré metric of gaussian curvature -1 and that we work with the heat equation $\frac{\partial}{\partial t} = \Delta_m$. Let W be the Wiener measure on the set of continuous paths starting at the origin. By applying Kingman's theorem to $(\Gamma, (a_t)_{t \geq 0}, W)$ and $H_t(\gamma) := d_{\mathbb{D}}(\gamma(0), \gamma(t))$, there exists $\alpha \in \mathbb{R}$ such that for W -almost every $\gamma \in \Gamma$:

$$\lim_{t \rightarrow \infty} \frac{1}{t} d_{\mathbb{D}}(0, \gamma(t)) = \alpha.$$

We want to show that $\alpha = 1$. Let $\varphi : \mathbb{D} \rightarrow \mathbb{R}^+$ defined by $\varphi(z) = \frac{1-|z|^2}{|1-z|^2}$. This is a harmonic function (it corresponds to the imaginary part in the hyperbolic plane) and the laplacian of $\log \varphi$ with respect to the Poincaré metric is identically -1 . Dynkin's formula asserts that

$$\forall t \geq 0, \mathbb{E}(\log \varphi(\gamma(t))) = \log \varphi(0) + \mathbb{E} \left(\int_0^t \Delta_{m_{\mathbb{D}}} \log \varphi(\gamma(s)) ds \right),$$

where the expectation \mathbb{E} holds on Γ with respect to W . We deduce

$$\forall t > 0, \frac{1}{t} \mathbb{E}(\log \varphi(\gamma(t))) = -1.$$

To complete the proof it suffices to show

$$(16) \quad \lim_{t \rightarrow +\infty} \frac{1}{t} \mathbb{E}(\log \varphi(\gamma(t))) = -\alpha.$$

Given a circle in \mathbb{D} centered at the origin, its hyperbolic radius $R \in \mathbb{R}^+$ is related to its euclidian radius $r \in [0, 1[$ by $r(R) = \frac{e^R - 1}{e^R + 1}$. For every $t \geq 0$, let W_t be the pushforward measure of W on $[0, 1[$ by the map $\gamma \mapsto r \circ d_{\mathbb{D}}(0, \gamma(t))$. Then $\mathbb{E}(\log \varphi(\gamma(t)))$ is equal to

$$(17) \quad \int_{[0,1[} \frac{1}{2\pi} \int_0^{2\pi} \log \varphi(re^{i\theta}) d\theta dW_t(r) =: \int_{[0,1[} V(r) dW_t(r).$$

Using the euclidian radius $r : \mathbb{R}^+ \rightarrow [0, 1[$, we define

$$(18) \quad I_{t,\epsilon} := [r(t(\alpha - \epsilon)), r(t(\alpha + \epsilon))].$$

We have $W_t(I_{t,\epsilon}) \geq 1 - \epsilon$ for t large enough, since $\lim_{t \rightarrow \infty} \frac{1}{t} d_{\mathbb{D}}(0, \gamma(t)) = \alpha$ for almost every γ . We shall decompose the integral of Equation (17) (right side) over $I_{t,\epsilon}$ and $[0, 1] \setminus I_{t,\epsilon}$. For that purpose observe that $V(r) = \log(1 - r^2)$ from the classical formula $\frac{1}{2\pi} \int_0^{2\pi} \log |1 - re^{i\theta}|^2 d\theta = 0$. Now, from

$$V(r) = \log(1 - r(R)^2) = \log \cosh^{-2}(R/2) \in [-R, -R + \log 4]$$

and the definition of $I_{t,\epsilon}$ in Equation (18), we get

$$-t(\alpha + \epsilon) \leq \int_{I_{t,\epsilon}} V(r) dW_t(r) \leq -t(\alpha - \epsilon) + \log 4.$$

Since W_t is absolutely continuous with respect to Lebesgue measure on $[0, 1]$ (from the heat kernel) and $V \in L^2[0, 1]$, Cauchy-Schwarz inequality yields

$$\int_{[0,1] \setminus I_{t,\epsilon}} V(r) dW_t(r) \leq \epsilon \int_{[0,1]} V^2(r) dW_t(r) =: c\epsilon.$$

That completes the proof of (16), and $\alpha = 1$ as desired.

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