Normalization of bundle holomorphic contractions and applications to dynamics

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Abstract

We establish a Poincaré-Dulac theorem for sequences $(G_n)_{n\in\mathbb{Z}}$ of holomorphic contractions whose differentials d_0G_n split regularly. The resonant relations determining the normal forms hold on the moduli of the exponential rates of contraction. Our results are actually stated in the framework of bundle maps.

Such sequences of holomorphic contractions appear naturally as iterated inverse branches of endomorphisms of $\mathbb{CP}^k$. In this context, our normalization result allows to estimate precisely the distortions of ellipsoids along typical orbits. As an application, we show how the Lyapunov exponents of the equilibrium measure are approximated in terms of the multipliers of the repulsive cycles.[1]

1 Introduction and results

As it is well known, any holomorphic function F which is invertible and contracting at the origin of $\mathbb{C}$ is conjugated to its linear part $A := F'(0)$. Moreover, the conjugation is realized by a function N obtained by a renormalization procedure : $N = \lim_n A^{-n}F^n$. Remarkably, the proof only relies on the fact that, by analyticity, F is tangent to A at the order 2. It turns out that these arguments may be adapted to the case of holomorphic mappings and yield to the Poincaré-Dulac theorem. We show in this paper that this strategy also works in a non-autonomous setting, that is for families of holomorphic contractions.

Our results will be expressed in the framework of bundle maps which we now briefly describe (see the subsection 2.1 for more details). Let X be a set and $\mathcal{X} := X \times \mathbb{C}^k$. For any function r on X, the *tube* $E(r) \subset \mathcal{X}$ is defined by

$$E(r) := \bigcup_{x\in X} \{(x, v) \in \mathcal{X} \ , \ |v| < r(x)\},$$

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where $|\,.\,|$ is the euclidean norm on $\mathbb{C}^k$. A *bundle map* over a bijective map $\tau : X \to X$ is a map of the form

$$\mathcal{K} : \quad \begin{array}{ccc} E(r) & \longrightarrow & \mathcal{X} \\ (x,v) & \longmapsto & (\tau(x), K_x(v)) \end{array}$$

where K_x is holomorphic and $K_x(0) = 0$. A bundle map $\mathcal{K}$ is *tame* when the coefficients of the Taylor expansion of $K_{\tau^n(x)}$ grow exponentially slowly when n tends to infinity. A function $\phi_\epsilon : X \to]0,1]$ is *ϵ-slow* if $\phi_\epsilon(\tau^{\pm 1}(x)) \geq e^{-\epsilon}\phi_\epsilon(x)$. We note $\mathcal{I}$ the identity map of $\mathcal{X}$ over Id_X.

Let us now present our results. To this purpose we will describe the autonomous case (Poincaré-Dulac theorem) and, step by step, precise the corresponding statements for bundle maps. Let $F : \mathbb{C}^k \to \mathbb{C}^k$ be an holomorphic map whose linear part A satisfies $m|v| \leq |A(v)| \leq M|v|$. If $M^{q+1} < m$, the sequence $(A^{-n}F^n)_n$ converges as soon as F and A are tangent at the order $q+1$. This means that a sufficently high tangency between F and A allows to overcome the fact that $\|A\| \neq \|A^{-1}\|^{-1}$ and implies the conjugation (when $k = 1$, then $m = M$ and one sees again that the tangency at the order 2 suffices). Moreover, the linear map A may actually be replaced by any automorphism N which is tangent to F at the order $q+1$. The same phenomenon actually occurs in a non-autonomous setting, the precise statement is the following :

Theorem 1.1 *Let $\mathcal{F}$ and $\mathcal{N}$ be two tame bundle maps over τ which are tangent at the order $q+1$, with $q \geq 1$. Assume that their linear part $\mathcal{A}$ satisfies $m|v| \leq |A_x(v)| \leq M|v|$ for any $x \in X$, where $0 < m \leq M < 1$ and $M^{q+1} < m$. Then $\mathcal{F}$ is conjugated to $\mathcal{N}$. More precisely, there exists a ϵ-slow function ρ_ϵ such that $\mathcal{F}$, $\mathcal{N}$ contract the tubes $E(\rho_\epsilon)$, $E(2\rho_\epsilon)$ and a bundle map $\mathcal{T} := \lim_{n\to+\infty} \mathcal{N}^{-n}\mathcal{F}^n$ which is κ-tangent to $\mathcal{I}$ at the order $q+1$ such that the following diagram commutes :*

$$\begin{array}{ccc} E(\rho_\epsilon) & \xrightarrow{\mathcal{F}} & E(\rho_\epsilon) \\ {\scriptstyle\mathcal{T}}\downarrow & & \downarrow{\scriptstyle\mathcal{T}} \\ E(2\rho_\epsilon) & \xrightarrow{\mathcal{N}} & E(2\rho_\epsilon) \end{array} .$$

Moreover, for any $M^{q+1}/m < \theta < 1$, the following estimate

$$\forall n \geq 1 \ , \ \forall v \in E_x(\rho_\epsilon) \ , \ |(N_x^n \circ T_x - F_x^n)(v)| \leq \varphi_\epsilon(x)(m\theta)^n |v|^{q+1}$$

occurs for some ϵ-fast function φ_ϵ.

Let us return to the autonomous case. To obtain a conjugation between F and A it thus would suffice to perform local changes of coordinates cancelling the non-linear terms of order at most q in the Taylor expansion of F. The determination of these changes of coordinates is a purely algebraic problem which yields to the so-called Poincaré's homological equations. As one easily sees, these equations have solutions when there are no resonance relations among the eigenvalues of A. In that case F is

thus linearizable. When resonances do occur, a finite number of monomials in the Taylor expansion of F can not be cancelled. However, as it turns out, these terms added to A define a triangular automorphism N. Then, the sequence $(N^{-n}F^n)_n$ converges and the map F is conjugated to N. This is the Poincaré-Dulac theorem for an holomorphic contraction.

This procedure may be used for a bundle map whose linear part is regular and contracting, a property which we state here formally :

Definition 1.2 *A linear bundle map $\mathcal{A}$ over τ is regular contracting if there exist*
- an integer $1 \leq l \leq k$ and a decomposition $k = k_1 + \ldots + k_l$,
- real numbers $\Lambda_l < \ldots < \Lambda_1 < 0$ and $\epsilon \ll |\Lambda_1|$ such that :

$$\mathcal{A}(\mathcal{L}^j) = \mathcal{L}^j \quad \textit{and} \quad e^{\Lambda_j - \epsilon}|v| \leq |A_x(v)| \leq e^{\Lambda_j + \epsilon}|v| \textit{ for any } (x,v) \in \mathcal{L}^j \textit{ and } 1 \leq j \leq l,$$

where $\mathcal{L}^j := X \times \left[\{0\} \times \ldots \times \mathbb{C}^{k_j} \times \ldots \times \{0\}\right]$ so that $\mathcal{X} = \mathcal{L}^1 \oplus \ldots \oplus \mathcal{L}^l$.

We say that a bundle map is regular contracting if its linear part is. In this setting, the resonances hold on the moduli of the contraction rates of $\mathcal{A}$, that is the Λ_j. These are relations of the form $\alpha \cdot \lambda = \Lambda_j$ where $\alpha := (\alpha_1, \ldots, \alpha_k) \in \mathbb{N}^k$ and $\lambda := (\Lambda_1, \ldots, \Lambda_1, \cdots, \Lambda_j, \ldots, \Lambda_j, \cdots, \Lambda_l, \ldots, \Lambda_l)$ (see the subsection 2.3 for more details). As in the autonomous case, the monomials $z_1^{\alpha_1} \cdots z_k^{\alpha_k}$ in the j-th component of the maps F_x may be cancelled by suitable changes of coordinates if $\alpha \cdot \lambda \neq \Lambda_j$. This is the meaning of the proposition 3.2 which, combined with the above theorem 1.1 gives the following version of Poincaré-Dulac theorem for bundle maps :

Theorem 1.3 *A tame and regular contracting bundle map $\mathcal{G}$ over τ is conjugated to a resonant bundle map $\mathcal{R}$. More precisely there exist a bundle map $\mathcal{V}$ which is κ-tangent to $\mathcal{I}$ and a ϵ-slow function r_ϵ such that the following diagram commutes :*

$$\begin{array}{ccc} E(r_\epsilon) & \xrightarrow{\mathcal{G}} & E(r_\epsilon) \\ \mathcal{V}\downarrow & & \downarrow\mathcal{V} \\ E(2r_\epsilon) & \xrightarrow{\mathcal{R}} & E(2r_\epsilon) \end{array}$$

and $\mathcal{G}, \mathcal{R}$ contract the tubes $E(r_\epsilon), E(2r_\epsilon)$.

The above theorem is a linearization statement ($\mathcal{R} = \mathcal{A}$) when there are no resonances among the Λ_j. Let us however stress that even in the resonant case, the "stability" properties of resonant bundle maps (see propositions 2.7 and 5.1) imply that the iterated bundle maps $\mathcal{R}^n$ and $\mathcal{A}^n$ behave similarly, a fact which is of great importance for our applications.

Similar results were proved by Guysinsky-Katok-Spatzier ([KS], [GK] theorem 1.2) for smooth bundle maps and by Jonsson-Varolin ([JV] theorem 2) in the holomorphic

case. In this article, Jonsson and Varolin proved the Bedford's conjecture on the complex structure of stable manifolds of holomorphic automorphisms in the non-uniform setting. The articles of Fornæss-Stensønes [FS] and Peters [P] are also dedicated to this conjecture.

The originality of our approach is the use of a renormalization technique. It has the advantage of being simpler and also provides an answer to a question asked by Jonsson and Varolin ([JV] final remarks). To our knowledge, this approach also gives the simplest proof of Poincaré-Dulac theorem for an holomorphic contraction (see the survey [B] for a precise exposition). Other simple proofs are due to Sternberg [St] and Rosay-Rudin [RR].

The Oseledec-Pesin reduction theorem (see theorem 4.2) opens a large field of applications for the theorem 1.3 in the setting of smooth ergodic dynamical systems. We will here investigate the case of holomorphic endomorphisms of the complex projective space $\mathbb{CP}^k$. Our aim is to precisely describe the asymptotic behaviour of typical iterated inverse branches for such endomorphisms. Let us recall that the works of Fornaess-Sibony [S] and Briend-Duval [BD] show that any holomorphic endomorphism $f : \mathbb{P}^k \to \mathbb{P}^k$ of algebraic degree $d \geq 2$ induces an ergodic dynamical system $(\mathbb{P}^k, f, \mu)$ where μ is the unique maximal entropy measure of f. This measure is mixing and its Lyapunov exponents $\chi_1 \leq \ldots \leq \chi_k$ are bounded from below by $\log \sqrt{d}$.
We aim to apply the Oseledec-Pesin reduction theorem and the normalization theorem 1.3 to a bundle map generated by the inverse branches of f. For that purpose we work in the set of orbits $\mathsf{O} := \{\hat{x} := (x_n)_{n\in\mathbb{Z}}\,,\ x_{n+1} = f(x_n)\}$. The right shift τ acts on O and leaves invariant a probability measure ν related to μ by $\nu(\pi^{-1}(A)) = \mu(A)$, π being the time zero projection $\pi(\hat{x}) = x_0$. A typical orbit $\hat{x}$ does not intersect the critical locus of f for all n and we may therefore define the inverse branch $f_{\hat{x}}^{-n}$ of f^n that sends x_0 to x_{-n}. Our result compares $f_{\hat{x}}^{-n}$ with its linear tangent map :

Theorem 1.4 *Let f be an holomorphic endomorphism of algebraic degree d on $\mathbb{P}^k$. Let Σ_s be the sum of the s largest Lyapounov exponents of the maximal entropy measure μ. Let ν be the measure induced by μ on the natural extension O of $(\mathbb{P}^k, f, \mu)$ and τ the right shift on O. There exist a full measure subset X of O, ϵ-slow functions $r_\epsilon, t_\epsilon : X \to]0,1]$, a resonant bundle map $\mathcal{R}$ over τ and an injective bundle map $\mathcal{S}$ over Id_X such that the following diagram commutes for all $n \geq 1$ (we note $R^n_{\hat{x}} = R_{\tau^n(\hat{x})} \circ \ldots \circ R_{\hat{x}}$) :*

$$\begin{array}{ccc} B_{x_0}(r_\epsilon(\hat{x})) & \xrightarrow{f_{\hat{x}}^{-n}} & f_{\hat{x}}^{-n}[B_{x_0}(r_\epsilon(\hat{x}))] \\ \downarrow{\scriptstyle S_{\hat{x}}} & & \downarrow{\scriptstyle S_{\tau^n(\hat{x})}} \\ \mathbb{B}(t_\epsilon(\hat{x})) & \xrightarrow{R^n_{\hat{x}}} & \mathbb{B}(t_\epsilon(\tau^n(\hat{x}))) \end{array}$$

There exist also constants $\alpha, M > 0$ and ϵ-fast functions $\beta_\epsilon, L_\epsilon, T_\epsilon : X \to [1, +\infty[$ such that for all $n \geq 0$:

1. $f_{\hat{x}}^{-n}[B_{x_0}(r_\epsilon(\hat{x}))] \subset B_{x_{-n}}(M)$,

2. $\forall (p,q) \in f_{\hat{x}}^{-n}[B_{x_0}(r_\epsilon(\hat{x}))]\,,\ \alpha\, d(p,q) \leq |S_{\tau^n(\hat{x})}(p) - S_{\tau^n(\hat{x})}(q)| \leq \beta_\epsilon(\tau^n(\hat{x}))\, d(p,q)$,

3. $\mathrm{Lip}\, f_{\hat{x}}^{-n} \leq L_\epsilon(\hat{x}) e^{-n\chi_1 + n\epsilon}$ *on* $B_{x_0}(r_\epsilon(\hat{x}))$,

4. for all $p \in f_{\hat{x}}^{-n}[B_{x_0}(r_\epsilon(\hat{x}))]$, $|\frac{1}{n}\log \| \bigwedge^s d_p f^n \| - \Sigma_s| \leq \frac{1}{n}\log T_\epsilon(\hat{x}) + \epsilon$.

The above result is useful for studying the properties of the maximal entropy measure μ. A weak version of it, corresponding to the case where all Lyapunov exponents are equal, has been used in [BDu] for characterizing the endomorphisms f for which μ is absolutely continuous with respect to the Lebesgue measure. In this article, we will use the theorem 1.4 for proving the following approximation formula for Lyapunov exponents :

Theorem 1.5 *We use the same notations than in the previous theorem, recall that* $\Sigma_s = \chi_{k-s+1} + \ldots + \chi_k$. *Let* $\mathbf{R}_n$ *(resp.* $\mathbf{R}_n^*$*) be the set of repulsive periodic points whose period divides* n *(resp. equals* n*). Then :* $\Sigma_s = \lim_{n\to+\infty} \frac{1}{d_t^n} \sum_{p\in\mathbf{R}_n} \frac{1}{n}\log \| \bigwedge^s d_p f^n \| = \lim_{n\to+\infty} \frac{1}{d_t^n} \sum_{p\in\mathbf{R}_n^*} \frac{1}{n}\log \| \bigwedge^s d_p f^n \|$.

Bedford-Lyubich-Smillie [BLS] proved a similar result for the positive Lyapunov exponent of a Hénon map $f : \mathbb{C}^2 \to \mathbb{C}^2$, with $\mathbf{R}_n$ replaced by the n-periodic saddle points. The theorem 1.5 was proved by Szpiro-Tucker for rational maps ($k = 1$) whose coefficients are in a number field ([ST], corollary 6.1). Observe also that for $s = k$, the exterior product $\bigwedge^k d_p f^n$ is the jacobian of f^n at p, which satisfies the multiplicative property $\bigwedge^k d_p f^{m+n} = \bigwedge^k d_{f^n(p)} f^m . \bigwedge^k d_p f^n$. We thus also have :

Corollary 1.6 $\lim_{n\to+\infty} \frac{1}{d_t^n} \sum_{p\in\mathbf{R}_n} \log |\mathrm{Jac}\, d_p f| = \chi_1 + \ldots + \chi_k$.

These approximation formulas have some importance in the study of bifurcations of holomorphic families of endomorphisms of $\mathbb{P}^k$. In particular the theorem 2.2 of Bassanelli-Berteloot [BB] is a consequence of the above corollary.

We may extend the theorems 1.4 and 1.5 to polynomial-like mappings i.e. holomorphic and proper maps $F : U \to V$ between two open sets $U \Subset V \subset \mathbb{C}^k$. The dynamical properties of these maps have been studied by Dinh-Sibony [DS]. They proved in particular the existence of an equilibrium measure μ when $d_t \geq 2$. This measure is mixing and its Lyapunov exponents are positive if μ is PLB (i.e. the plurisubharmonic functions are in $L^1(\mu)$). The theorem 1.5 remains valid if the cardinal of the n-periodic points is asymptotically bounded from above by d_t^n ([DS] subsection 3.5). These conditions are fullfilled e.g. for perturbations of polynomial lifts of endomorphisms of $\mathbb{P}^k$.

2 Generalities

2.1 Slow and fast functions, tube and bundle maps

We introduce in this subsection several notations and definitions. We consider a set X and a fixed bijective map $\tau : X \to X$. A function $\phi_\epsilon : X \to]0,1]$ is ϵ-*slow* if

$\phi_\epsilon(\tau^{\pm 1}(x)) \geq e^{-\epsilon}\phi_\epsilon(x)$. A function $\phi : X \to]0,1]$ is *slow* if ϕ is bounded from below by a ϵ-slow function for all $\epsilon \ll 1$. A function $\psi_\epsilon : X \to [1,+\infty[$ is *ϵ-fast* if $1/\psi_\epsilon$ is ϵ-slow and a function $\psi : X \to [1,+\infty[$ is fast if $1/\psi$ is slow.

Lemma 2.1 *Assume that X is endowed with a τ-invariant probability measure ν. Let $\epsilon > 0$ and $u : X \to]0,1]$ satisfying $\log u \in L^1(\nu)$. Then there exists a ϵ-slow function u_ϵ such that $u_\epsilon \leq u$ for ν-almost every $x \in X$.*

Similarly, if $v : X \to [1,+\infty[$ satisfies $\log v \in L^1(\nu)$ then there exists a ϵ-fast function v_ϵ such that $v \leq v_\epsilon$ for ν-almost every $x \in X$.

Proof : By Birkhoff ergodic theorem, we have $\lim_{n\to\pm\infty} \frac{1}{n}\log u(\tau^n(x)) = 0$ for ν-almost every $x \in X$. So there exists $V : X \to]0,1]$ such that $u(\tau^n(x)) \geq e^{-|n|\epsilon}V(x)$ for all $n \in \mathbb{Z}$. Let $u_\epsilon(x) := \inf_{n\in\mathbb{Z}}\{u(\tau^n(x))e^{|n|\epsilon}\}$. We have $V(x) \leq u_\epsilon(x) \leq u(x)$ and

$$\begin{array}{rcl} u_\epsilon(\tau(x)) & = & \inf_{n\in\mathbb{Z}}\{u(\tau^{n+1}(x))e^{|n|\epsilon}\} \\ & = & e^{-\epsilon}\inf_{n\in\mathbb{Z}}\{\ldots, u(\tau^{-1}(x))e^{3\epsilon}, u(x)e^{2\epsilon}, u(\tau(x))e^{\epsilon}, u(\tau^2(x))e^{2\epsilon}, \ldots\} \\ & \geq & e^{-\epsilon}\inf_{n\in\mathbb{Z}}\{\ldots, u(\tau^{-1}(x))e^{\epsilon}, u(x), u(\tau(x))e^{\epsilon}, u(\tau^2(x))e^{2\epsilon}, \ldots\} \\ & = & e^{-\epsilon}u_\epsilon(x). \end{array}$$

We prove similarly $u_\epsilon(\tau^{-1}(x)) \geq e^{-\epsilon}u_\epsilon(x)$: the function u_ϵ is therefore ϵ-slow. The analogous property for v is obtained by considering $u := 1/v$. $\square$

We set $\mathcal{X} = X \times \mathbb{C}^k$. Let $E_x = \{(x,v) \in \mathcal{X}\ ,\ v \in \mathbb{C}^k\}$ and $E_x(t) := \{(x,v) \in \mathcal{X}\ ,\ |v| < t\}$. For any function $r : X \to [0,+\infty[$, we note $E_x(r) := E_x(r(x))$ and $E(r) = \bigcup_{x\in X} E_x(r(x))$. The subset $E(r) \subset \mathcal{X}$ is the *tube* of radius r. We say that $E(r)$ is *slow* (resp. ϵ-slow) if r is slow (resp. ϵ-slow). Idem with the "fast" terminology.

Let $\sigma \in \{\mathrm{Id}_X, \tau\}$. A holomorphic bundle map $\mathcal{K} : E(r) \to \mathcal{X}$ over σ is a map satisfying $\mathcal{K}(x,v) = (\sigma(x), K_x(v))$, where $K_x : E_x(r) \to E_{\sigma(x)}$ is holomorphic and $K_x(0) = 0$. We say that $\mathcal{K}$ is *tame* if there exists $\epsilon_0 > 0$ such that for all $\epsilon < \epsilon_0$ there exist a ϵ-slow function r_ϵ and a ϵ-fast function s_ϵ with $\mathcal{K} : E(r_\epsilon) \to E(s_\epsilon)$.

A *stable* bundle map, that is a map of the form $\mathcal{K} : E(r) \to E(r)$, may be iterated. The *n-th iterate* $\mathcal{K}^n$ is defined by $K_x^n := K_{\sigma^{n-1}(x)} \circ \cdots \circ K_x$.

When it makes sense, we consider $\mathcal{K}^{-1}$ the *inverse* bundle map of $\mathcal{K}$. We note $\mathcal{K}^{-1}(x,v) = (\sigma^{-1}(x), K_x^{-1}(v))$ where $K_x^{-1} := (K_{\sigma^{-1}(x)})^{-1} : E_x \to E_{\sigma^{-1}(x)}$ (this map is defined in a neighboorhood of $0 \in E_x$).

We note $\mathrm{Lip}\,(\mathcal{K}) := \sup_{x\in X}\mathrm{Lip}\,(K_x)$ and say that $\mathcal{K}$ *contracts* $E(r)$ if $\mathrm{Lip}\,(\mathcal{K}) < 1$ on $E(r)$. The tube $E(r)$ is *stable* if $\mathcal{K}(E(r)) \subset E(r)$. Let $\mathcal{I}$ be the identity map of $\mathcal{X}$ over Id_X and $\mathcal{O}$ the zero bundle map over τ, which sends $\mathcal{X}$ to $X \times \{0\}$. We say that a bundle map $\mathcal{K}$ over Id_X is *κ-tangent* to $\mathcal{I}$ if $\mathrm{Lip}\,(\mathcal{K} - \mathcal{I}) < \kappa$.

Let $m \geq 1$. A bundle map $\mathcal{K}$ is *homogeneous of degree m* (or *m-homogeneous*) if the map K_x is homogeneous of degree m for all $x \in X$. Let $\sum_{m\geq 1}\mathcal{K}^{(m)}$ be the Taylor

expansion of $\mathcal{K}$, where $\mathcal{K}^{(m)}$ is m-homogeneous. The *linear part* of $\mathcal{K}$ is $\mathcal{K}^{(1)}$. A bundle map $\mathcal{K}$ is *polynomial* if there exists $m_0 \geq 1$ such that $\mathcal{K} = \sum_{m=1}^{m_0} \mathcal{K}^{(m)}$.

Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be two bundle maps over $\sigma \in \{\mathrm{Id}_X, \tau\}$. We say that $\mathcal{K}_1$ is *tangent to $\mathcal{K}_2$ at the order $m+1$* if $\mathcal{K}_1$ and $\mathcal{K}_2$ share the same Taylor expansion up to the order m. We then note $\mathcal{K}_1 = \mathcal{K}_2 + O(m+1)$. When $m = 1$, we just say that $\mathcal{K}_1$ is *tangent* to $\mathcal{K}_2$. If $\mathcal{K}_1$ and $\mathcal{K}_2$ are bundle maps over τ, we say that $\mathcal{K}_1$ is *conjugated to $\mathcal{K}_2$* if there exist a bundle map $\mathcal{W}$ tangent to $\mathcal{I}$ and positive functions r, s such that the following diagram is commutative :

$$\begin{array}{ccc} E(r) & \xrightarrow{\mathcal{K}_1} & E(r) \\ \mathcal{W}\downarrow & & \downarrow\mathcal{W} \\ E(s) & \xrightarrow{\mathcal{K}_2} & E(s) \end{array}$$

We say that $\mathcal{K}_1$ is *conjugated to $\mathcal{K}_2$ at the order $m+1$* if $\mathcal{K}_1$ is conjugated to a bundle map $\tilde{\mathcal{K}}_2$ which is tangent to $\mathcal{K}_2$ at the order $m+1$.

Let $|\mathcal{K}^{(m)}|(x) := \max_{v\neq 0} |K_x^{(m)}(v)|/|v|^m$ (also denoted $|K_x^{(m)}|$). For any $\alpha = (\alpha_1, \ldots, \alpha_k) \in \mathbb{N}^k$, we note $|\alpha| := \alpha_1 + \ldots + \alpha_k$ and $P_\alpha(z_1, \ldots, z_k) = z_1^{\alpha_1} \ldots z_k^{\alpha_k}$. Then $|\mathcal{K}^{(m)}|(x)$ is equal (up to a constant) to the maximum of the coefficients of $K_x^{(m)}$ with respect to the basis $\{P_\alpha e_i\,,\ |\alpha| = m\,,\ 1 \leq i \leq k\}$, $(e_i)_{1\leq i\leq k}$ being the canonical basis of $\mathbb{C}^k$. The notation $|\,.\,|$ will also be used for the standard hermitian norm on the spaces $(\bigwedge^s \mathbb{C}^k)_{1\leq s\leq k}$ and for the norm of operators : $\bigwedge^s \mathbb{C}^k \to \bigwedge^s \mathbb{C}^k$.

2.2 Two simple lemmas on bundle maps

Lemma 2.2 *Let $\sigma \in \{\mathrm{Id}_X, \tau\}$ and $\mathcal{K} : E(r) \to E(s)$ be a bundle map over σ. For all $m \geq 1$, $|\mathcal{K}^{(m)}| \leq s \circ \sigma / r^m$. In particular $|\mathcal{K}^{(m)}|$ is a fast function when $\mathcal{K}$ is tame.*

Proof : Let $K_x : E_x(r) \to E_{\sigma(x)}(s)$. For all $v \neq 0$ and $\rho < r(x)$, we have :

$$\frac{\rho^m}{|v|^m} K_x^{(m)}(v) = K_x^{(m)}\left(\rho \frac{v}{|v|}\right) = \frac{1}{2\pi} \int_0^{2\pi} K_x\left(\rho \frac{v}{|v|} e^{i\theta}\right) e^{-im\theta}\, d\theta.$$

The lemma follows by taking the norm and the limits when ρ tends to $r(x)$. $\square$

Lemma 2.3 *Let $\sigma \in \{\mathrm{Id}_X, \tau\}$ and $\mathcal{K}$ be a tame bundle map over σ. Let $\mathcal{D}$ be the linear part of $\mathcal{K}$. Assume that there exist $0 < a \leq b$ such that $a|v| \leq |D_x(v)| \leq b|v|$. Let $\gamma, \kappa > 0$. Then for $\epsilon \ll 1$ there exists a ϵ-slow function ϕ_ϵ such that :*

1. $\forall (u, v) \in E_x(\phi_\epsilon), ae^{-\gamma}|u - v| \leq |K_x(u) - K_x(v)| \leq be^{\gamma}|u - v|$.

   In particular if $be^{\gamma} \leq e^{-\epsilon}$ then the tube $E(\phi_\epsilon)$ is stable by $\mathcal{K}$.

2. $\mathrm{Lip}\,(\mathcal{K} - \mathcal{D}) \leq \kappa$ *on* $E(\phi_\epsilon)$.

3. If $\mathcal{D}=\mathcal{I}$, $|\bigwedge^s \mathrm{Id}_{\mathbb{C}^k} - \bigwedge^s (d_t K_x)^{\pm 1}| \leq \frac{1}{10}$ for all $t \in E_x(\phi_\epsilon)$ and $1 \leq s \leq k$.

PROOF : Let $\epsilon' = \epsilon/3$ with $\epsilon \ll 1$. As $\mathcal{K}$ is tame, there exist a ϵ'-slow function $r_{\epsilon'}$ and a ϵ'-fast function $s_{\epsilon'}$ such that $\mathcal{K} : E(r_{\epsilon'}) \to E(s_{\epsilon'})$. Let $x \in X$. The Cauchy's estimates on $E_x(r_{\epsilon'}/2)$ bound the second derivatives of K_x by $c\, s_{\epsilon'}(\sigma(x))/r_{\epsilon'}(x)^2$, where c is a constant depending only on the dimension k. We deduce that for all $0 \leq \rho \leq r_{\epsilon'}(x)/2$:

$$\forall t \in E_x(\rho)\ ,\ |d_t(D_x - K_x)| = |d_0 K_x - d_t K_x| \leq \frac{c\, s_{\epsilon'}(\sigma(x))}{r_{\epsilon'}(x)^2}\rho. \tag{1}$$

Let $\eta < 1$ be such that for any $1 \leq s \leq k$ and any linear map $L : \mathbb{C}^k \to \mathbb{C}^k$, $|\mathrm{Id}_{\mathbb{C}^k} - L| < \eta$ implies $|\bigwedge^s \mathrm{Id}_{\mathbb{C}^k} - \bigwedge^s L^{\pm 1}| \leq \frac{1}{10}$. Define the ϵ-slow function ϕ_ϵ by:

$$\phi_\epsilon := \frac{r_{\epsilon'}^2}{c\, s_{\epsilon'} \circ \sigma} \min\{(e^\gamma - 1)b\, ,\, (1 - e^{-\gamma})a\, ,\, \kappa, \eta\}.$$

As $\phi_\epsilon \leq \frac{r_{\epsilon'}^2}{c\, s_{\epsilon'} \circ \sigma}$ and $r_{\epsilon'} \leq 1 \leq s_{\epsilon'}$, we may assume that $\phi_\epsilon \leq r_{\epsilon'}/2$ by taking $c \geq 2$. Making $\rho = \phi_\epsilon$ in (1) we get the following estimates on $E_x(\phi_\epsilon)$:

$$\begin{array}{rcl} |K_x(u) - K_x(v)| & \leq & |D_x(u) - D_x(v)| + |(D_x - K_x)(u) - (D_x - K_x)(v)| \\ & \leq & b|u - v| + \frac{c\, s_{\epsilon'}(\sigma(x))}{r_{\epsilon'}(x)^2}\phi_\epsilon(x)|u - v| \\ & \leq & b e^\gamma |u - v|. \end{array}$$

We have similarly $|K_x(u) - K_x(v)| \geq a e^{-\gamma}|u - v|$. If $b e^\gamma \leq e^{-\epsilon}$, then $|K_x(u)| \leq b e^\gamma \phi_\epsilon(x) \leq e^{-\epsilon}\phi_\epsilon(x) \leq \phi_\epsilon(\sigma(x))$ on $E(\phi_\epsilon)$, and the tube $E(\phi_\epsilon)$ is stable by $\mathcal{K}$. The points 2 and 3 are also a consequences of (1) with $\rho = \phi_\epsilon$, using respectively the estimates $\phi_\epsilon \leq \kappa\, r_{\epsilon'}^2/c\, s_{\epsilon'} \circ \sigma$ and $\phi_\epsilon \leq \eta\, r_{\epsilon'}^2/c\, s_{\epsilon'} \circ \sigma$. □

2.3 Resonances

In this subsection, we will consider a linear bundle map $\mathcal{A}$ over τ which is regular contracting. Our aim is to discuss the resonances associated to such a bundle map. According to definition 1.2, there exist $k = k_1 + \ldots + k_l$, $\Lambda_l < \ldots < \Lambda_1 < 0$ and $\epsilon \ll |\Lambda_1|$ such that

$$\forall 1 \leq j \leq l\ ,\ \mathcal{L}^j := X \times \left[\{0\} \times \ldots \times \mathbb{C}^{k_j} \times \ldots \times \{0\}\right],$$

$$\mathcal{A}(\mathcal{L}^j) = \mathcal{L}^j \ \text{ and } \ \forall (x, v) \in \mathcal{L}^j\ ,\ e^{\Lambda_j - \epsilon}|v| \leq |A_x(v)| \leq e^{\Lambda_j + \epsilon}|v|.$$

Observe that the matrix of A_x in the canonical basis is block diagonal.

Let us now consider a bundle map $\mathcal{K} : E(r) \to \mathcal{X}$ whose linear part is equal to $\mathcal{A}$. Let $\pi_j(\mathcal{K}) : E(r) \to \mathcal{L}^j$ be the j-th component of $\mathcal{K}$ with respect to the splitting $\mathcal{X} = \oplus_{1 \leq j \leq l}\mathcal{L}^j$. If $\alpha \in \mathbb{N}^k$, we note $\pi_j^\alpha(\mathcal{K}) : \mathcal{X} \to \mathcal{L}^j$ the homogeneous part of degree α in $\pi_j(\mathcal{K})$.

We set $(\lambda_1, \ldots, \lambda_k) := (\Lambda_1, \ldots, \Lambda_1, \cdots, \Lambda_j, \ldots, \Lambda_j, \cdots, \Lambda_l, \ldots, \Lambda_l)$, where Λ_j appears k_j times. If $\alpha = (\alpha_1, \ldots, \alpha_k) \in \mathbb{N}^k$, we note $\alpha \cdot \lambda := \alpha_1\lambda_1 + \ldots + \alpha_k\lambda_k$. We shall denote by $\tilde{q} \geq 1$ the entire part of Λ_l/Λ_1.

Definition 2.4 *Let $\mathcal{A}$ be the above linear bundle map. For any $1 \leq j \leq l$, the set $\mathfrak{R}_j$ of j-*resonant degrees *is defined by :*

$$\mathfrak{R}_j := \{\alpha \in \mathbb{N}^k \ , \ |\alpha| \geq 2 \ \ \textit{and} \ \ \alpha \cdot \lambda = \Lambda_j\}.$$

*The set $\mathfrak{B}_j$ of j-*subresonant degrees *is defined similarly with $\alpha \cdot \lambda < \Lambda_j$ and the set $\mathfrak{P}_j$ of j-*superresonant degrees *with $\alpha \cdot \lambda > \Lambda_j$. We set $\mathfrak{B} := \cup_{j=1}^l \mathfrak{B}_j$, $\mathfrak{R} := \cup_{j=1}^l \mathfrak{R}_j$ and $\mathfrak{P} := \cup_{j=1}^l \mathfrak{P}_j$.*

It should be observed that the sets $\mathfrak{R}$, $\mathfrak{B}$ and $\mathfrak{P}$ are unchanged if $\mathcal{A}$ is replaced by $\mathcal{A}^n$. As $\Lambda_l < \ldots < \Lambda_1 < 0$ one sees that $\{|\alpha| \geq \tilde{q}+1\} \subset \mathfrak{B}$ and $\mathfrak{R} \cup \mathfrak{P} \subset \{2 \leq |\alpha| \leq \tilde{q}\}$. In particular, $\mathfrak{R}$ and $\mathfrak{P}$ are finite sets. Observe also that the set $\mathfrak{R}_1$ is empty, and that $\alpha_i = 0$ for $\alpha \in \mathfrak{R}_j$ and $i \geq k_1 + \ldots + k_{j-1} + 1$. In particular for any $j \geq 2$ and $\alpha \in \mathfrak{R}_j$, the bundle map $\pi_j^\alpha(\mathcal{K})$ may be viewed as a bundle map from $\mathcal{L}^1 \oplus \ldots \oplus \mathcal{L}^{j-1}$ to $\mathcal{L}^j$:

$$\pi_j^\alpha(\mathcal{K}) \ : \ E(r) \cap \left[\mathcal{L}^1 \oplus \ldots \oplus \mathcal{L}^{j-1}\right] \longrightarrow \mathcal{L}^j. \tag{2}$$

The following lemma will be used in subsection 3.2, the proof is left to the reader. We recall that $\epsilon \ll |\Lambda_1|$.

Lemma 2.5 *There exists $\zeta > 0$ such that for all $1 \leq j \leq l$:*

1. *if $\alpha \in \mathfrak{B}_j$ then $\alpha \cdot \lambda - \Lambda_j + (|\alpha| + 2)\epsilon \leq -\zeta$.*
2. *if $\alpha \in \mathfrak{P}_j$ then $\alpha \cdot \lambda - \Lambda_j - (|\alpha| + 2)\epsilon \geq \zeta$.*

The disjoint sets $\mathfrak{R}, \mathfrak{B}, \mathfrak{P}$ lead to the following decomposition for $\mathcal{K} : E(r) \to \mathcal{X}$:

$$\mathcal{K} = \mathcal{K}^{(1)} + \sum_{1 \leq j \leq l \, , \, \alpha \in \mathfrak{R}_j} \pi_j^\alpha(\mathcal{K}) + \sum_{1 \leq j \leq l \, , \, \alpha \in \mathfrak{B}_j} \pi_j^\alpha(\mathcal{K}) + \sum_{1 \leq j \leq l \, , \, \alpha \in \mathfrak{P}_j} \pi_j^\alpha(\mathcal{K}).$$

These three sums are respectively denoted $\mathfrak{R}(\mathcal{K})$, $\mathfrak{B}(\mathcal{K})$, $\mathfrak{P}(\mathcal{K})$ and called the *resonant part* and the *sub/superresonant parts* of $\mathcal{K}$.

Definition 2.6 *A bundle map $\mathcal{K}$ over τ is resonant if $\mathfrak{B}(\mathcal{K}) = \mathfrak{P}(\mathcal{K}) = \mathcal{O}$ (the zero bundle map over τ). In other words, $\mathcal{K}$ is resonant if $\mathcal{K} = \mathcal{A} + \mathfrak{R}(\mathcal{K}) = \mathcal{K}^{(1)} + \mathfrak{R}(\mathcal{K})$.*

The following classical result will be crucial. It asserts that resonant bundle maps enjoy strong stability properties under iteration (see section 5 for a proof).

Proposition 2.7 *Let $\mathcal{K}$ be a resonant bundle map. For any $n \geq 1$, we have $\mathfrak{B}(\mathcal{K}^n) = \mathfrak{P}(\mathcal{K}^n) = \mathcal{O}$. In particular $\mathcal{K}^n = \mathcal{A}^n + \mathcal{R}(\mathcal{K}^n)$ and the degree of $\mathcal{K}^n$ is bounded by $\tilde{q}$.*

Remark 2.8 *Let $\mathcal{K}$ be a resonant bundle map. It follows from (2) and the proposition 2.7 that for any $x \in X$, $w \in E_x$ and $n \geq 1$, the matrix of the differential $d_w K_x^n$ is "block lower triangular", with a block diagonal part equal to A_x^n.*

The following result compares $|\bigwedge^s d_w K_x^n|$ and $|\bigwedge^s A_x^n|$ (see section 5).

Proposition 2.9 *Let $\mathcal{K} : E(\rho_\epsilon) \to E(\rho_\epsilon)$ be a resonant bundle map where ρ_ϵ is a ϵ-slow function. There exists $\eta > 0$ (depending only on $\tilde{q}, k$ and not on $\mathcal{K}$) and a $\eta\epsilon$-fast function $H_{\eta\epsilon} : X \to [1, +\infty[$ such that for all $w \in E_x(\rho_\epsilon)$, $s \in \{1, \ldots, k\}$ and $n \geq 1$:*

$$\Big|\frac{1}{n}\log|\bigwedge^s d_w K_x^n| - (\lambda_1 + \ldots + \lambda_s)\Big| \leq \frac{1}{n}\log H_{\eta\epsilon}(x) + \eta\epsilon.$$

3 Normalization of contracting bundle maps

This section is devoted to the proof of theorem 1.3. We essentially proceed in two steps. We first prove that sufficently tangent bundle maps are conjugated. We then show that any bundle map satisfying the assumptions of theorem 1.3 is tangent to a resonant bundle map.

3.1 High tangency implies conjugation

The aim of this subsection is to prove the theorem 1.1 stated in the introduction.

Proof : Let $M^{q+1}/m < \theta < 1$ and $\kappa > 0$. We set $\sigma := \sum_{j\geq 0}\theta^j$ and $\beta := e^{(q+2)\epsilon}M^{q+1}/m$ where ϵ is so small that $Me^\epsilon \leq e^{-\epsilon}$ and $\beta e^\epsilon \leq \theta$.

To establish the theorem, we will show that there exist a ϵ-slow function r_ϵ and a ϵ-fast function D_ϵ such that the map $T_{x,n} := N_{\tau^n(x)}^{-n} F_x^n$ is well defined on $E_x(r_\epsilon)$ and the following estimates hold on $E_x(r_\epsilon)$:

$$\max\{|F_x(v)|\ ,\ |N_x(v)|\} \leq Me^\epsilon|v|, \tag{3}$$

$$|T_{x,n+1}(v) - T_{x,n}(v)| \leq \beta^n D_\epsilon(\tau^n(x))|v|^{q+1}. \tag{4}$$

Let us note that for any ϵ-slow function $\rho_\epsilon \leq \frac{r_\epsilon}{2}$ the inequality $Me^\epsilon \leq e^{-\epsilon}$ and (3) implies that $E(\rho_\epsilon)$ and $E(2\rho_\epsilon)$ are stable by $\mathcal{F}$ and $\mathcal{N}$.

We start by showing how the theorem 1.1 follows from (3) and (4). We check the convergence of $T_{x,n}$ on $E(r_\epsilon)$ and define the functions φ_ϵ and $\rho_\epsilon \leq r_\epsilon$. The estimate (4) implies for all $v \in E_x(r_\epsilon)$:

$$|T_{x,n+1}(v) - T_{x,n}(v)| \leq \beta^n D_\epsilon(x)e^{n\epsilon}|v|^{q+1} \leq \theta^n D_\epsilon(x)|v|^{q+1}. \tag{5}$$

Therefore $T_x := \lim_{n\to+\infty} T_{x,n}$ exists on $E_x(r_\epsilon)$ and satisfies $T_x = N_{\tau(x)}^{-1} T_{\tau(x)} F_x$. We define $\varphi_\epsilon := \sigma D_\epsilon$ and $\rho_\epsilon := \min\{\frac{r_\epsilon}{2}, \kappa/\varphi_\epsilon\}$. Observe that (5) implies on $E_x(\rho_\epsilon)$:

$$|T_x(v) - v| \leq \sum_{n\geq 0}\theta^n D_\epsilon(x)|v|^{q+1} = \varphi_\epsilon(x)|v|^{q+1}.$$

Thus, by the very definition of ρ_ϵ, the commutative diagram holds and $\mathcal{T}$ is κ-tangent to $\mathcal{I}$. The commutative relation $T_x = N_{\tau(x)}^{-1} T_{\tau(x)} F_x$ implies for all $v \in E_x(\rho_\epsilon)$:

$$|N_x^n T_x(v) - F_x^n(v)| = |T_{\tau^n(x)} F_x^n(v) - F_x^n(v)| \leq \varphi_\epsilon(\tau^n(x)) |F_x^n(v)|^{q+1}.$$

By the ϵ-fast property of φ_ϵ, (3) and the stability of $E(\rho_\epsilon)$, the right hand side is lower than :

$$\varphi_\epsilon(x) e^{n\epsilon} (M^n e^{n\epsilon})^{q+1} |v|^{q+1} = \varphi_\epsilon(x)(m\beta)^n |v|^{q+1} \leq \varphi_\epsilon(x)(m\theta)^n |v|^{q+1},$$

which completes the proof of the theorem 1.1.

We shall now define the functions r_ϵ, D_ϵ and establish the estimates (3) and (4). Let $\epsilon' := \frac{\epsilon}{q+2}$. By the lemma 2.3(1), there exists a ϵ'-slow function $\phi_{\epsilon'}$ such that (3) and the following property hold on $E(\phi_{\epsilon'})$:

$$\forall (u,v) \in E_x(\phi_{\epsilon'}) \ , \ me^{-\epsilon}|u-v| \leq |N_x(u) - N_x(v)| \leq Me^{\epsilon}|u-v|. \tag{6}$$

Define $C_\epsilon := 4/\phi_{\epsilon'}^{q+1}$, $D_\epsilon := C_\epsilon / me^{-\epsilon}$ and

$$r_\epsilon := \min \left\{ \frac{\phi_{\epsilon'}}{2} \ , \ \frac{me^{-\epsilon}\phi_{\epsilon'}}{\sigma C_\epsilon} \ , \ \frac{me^{-\epsilon}\phi_{\epsilon'}}{Me^{\epsilon}} \right\}.$$

The functions C_ϵ, D_ϵ are $\frac{q+1}{q+2}\epsilon$-fast and r_ϵ is ϵ-slow. Since $r_\epsilon \leq \phi_{\epsilon'}$, the estimate (3) is satisfied on $E(r_\epsilon)$. In particular, as $Me^{\epsilon} \leq e^{-\epsilon}$, $E(r_\epsilon)$ is stable by $\mathcal{F}$ and $\mathcal{N}$. Also it follows from (6) that N_x is invertible on $E_{\tau(x)}(me^{-\epsilon}\phi_{\epsilon'}(x))$ and satisfies :

$$\forall (u',v') \in E_{\tau(x)}(me^{-\epsilon}\phi_{\epsilon'}(x)) \ , \ |N_{\tau(x)}^{-1}(u') - N_{\tau(x)}^{-1}(v')| \leq \frac{1}{me^{-\epsilon}}|u'-v'|. \tag{7}$$

In order to establish (4) we need the following lemma. The last point relies on the tangency of $\mathcal{F}$ and $\mathcal{N}$ at the order $q+1$.

Lemma 3.1 *For all $v \in E_x(r_\epsilon)$ and $n \geq 0$:*

1. $\left[\sum_{j=0}^n C_\epsilon(\tau^j(x))\beta^j\right] |v| \leq me^{-\epsilon}\phi_{\epsilon'}(x)$.
2. $\max\{|F_x(v)| \ , \ |N_x(v)|\} \leq me^{-\epsilon}\phi_{\epsilon'}(x)$.
3. $|F_x(v) - N_x(v)| \leq C_\epsilon(x)|v|^{q+1}$.

Proof : By the ϵ-fast property of C_ϵ, the left hand side in the point 1 is lower than

$$\sum_{j=0}^n (\beta e^{\epsilon})^j C_\epsilon(x)|v| \leq \sigma C_\epsilon(x) r_\epsilon(x) \leq \sigma C_\epsilon(x) \frac{me^{-\epsilon}\phi_{\epsilon'}(x)}{\sigma C_\epsilon(x)} = me^{-\epsilon}\phi_{\epsilon'}(x).$$

The point 2 is a consequence of (3) :

$$\max\{|F_x(v)|\ ,\ |N_x(v)|\} \leq Me^{\epsilon} r_\epsilon(x) \leq Me^{\epsilon}\frac{me^{-\epsilon}\phi_{\epsilon'}(x)}{Me^{\epsilon}} = me^{-\epsilon}\phi_{\epsilon'}(x).$$

We now come to the last point. By lemma 2.2 and the stability of $E(r_\epsilon)$ by $\mathcal{F}$ and $\mathcal{N}$, we have for all $j \geq 1$ and $v \in E_x(r_\epsilon)$:

$$\max\left\{|F_x^{(j)}(v)|\ ,\ |N_x^{(j)}(v)|\right\} \leq \frac{\phi_{\epsilon'}(\tau(x))}{\phi_{\epsilon'}^j(x)}|v|^j \leq \left(\frac{|v|}{\phi_{\epsilon'}(x)}\right)^j \leq \left(\frac{r_\epsilon(x)}{\phi_{\epsilon'}(x)}\right)^j \leq \frac{1}{2^j}.$$

As $\mathcal{F}$ and $\mathcal{N}$ are tangent at the order $q+1$, we deduce that :

$$|F_x(v) - N_x(v)| \leq 2\left(\frac{|v|}{\phi_{\epsilon'}(x)}\right)^{q+1}\sum_{m\geq 0}\frac{1}{2^m} = 4\frac{|v|^{q+1}}{\phi_{\epsilon'}^{q+1}(x)} = C_\epsilon(x)|v|^{q+1}. \qquad \square$$

To end the proof of the theorem, we establish (4) by induction. Let us rewrite (4) explicitely :

$$(p_n)\ :\ \forall v \in E_x(r_\epsilon)\ ,\ |N_{\tau^{n+1}(x)}^{-(n+1)}F_x^{n+1}(v) - N_{\tau^n(x)}^{-n}F_x^n(v)| \leq \beta^n D_\epsilon(\tau^n(x))|v|^{q+1}.$$

The proof of $(p_n) \Rightarrow (p_{n+1})$ will use the following auxiliary inequality :

$$(q_n)\ :\ \forall v \in E_x(r_\epsilon)\ ,\ |N_{\tau^{n+1}(x)}^{-n}F_x^{n+1}(v)| \leq \left[\sum_{j=0}^{n} C_\epsilon(\tau^j(x))\beta^j\right]|v|.$$

The assertion (p_0) is a consequence of (7) and the two last points of lemma 3.1 :

$$|N_{\tau(x)}^{-1}(F_x(v)) - N_{\tau(x)}^{-1}(N_x(v))| \leq \frac{1}{me^{-\epsilon}}|F_x(v) - N_x(v)| \leq D_\epsilon(x)|v|^{q+1}.$$

The assertion (q_0) follows from (3) and the observation $Me^{\epsilon} \leq 1 \leq C_\epsilon$.

Assume now that (p_n) and (q_n) are satisfied. Let $v' \in E_{x'}(r_\epsilon)$, $v := F_{x'}(v')$ and $x := \tau(x')$. Observe that $v \in E_x(r_\epsilon)$ because the tube $E(r_\epsilon)$ is stable by $\mathcal{F}$. Using $D_\epsilon = C_\epsilon/me^{-\epsilon}$ and (3), the assertion (p_n) yields :

$$\begin{aligned} |N_{\tau^{n+2}(x')}^{-(n+1)}F_{x'}^{n+2}(v') - N_{\tau^{n+1}(x')}^{-n}F_{x'}^{n+1}(v')| &\leq \beta^n\tfrac{C_\epsilon(\tau^{n+1}(x'))}{me^{-\epsilon}}|F_{x'}(v')|^{q+1} \\ &\leq \beta^n\tfrac{C_\epsilon(\tau^{n+1}(x'))}{me^{-\epsilon}}(Me^{\epsilon})^{q+1}|v'|^{q+1}, \end{aligned}$$

which, by the definition of β, leads to :

$$|N_{\tau^{n+2}(x')}^{-(n+1)}F_{x'}^{n+2}(v') - N_{\tau^{n+1}(x')}^{-n}F_{x'}^{n+1}(v')| \leq \beta^{n+1}C_\epsilon(\tau^{n+1}(x'))|v'|^{q+1}. \tag{8}$$

We now deduce (q_{n+1}) from (q_n) and (8). For all $v' \in E_{x'}(r_\epsilon)$ we have :

$$\begin{aligned} |N_{\tau^{n+2}(x')}^{-(n+1)}F_{x'}^{n+2}(v')| &\leq \beta^{n+1}C_\epsilon(\tau^{n+1}(x'))|v'|^{q+1} + \left[\textstyle\sum_{j=0}^{n} C_\epsilon(\tau^j(x'))\beta^j\right]|v'| \\ &\leq \left[\textstyle\sum_{j=0}^{n+1} C_\epsilon(\tau^j(x'))\beta^j\right]|v'|. \end{aligned}$$

It follows from (q_n), (q_{n+1}) and the lemma 3.1-(1) that $|N^{-(n+1)}_{\tau^{n+2}(x')}F^{n+2}_{x'}(v')|$ and $|N^{-n}_{\tau^{n+1}(x')}F^{n+1}_{x'}(v')|$ are lower than $me^{-\epsilon}\phi_{\epsilon'}(x')$. We may thus compose (8) by $N^{-1}_{\tau(x')}$ and, using (7), get :

$$\begin{aligned} |N^{-(n+2)}_{\tau^{n+2}(x')}F^{n+2}_{x'}(v') - N^{-(n+1)}_{\tau^{n+1}(x')}F^{n+1}_{x'}(v')| &\leq \beta^{n+1}\frac{C_\epsilon(\tau^{n+1}(x'))}{me^{-\epsilon}}|v'|^{q+1} \\ &\leq \beta^{n+1}D_\epsilon(\tau^{n+1}(x'))|v'|^{q+1}, \end{aligned}$$

which is the assertion (p_{n+1}). □

3.2 Normalization at a finite order

We prove here that a bundle map which satisfies the assumptions of theorem 1.3 is, for any prescribed order, tangent to some resonant bundle map :

Proposition 3.2 *Let $\mathcal{G}$ be a tame and regular contracting bundle map over τ. For any $p \geq 1$, there exists a resonant bundle map $\mathcal{R}$ such that $\mathcal{G}$ is tangent to $\mathcal{R}$ at the order $p+1$. More precisely, for any $\epsilon > 0$ and $0 < \kappa < 1$, there exist a ϵ-slow function σ_ϵ, a bundle map $\mathcal{U}$ which is κ-tangent to $\mathcal{I}$ and a bundle map $\mathcal{F} = \mathcal{R} + O(p+1)$ such that the following diagram commutes :*

$$\begin{array}{ccc} E(\sigma_\epsilon) & \xrightarrow{\mathcal{G}} & E(\sigma_\epsilon) \\ \mathcal{U}\downarrow & & \downarrow\mathcal{U} \\ E(2\sigma_\epsilon) & \xrightarrow{\mathcal{F}} & E(2\sigma_\epsilon) \end{array}$$

and the tubes $E(\sigma_\epsilon), E(2\sigma_\epsilon)$ are contracted.

Following Jonsson-Varolin ([JV], lemma 5.2), we first solve the Poincaré's homological equation in a non-autonomous setting.

Proposition 3.3 *Let $\mathcal{A}$ be a linear regular contracting bundle map (see definition 1.2). Let $\mathcal{H} = \mathcal{H}^{(m)}$ be a m-homogeneous bundle map over τ and $\mathfrak{R}(\mathcal{H})$ be its resonant part. Assume that $|\mathcal{H}|$ is a fast function. Then there exists a m-homogeneous bundle map $\mathcal{Q}$ over Id_X such that $|\mathcal{Q}|$ is a fast function and $\mathcal{H} + (\mathcal{Q}\circ\mathcal{A} - \mathcal{A}\circ\mathcal{Q}) = \mathfrak{R}(\mathcal{H})$.*

Proof : It suffices to consider a bundle map $\mathcal{M} := \pi^\alpha_j(\mathcal{H})$, where $|\alpha| = m$ and $1 \leq j \leq k$. We treat the three cases $\alpha \in \mathfrak{R}_j, \mathfrak{B}_j$ and $\mathfrak{P}_j$ separately. If $\alpha \in \mathfrak{R}_j$, we let $\mathcal{Q} = 0$. If $\alpha \in \mathfrak{B}_j$, we set $\mathcal{Q} := \sum_{n\geq 0}\mathcal{A}^{-(n+1)}\mathcal{M}\mathcal{A}^n$, i.e.

$$\forall x \in X\ ,\ Q_x = A^{-1}_{\tau(x)}M_x + \sum_{n\geq 1}\left[A^{-1}_{\tau(x)}\dots A^{-1}_{\tau^{n+1}(x)}\,M_{\tau^n(x)}\,A_{\tau^{n-1}(x)}\dots A_x\right]. \qquad (9)$$

A formal computation shows that $\mathcal{Q}$ is m-homogeneous and satisfies $M_x + (Q_{\tau(x)}\circ A_x - A_x\circ Q_x) = 0 = \mathfrak{R}(M_x)$ for all $x \in X$. We now check the convergence of the series (9), which we denote $Q_x := \sum_{n\geq 0}Q_{n,x}$. Let $\psi_\epsilon \geq |\mathcal{H}| \geq |\mathcal{M}|$ be a ϵ-fast function. Observe

that each of the k_j coordinates of $\mathcal{M} : \mathcal{X} \to \mathcal{L}^j$ is a multiple of the monomial P_α. We have therefore by the ϵ-fast property of ψ_ϵ and the regular contracting property of $\mathcal{A}$:

$$\begin{array}{rcl} \forall n \geq 1 \ , \ |Q_{n,x}(v)| & \leq & e^{-(n+1)(\Lambda_j-\epsilon)} \left[\psi_\epsilon(x) e^{n\epsilon}\right] e^{n(\lambda\cdot\alpha+|\alpha|\epsilon)} |v|^m \\ & \leq & e^{n[\lambda\cdot\alpha-\Lambda_j+(|\alpha|+2)\epsilon]} e^{-\Lambda_j+\epsilon} \psi_\epsilon(x)|v|^m \\ & \leq & e^{-n\zeta} e^{-\Lambda_j+\epsilon}\psi_\epsilon(x)|v|^m, \end{array}$$

where $\zeta > 0$ is defined in the lemma 2.5. Besides the convergence of (9) this shows that $|\mathcal{Q}| \leq [e^{-\Lambda_j+\epsilon} \sum_{n\geq 0} e^{-n\zeta}]\psi_\epsilon(x)$, so that $|\mathcal{Q}|$ is fast.

Finally, if $\alpha \in \mathfrak{P}_j$, we set $\mathcal{Q} := -\sum_{n\geq 0} \mathcal{A}^n \mathcal{M} \mathcal{A}^{-(n+1)}$, i.e. :

$$Q_x = -M_{\tau^{-1}(x)} A_x^{-1} - \sum_{n\geq 1} \left[A_{\tau^{-1}(x)} \dots A_{\tau^{-n}(x)} \, M_{\tau^{-(n+1)}(x)} \, A^{-1}_{\tau^{-n}(x)} \dots A_x^{-1} \right].$$

We obtain in that case :

$$\begin{array}{rcl} |Q_{n,x}(v)| & \leq & e^{n(\Lambda_j+\epsilon)} \left[\psi_\epsilon(x) e^{(n+1)\epsilon}\right] e^{(n+1)(-\lambda\cdot\alpha+|\alpha|\epsilon)} |v|^m \\ & \leq & e^{(n+1)[-\lambda\cdot\alpha+\Lambda_j+(|\alpha|+2)\epsilon]} e^{-(\Lambda_j+\epsilon)} \psi_\epsilon(x)|v|^m \\ & \leq & e^{-(n+1)\zeta} e^{-(\Lambda_j+\epsilon)}\psi_\epsilon(x)|v|^m, \end{array}$$

and we conclude as before. □

Corollary 3.4 *Let $\mathcal{K} : E(s_\epsilon) \to E(s_\epsilon)$ be a regular contracting bundle map, where s_ϵ is ϵ-slow. Suppose that* $\mathrm{Lip}\,(\mathcal{K}) \leq c < 1$ *on $E(s_\epsilon)$. Let $m \geq 2$ and $\gamma > 0$ small (depending on c, ϵ). There exist a ϵ-slow function $\tilde{r}_\epsilon \leq s_\epsilon$, a bundle map $\mathcal{S}_m$ tangent to $\mathcal{I}$ and a bundle map $\widehat{\mathcal{K}}$ tangent to $\mathcal{A} + \mathcal{K}^{(2)} + \ldots + \mathcal{K}^{(m-1)} + \mathfrak{R}(\mathcal{K}^{(m)})$ at the order $m+1$, such that the following diagram commutes for any ϵ-slow function $r_\epsilon \leq \tilde{r}_\epsilon$:*

$$\begin{array}{ccc} E(e^{-\gamma} r_\epsilon) & \xrightarrow{\mathcal{K}} & E(e^{-\gamma} r_\epsilon) \\ {\scriptstyle \mathcal{S}_m}\downarrow & & \downarrow{\scriptstyle \mathcal{S}_m} \\ E(r_\epsilon) & \xrightarrow{\widehat{\mathcal{K}}} & E(r_\epsilon) \ . \end{array}$$

Moreover $\mathrm{Lip}\,(\widehat{K}) \leq ce^{2\gamma}$.

Proof : Let $\gamma > 0$ such that $ce^{2\gamma} \leq e^{-\epsilon}$. The bundle map $\mathcal{K}$ is tame, so the function $|\mathcal{K}^{(m)}|$ is fast (see lemma 2.2) and we may apply the proposition 3.3 : there exists a m-homogeneous bundle map $\mathcal{Q}$ over Id_X such that $|\mathcal{Q}|$ is fast and

$$K_x^{(m)} + (Q_{\tau(x)} \circ A_x - A_x \circ Q_x) = \mathfrak{R}(K_x^{(m)}). \tag{10}$$

We let $\mathcal{S} := \mathcal{I} + \mathcal{Q}$, this bundle map is tame. As $\mathcal{S}$ is tangent to $\mathcal{I}$, one may decrease $\tilde{r}_\epsilon$ to obtain (see lemma 2.3(1)) :

$$\forall (u,v) \in E_x(e^\gamma \tilde{r}_\epsilon) \ , \ e^{-\gamma}|u-v| \leq |S_x(u) - S_x(v)| \leq e^\gamma |u-v|.$$

In particular, $\mathcal{S}^{-1}$ exists on $E(\tilde{r_\epsilon})$. Thus, for any ϵ-slow function $r_\epsilon \leq \tilde{r_\epsilon}$, the bundle map $\widehat{\mathcal{K}} := \mathcal{S} \circ \mathcal{K} \circ \mathcal{S}^{-1}$ is well defined on $E(r_\epsilon)$ and is $ce^{2\gamma}$-Lipschitz. Furthermore, $E(r_\epsilon)$ is stable by $\widehat{\mathcal{K}}$ since $ce^{2\gamma} \leq e^{-\epsilon}$. It remains to prove that $\widehat{\mathcal{K}}$ is tangent to $\mathcal{A} + \mathcal{K}^{(2)} + \ldots + \mathcal{K}^{(m-1)} + \mathfrak{R}(\mathcal{K}^{(m)})$ at the order $m+1$. Let us write $U_x \simeq V_x$ when $U_x - V_x = O(m+1)$. Observe first that $S_x^{-1} \simeq \mathrm{Id}_{\mathbb{C}^k} - Q_x$. Moreover, as $\mathcal{Q}$ is homogeneous of degree $m \geq 2$, we have :

$$\begin{aligned} K_x S_x^{-1} &\simeq \left[(A_x + K_x^{(2)} + \ldots + K_x^{(m-1)}) + K_x^{(m)}\right] \circ (\mathrm{Id}_{\mathbb{C}^k} - Q_x) \\ &\simeq (A_x + K_x^{(2)} + \ldots + K_x^{(m-1)}) + K_x^{(m)} - A_x \circ Q_x \end{aligned}$$

and then :

$$\begin{aligned} S_{\tau(x)} K_x S_x^{-1} &\simeq \left(\mathrm{Id}_{\mathbb{C}^k} + Q_{\tau(x)}\right) \circ \left[(A_x + K_x^{(2)} + \ldots + K_x^{(m-1)}) + K_x^{(m)} - A_x \circ Q_x\right] \\ &\simeq (A_x + K_x^{(2)} + \ldots + K_x^{(m-1)}) + K_x^{(m)} + Q_{\tau(x)} \circ A_x - A_x \circ Q_x \\ &\simeq (A_x + K_x^{(2)} + \ldots + K_x^{(m-1)}) + \mathfrak{R}(K_x^{(m)}), \end{aligned}$$

where the last line follows from (10). □

Proof of the proposition 3.2 : If $p = 1$, we just take $\mathcal{R} = \mathcal{A}$, so let $p \geq 2$. By the lemma 2.3(1), there exists a ϵ-slow function s_ϵ such that $\mathcal{G} : E(s_\epsilon) \to E(s_\epsilon)$ and $\mathrm{Lip}\,\mathcal{G} \leq e^{\Lambda_1+\epsilon}e^{\epsilon} < 1$ (take $b = e^{\Lambda_1+\epsilon}$ and $\gamma = \epsilon$). Let $c' := e^{\Lambda_1+2\epsilon}$ and γ small enough such that $1/2 \leq e^{-(p-1)\gamma}$ and $c'e^{2(p-1)\gamma} < 1$. We apply successively the corollary 3.4 with $(m, c) = (2, c'), (3, c'e^{2\gamma}), \ldots, (p, c'e^{2(p-2)\gamma})$. We obtain bundle maps $\mathcal{S}_2, \mathcal{S}_3, \ldots, \mathcal{S}_p$ tangent to $\mathcal{I}$ and a ϵ-slow function r_ϵ such that the following diagram is commutative, where $\mathcal{U} := \mathcal{S}_p \circ \ldots \circ \mathcal{S}_2$:

$$\begin{tikzcd} E(e^{-(p-1)\gamma} r_\epsilon) \arrow[r, "\mathcal{G}"] \arrow[d, "\mathcal{U}"'] & E(e^{-(p-1)\gamma} r_\epsilon) \arrow[d, "\mathcal{U}"] \\ E(r_\epsilon) \arrow[r, "\mathcal{F}"] & E(r_\epsilon) \end{tikzcd}$$

The bundle map $\mathcal{F}$ is tangent to $\mathcal{R}$ at the order $p+1$ and satisfies $\mathrm{Lip}\,(\mathcal{F}) \leq c'e^{2(p-1)\gamma} < 1$ on $E(r_\epsilon)$. We have $\mathcal{U} : E(r_\epsilon/2) \to E(r_\epsilon)$ because $1/2 \leq e^{-(p-1)\gamma}$. We set $\sigma_\epsilon := r_\epsilon/2$. The bundle map $\mathcal{U}$ is κ-tangent to $\mathcal{I}$ on $E(\sigma_\epsilon)$ (decrease σ_ϵ if necessary, see lemma 2.3(2)). □

3.3 Proof of theorem 1.3

Let $\mathcal{G}$ be a tame and regular contracting bundle map over τ, let $m := e^{\Lambda_l - \epsilon}$, $M := e^{\Lambda_1+\epsilon}$ and $\tilde{q}$ be the entire part of Λ_l/Λ_1 (so that $M^{\tilde{q}+1} < m$). We apply successively the proposition 3.2 and the theorem 1.1 with $p = q = \tilde{q}$. There exists bundle maps $\mathcal{T}$,$\mathcal{U}$

tangent to $\mathcal{I}$, a resonant bundle map $\mathcal{R}$ and a ϵ-slow function σ_ϵ such that the following diagram commutes :

$$\begin{array}{ccc} E(\sigma_\epsilon) & \xrightarrow{\mathcal{G}} & E(\sigma_\epsilon) \\ \mathcal{U}\downarrow & & \downarrow\mathcal{U} \\ E(2\sigma_\epsilon) & \xrightarrow{\mathcal{R}+O(\tilde{q}+1)} & E(2\sigma_\epsilon) \\ \mathcal{T}\downarrow & & \downarrow\mathcal{T} \\ E(4\sigma_\epsilon) & \xrightarrow{\mathcal{R}} & E(4\sigma_\epsilon) \end{array}$$

We let $\mathcal{V} := \mathcal{T} \circ \mathcal{U}$ and $\rho_\epsilon \leq \sigma_\epsilon$ be a ϵ-slow function such that $\mathcal{V}$ is κ-tangent to $\mathcal{I}$ on $E(\rho_\epsilon)$ (see lemma 2.3). We obtain $\mathcal{V} : E(\rho_\epsilon) \to E(2\rho_\epsilon)$ by choosing $\kappa < 1$. $\square$

4 Application to holomorphic dynamics

We prove in this section the theorems 1.4 and 1.5, let us first recall the setting in which we will work. We consider an holomorphic endomorphism $f : \mathbb{P}^k \to \mathbb{P}^k$ of algebraic degree $d \geq 2$. The topological degree of f is $d_t := d^k$. The equilibrium measure μ of f is given by $\mu = \lim_{n\to\infty} \frac{1}{d_t^n} f^{n*}\omega^k$, where ω is the Fubini-Study $(1,1)$ form on $\mathbb{P}^k$. This measure is mixing, satisfies $\mu(f(B)) = d_t\mu(B)$ whenever f is injective on the borel set B, and does not charge any analytic subset of $\mathbb{P}^k$. As Briend-Duval [BD] proved, the Lyapunov exponents $\chi_1 \leq \ldots \leq \chi_k$ of μ are bounded from below by $\log\sqrt{d}$. We are interested in the following quantities :

$$\Sigma_s := \chi_{k-s+1} + \ldots + \chi_k.$$

Let $\mathsf{O} := \{\hat{x} := (x_n)_{n\in\mathbb{Z}}\,,\, x_{n+1} = f(x_n)\}$ be the set of orbits. We note $\hat{f}$ the left shift sending $(\ldots, x_{-1}, x_0, x_1, \ldots)$ to $(\ldots, x_0, x_1, x_2, \ldots)$ and $\tau := \hat{f}^{-1}$. Let π be the projection $\hat{x} \mapsto x_0$ and ν be the unique probability measure on O invariant by τ and satisfying $\nu(\pi^{-1}(B)) = \mu(B)$ on the borel sets $B \subset \mathbb{P}^k$. The measure ν is mixing. We will work with the subset $X := \{\hat{x} \in \mathsf{O}\ ,\ x_n \notin \mathcal{C}_f\ ,\ \forall n \in \mathbb{Z}\}$ where $\mathcal{C}_f$ is the critical set of f. This subset has full ν-mesure, because $\mu(\mathcal{C}_f) = 0$. For all $\hat{x} \in X$, the inverse branch of f^n that sends x_0 to x_{-n} is denoted $f_{\hat{x}}^{-n}$.

We note $d(.,.)$ the distance on $\mathbb{P}^k$ induced by the Fubini-Study metric, $B_x(r) \subset \mathbb{P}^k$ the ball centered at x of radius r for this metric and $\mathbb{B}(r)$ the ball $\{|z| < r\} \subset \mathbb{C}^k$ for the standard metric.

4.1 Normalization along orbits of endomorphisms of $\mathbb{P}^k$

Our aim here is to prove the theorem 1.4. To this purpose we will first construct the bundle map $\mathcal{F}^{-1}$ of the inverse branches of an endomorphism $f : \mathbb{P}^k \to \mathbb{P}^k$. We will then apply to $\mathcal{F}^{-1}$ the Oseledec-Pesin reduction theorem and our theorem 1.3.

For defining this bundle map we shall use a family of charts $(\psi_x)_{x\in\mathbb{P}^k}$ satisfying the following properties, where $0 < M_0 < 1$ is a constant independant of $x \in \mathbb{P}^k$:

(P1) $\psi_x : \mathbb{C}^k \to \mathbb{P}^k$ is a biholomorphism onto its image and $\psi_x(0) = x$.

(P2) for all $(z_1, z_2) \in \mathbb{B}(M_0)$, $\frac{1}{2}|z_1 - z_2| \leq d(\psi_x(z_1), \psi_x(z_2)) \leq 2|z_1 - z_2|$.

We also require the following condition where $|\,.\,|_{x,u}$ is the norm on the spaces $(\bigwedge^s \mathbb{C}^k)_{1\leq s\leq k}$ inherited by the strictly positive $(1,1)$ form $(\psi_x^*\omega)(u)$:

(P3) for all $u \in \mathbb{B}(M_0)$, $\frac{1}{10}|\,.\,| \leq |\,.\,|_{x,u} \leq 10|\,.\,|$ and $|\,.\,|_{x,0} = |\,.\,|$.

For $\hat{x} \in X$, we identify $E_{\hat{x}} = \{(\hat{x}, v)\ ,\ v \in \mathbb{C}^k\}$ with $\mathbb{C}^k$ and let $\psi_{\hat{x}} : E_{\hat{x}} \to \mathbb{P}^k$ be the map $\psi_{\hat{x}} := \psi_{x_0} : \mathbb{C}^k \to \mathbb{P}^k$. We define

$$F_{\hat{x}} := \psi_{\hat{f}(\hat{x})}^{-1} \circ f \circ \psi_{\hat{x}}.$$

This map satisfies $F_{\hat{x}}(0) = 0$. By the uniform continuity of f, there exists $0 < M_1 \leq M_0$ such that the following bundle map is well defined :

$$\mathcal{F} : \begin{array}{ccc} E(M_1) & \longrightarrow & E(M_0) \\ (\hat{x}, z) & \longmapsto & \left(\hat{f}(\hat{x}), F_{\hat{x}}(z)\right). \end{array}$$

Recall that $X = \{(x_n)_{n\in\mathbb{Z}}\ ,\ x_{n+1} = f(x_n)\ ,\ x_n \notin \mathcal{C}_f\}$. So for any $\hat{x} \in X$ the map $F_{\tau(\hat{x})}$ is invertible in a neighbourhood of the origin. As the next lemma shows, the bundle map $\mathcal{F}^{-1}$ is actually tame.

Lemma 4.1 *There exists a slow function $\alpha : X \to]0,1]$ and a (tame) bundle map*

$$\mathcal{F}^{-1} : \begin{array}{ccc} E(\alpha) & \longrightarrow & E(M_1) \\ (\hat{x}, z) & \longmapsto & \left(\tau(\hat{x}), F_{\hat{x}}^{-1}(z)\right), \end{array}$$

where $F_{\hat{x}}^{-1} := (F_{\tau(\hat{x})})^{-1} = \left(\psi_{\hat{x}}^{-1} \circ f \circ \psi_{\tau(\hat{x})}\right)^{-1}$.

Proof : Let $t(\hat{x}) := \left\| (d_{x_{-1}} f)^{-1} \right\|^{-2}$. There exists a constant $c > 0$ depending only on the first and second derivatives of f such that the map $F_{\hat{x}}^{-1}$ exists on $E_{\hat{x}}(ct)$. We use here a quantitative version of the inverse local theorem (see e.g. [BD], lemma 2). We let $\alpha := \min\{ct, 1\}$. As $\log \alpha$ is ν-integrable (see [S], subsection 3.7), there exists a ϵ-slow function $\alpha_\epsilon : X \to]0,1]$ such that $\alpha_\epsilon \leq \alpha$ (see lemma 2.1). The function α is therefore slow. $\square$

Let $\mathcal{D}$ be the linear part of $\mathcal{F}^{-1}$. The Oseledec-Pesin theorem asserts that $\mathcal{D}$ is regular contracting after conjugation by a "tempered" family $\mathcal{C}$ of linear maps :

Theorem 4.2 (Oseledec-Pesin ϵ-reduction [KH]) *There exist a linear bundle map $\mathcal{C}$ over Id_X and a function $h_\epsilon : X \to [1, +\infty[$ such that :*

1. *$\mathcal{A} := \mathcal{C} \circ \mathcal{D} \circ \mathcal{C}^{-1}$ is regular contracting.*

2. $\forall \hat{x} \in X$, $|v| \leq |C_{\hat{x}}(v)| \leq h_\epsilon(\hat{x})|v|$ *and* $e^{-\epsilon} h_\epsilon(\hat{x}) \leq h_\epsilon(\tau(\hat{x})) \leq e^{\epsilon} h_\epsilon(\hat{x})$.

The contraction rates $\Lambda_l < \ldots < \Lambda_1 < 0$ of $\mathcal{A}$ are in our context the distinct opposite Lyapunov exponents of $(\mathbb{P}^k, f, \mu)$ (i.e. the distinct $-\chi_i$) and are negative. The integers k_j are the multiplicities of these exponents.

As the reader may easily check, the following proposition is a version of the theorem 1.4. We stress that the points 2 and 3 are consequences of the algebraic properties of the resonant maps (see proposition 2.9). We recall that $\epsilon \ll |\Lambda_1|$.

Proposition 4.3 *With the preceding notations, there exist* ϵ*-slow functions* $\eta_\epsilon, t_\epsilon : X \to]0, M_0]$*, a resonant bundle map* $\mathcal{R}$ *over* τ *and an injective bundle map* $\mathcal{W}$ *over* Id_X *(tangent to* $\mathcal{C}$*) such that the following diagram commutes for all* $n \geq 1$ *:*

$$\begin{array}{ccc} E(\eta_\epsilon) & \xrightarrow{\mathcal{F}^{-n}} & \mathcal{F}^{-n} E(\eta_\epsilon) \\ \mathcal{W} \downarrow & & \downarrow \mathcal{W} \\ E(t_\epsilon) & \xrightarrow{\mathcal{R}^n} & E(t_\epsilon) \end{array} .$$

We have $\mathcal{F}^{-n} E(\eta_\epsilon) \subset E(M_0)$*. There exist* $\alpha' > 0$ *and* ϵ*-fast functions* $\beta'_\epsilon, L'_\epsilon, T'_\epsilon : X \to [1, +\infty[$ *such that :*

1. $\forall (y, y') \in F_{\hat{x}}^{-n}(E_{\hat{x}}(\eta_\epsilon))$, $\alpha' \, |y - y'| \leq |W_{\tau^n(\hat{x})}(y) - W_{\tau^n(\hat{x})}(y')| \leq \beta'_\epsilon(\tau^n(\hat{x}))|y - y'|$.
2. $\mathrm{Lip}\, F_{\hat{x}}^{-n} \leq L'_\epsilon(\hat{x}) e^{-n\chi_1 + n\epsilon}$ *on* $E_{\hat{x}}(\eta_\epsilon)$,
3. *for all* $z \in F_{\hat{x}}^{-n}(E_{\hat{x}}(\eta_\epsilon))$, $|\frac{1}{n} \log |\bigwedge^s d_z F^n_{\tau^n(\hat{x})}| - \Sigma_s| \leq \frac{1}{n} \log T'_\epsilon(\hat{x}) + \epsilon$.

Proof : Let $\epsilon' = \epsilon/2(k + \eta)$ (the constant η is defined in the proposition 2.9 and depends only on $\tilde{q}, k$). By the lemma 4.1 and the theorem 4.2, there exist a ϵ'-slow function $\alpha_{\epsilon'} \leq \alpha$ and a ϵ'-fast function $h_{\epsilon'}$ such that the bundle map $\mathcal{G} := \mathcal{C} \circ \mathcal{F}^{-1} \circ \mathcal{C}^{-1}$ is well defined $E(\alpha_{\epsilon'}) \to E(M_1 h_{\epsilon'})$ and is regular contracting (observe that $E(\rho) \subset \mathcal{C}(E(\rho)) \subset E(\rho.h_{\epsilon'})$). In particular $\mathcal{G}$ is tame. By the theorem 1.3, there exists a ϵ'-slow function $\xi_{\epsilon'}$, a bundle map $\mathcal{V}$ 1/2-tangent to $\mathcal{I}$ and a resonant bundle map $\mathcal{R}$ such the following diagram commutes :

$$\begin{array}{ccc} E(\eta_\epsilon) & \xrightarrow{\mathcal{F}^{-n}} & \mathcal{F}^{-n} E(\eta_\epsilon) \\ \mathcal{C} \downarrow & & \downarrow \mathcal{C} \\ E(\xi_{\epsilon'}) & \xrightarrow{\mathcal{G}^n} & E(\xi_{\epsilon'}) \\ \mathcal{V} \downarrow & & \downarrow \mathcal{V} \\ E(2\xi_{\epsilon'}) & \xrightarrow{\mathcal{R}^n} & E(2\xi_{\epsilon'}) \end{array} .$$

We may assume that $\xi_{\epsilon'} \leq M_0/2$ and that $|\bigwedge^s \mathrm{Id}_{\mathbb{C}^k} - \bigwedge^s (d_t V_{\hat{x}})^{\pm 1}| \leq 1/10$ for any $t \in E_{\hat{x}}(\xi_{\epsilon'})$ and $\hat{x} \in X$ (lemma 2.3(3)). Let $\eta_\epsilon = \xi_{\epsilon'}/h_{\epsilon'}$ and $t_\epsilon := 2\xi_{\epsilon'}$. These functions

are ϵ-slow and satisfy $\eta_\epsilon, t_\epsilon \leq M_0$. Moreover $\mathcal{F}^{-n}E(\eta_\epsilon) \subset \mathcal{C}^{-1}E(\xi_{\epsilon'}) \subset E(\xi_{\epsilon'}) \subset E(M_0)$. We let $\mathcal{W} := \mathcal{V} \circ \mathcal{C}$. Observe that :

$$\forall (y, y') \in F_{\hat{x}}^{-n}(E_{\hat{x}}(\eta_\epsilon)) \ , \ \frac{1}{2}|y - y'| \leq |W_{\tau^n(\hat{x})}(y) - W_{\tau^n(\hat{x})}(y')| \leq \frac{3}{2} h_{\epsilon'}(\tau^n(\hat{x}))|y - y'|.$$

The point 1 follows with $\alpha' = 1/2$ and $\beta'_\epsilon = \frac{3}{2}h_{\epsilon'}$. Now we prove the points 2 and 3. Let $y \in E_{\hat{x}}(\eta_\epsilon)$ and $t = C_{\hat{x}}(y)$. We obtain by the theorem 4.2(2) and the commutative diagram above :

$$|\bigwedge^s d_y F_{\hat{x}}^{-n}| \leq |C_{\tau^n(\hat{x})}^{-1}|^s |\bigwedge^s d_t G_{\hat{x}}^n||C_{\hat{x}}|^s \leq |\bigwedge^s d_t G_{\hat{x}}^n| h_{\epsilon'}(\hat{x})^s,$$

$$|\bigwedge^s d_t G_{\hat{x}}^n| \leq |C_{\tau^n(\hat{x})}|^s |\bigwedge^s d_y F_{\hat{x}}^{-n}||C_{\hat{x}}^{-1}|^s \leq (h_{\epsilon'}(\hat{x}) e^{n\epsilon'})^s |\bigwedge^s d_y F_{\hat{x}}^{-n}|.$$

We deduce the estimate :

$$\Big|\frac{1}{n}\log|\bigwedge^s d_y F_{\hat{x}}^{-n}| - \frac{1}{n}\log|\bigwedge^s d_t G_{\hat{x}}^n|\Big| \leq \frac{1}{n}\log h_{\epsilon'}^k(\hat{x}) + k\epsilon'. \tag{11}$$

Now let $w := V_{\hat{x}}(t)$, $G^n := G_{\hat{x}}^n$, $R^n := R_{\hat{x}}^n$, $V := V_{\hat{x}}$, $V_n := V_{\tau^n(\hat{x})}$ and write :

$$\bigwedge^s d_t G^n = \bigwedge^s (d_{G^n(t)} V_n)^{-1} \bigwedge^s d_w R^n \bigwedge^s d_t V := (\bigwedge^s \mathrm{Id}_{\mathbb{C}^k} + \Omega_1) \bigwedge^s d_w R^n (\bigwedge^s \mathrm{Id}_{\mathbb{C}^k} + \Omega_2),$$

where $\Omega_i : \bigwedge^s \mathbb{C}^k \to \bigwedge^s \mathbb{C}^k$ satisfy $|\Omega_i| \leq 1/10$. This implies that

$$\frac{1}{2}|\bigwedge^s d_w R^n| \leq |\bigwedge^s d_t G^n| \leq 2|\bigwedge^s d_w R^n|. \tag{12}$$

The proposition 2.9 gives a $\eta\epsilon'$-fast function $H_{\eta\epsilon'}$ such that for all $s \in \{1, \ldots, k\}$:

$$\Big|\frac{1}{n}\log|\bigwedge^s d_w R_x^n| - (\lambda_1 + \ldots + \lambda_s)\Big| \leq \frac{1}{n}\log H_{\eta\epsilon'}(x) + \eta\epsilon'.$$

We deduce with (11), (12) and $\lambda_i = -\chi_i$:

$$\Big|\frac{1}{n}\log|\bigwedge^s d_y F_{\hat{x}}^{-n}| + (\chi_1 + \ldots + \chi_s)\Big| \leq \frac{1}{n}\log[2H_{\eta\epsilon'} h_{\epsilon'}^k](x) + (k + \eta)\epsilon'. \tag{13}$$

The function $L'_\epsilon := 2H_{\eta\epsilon'} h_{\epsilon'}^k$ is $(k+\eta)\epsilon'$-fast (therefore ϵ-fast). The point 2 follows from (13) with $s = 1$. For the point 3, observe that $|\bigwedge^s d_z F_{\tau^n(\hat{x})}^n| = |\bigwedge^{k-s} d_y F_{\hat{x}}^{-n}||\bigwedge^k d_y F_{\hat{x}}^{-n}|^{-1}$ where $z = F_{\hat{x}}^{-n}(y)$. We obtain by using (13) twice :

$$\Big|\frac{1}{n}\log|\bigwedge^{k-s} d_y F_{\hat{x}}^{-n}| + (\chi_1 + \ldots + \chi_{k-s})\Big| \leq \frac{1}{n}\log L'_\epsilon(x) + (k + \eta)\epsilon',$$

$$\Big|\frac{1}{n}\log|\bigwedge^k d_y F_{\hat{x}}^{-n}| + (\chi_1 + \ldots + \chi_k)\Big| \leq \frac{1}{n}\log L'_\epsilon(x) + (k + \eta)\epsilon'.$$

These estimates imply :

$$\Big|\frac{1}{n}\log|\bigwedge^s d_z F_{\tau^n(\hat{x})}^n| - (\chi_{k-s+1} + \ldots + \chi_k)\Big| \leq \frac{1}{n}\log L'^2_\epsilon(x) + 2(k + \eta)\epsilon'.$$

We finally let $T'_\epsilon := L'^2_\epsilon$, which is a ϵ-slow function. $\square$

4.2 An approximation formula for sums of Lyapunov exponents

This subsection is devoted to the proof of theorem 1.5. Let $\mathbf{R}_n$ (resp. $\mathbf{R}_n^*$) be the set of repulsive periodic points whose period divides n (resp. equals n). Let us consider the function φ_n defined on $\mathbb{P}^k$ by

$$\varphi_n(z) := \frac{1}{n} \log \left\| \bigwedge^s d_z f^n \right\|.$$

Then, the theorem 1.5 may be stated as follows:

$$\lim_{n\to+\infty} \frac{1}{d_t^n} \sum_{p\in\mathbf{R}_n} \varphi_n(z) = \lim_{n\to+\infty} \frac{1}{d_t^n} \sum_{p\in\mathbf{R}_n^*} \varphi_n(z) = \Sigma_s. \tag{14}$$

Let $\mathbf{F}_n := \{z \in \mathbb{P}^k \,,\, f^n(z) = z\}$. Since the number of fixed points of f^n *counted with multiplicity* is, by Bezout's theorem, equal to $1 + d^n + \ldots + d^{nk}$ (see [S], subsection 1.3) we have $\mathsf{Card}(\mathbf{F}_n) \leq (k+1)d_t^n$. This implies that for any $n \geq 1$, $\mathsf{Card}(\mathbf{R}_n) - \mathsf{Card}(\mathbf{R}_n^*) \leq \sum_m \mathsf{Card}(\mathbf{F}_m) \leq n(k+1)d_t^{n/2}$, where the sum runs over the integers $1 \leq m \leq n/2$ which divide n. The first equality in (14) is then a consequence of the following lemma.

Lemma 4.4 *There exists $\Gamma \geq 0$ such that for all $p \in \mathbf{R}_n$ and $n \geq 1$, $0 \leq \varphi_n(p) \leq \Gamma$.*

PROOF : Let $p \in \mathbf{R}_n$ and $(p_0, p_1, \ldots, p_{n-1})$ be the repulsive cycle generated by $p_0 := p$. Let $\Gamma := k.\max_{z\in\mathbb{P}^k} \log^+ \| d_z f \|$. The observations $\| \bigwedge^s d_p f^n \| \leq \| d_p f^n \|^k \leq \prod_{i=0}^{n-1} \| d_{p_i} f \|^k$ and $0 \leq \varphi_n(p) = \frac{1}{n} \log^+ \| \bigwedge^s d_p f^n \|$ imply the inequalities : $0 \leq \varphi_n(p) \leq k.\frac{1}{n} \sum_{i=0}^{n-1} \log^+ \| d_{p_i} f \| \leq \Gamma$. □

The proof of the theorem 1.5 basically consists in producing repulsive cycles by Briend-Duval's method taking into account the information on inverse branches given by our theorem 1.4.

Let $0 < \epsilon \ll \chi_1$. We introduce :

$$\mathbf{R}_n^\epsilon := \{p \in \mathbf{R}_n \ , \ |\varphi_n(p) - \Sigma_s| \leq 2\epsilon\}$$

and write $\frac{1}{d_t^n} \sum_{p\in\mathbf{R}_n} \varphi_n(p) - \Sigma_s = \frac{1}{d_t^n}(u_n + v_n + w_n)$, where

$$u_n := \sum_{p\in\mathbf{R}_n^\epsilon} (\varphi_n(p) - \Sigma_s) \quad , \quad v_n := \sum_{p\in\mathbf{R}_n\setminus\mathbf{R}_n^\epsilon} (\varphi_n(p) - \Sigma_s) \quad , \quad w_n := (\mathsf{Card}(\mathbf{R}_n) - d_t^n)\Sigma_s.$$

We show that for n sufficiently large the sequences u_n, v_n and w_n are essentially bounded by ϵd_t^n. The key estimate is given by the following lemma whose proof is postponed to the end of the subsection.

Lemma 4.5 *There exists $n_1 \geq 1$ such that :*

$$\forall n \geq n_1 \ , \ \mathsf{Card}(\mathbf{R}_n^\epsilon) \geq d_t^n (1-\epsilon)^3.$$

Let us now give the expected bounds on u_n, v_n, w_n. As $\mathbf{R}_n \subset \mathbf{F}_n$ and $\mathsf{Card}(\mathbf{F}_n) \leq (d_t^n d^n - 1)(d^n - 1)^{-1}$ (recall that $d_t = d^k$), there exists $n_2 \geq 1$ such that :

$$\forall n \geq n_2 \ , \ \mathsf{Card}(\mathbf{R}_n) \leq d_t^n (1+\epsilon). \tag{15}$$

Using (15) and the lemma 4.5 we see that for any $n \geq \max\{n_1, n_2\}$:

$$|u_n| \leq \mathsf{Card}(\mathbf{R}_n^\epsilon) 2\epsilon \leq 2\epsilon(1+\epsilon) d_t^n \leq 4\epsilon d_t^n \quad \text{and} \quad |w_n| \leq \ 4\Sigma_s \cdot \epsilon d_t^n.$$

The lemmas 4.4, 4.5 and the estimate (15) finally give :

$$|v_n| \leq (\Gamma + \Sigma_s)\left(\mathsf{Card}(\mathbf{R}_n) - \mathsf{Card}(\mathbf{R}_n^\epsilon)\right) \leq (\Gamma + \Sigma_s)[(1+\epsilon) - (1-\epsilon)^3] d_t^n,$$

which is bounded by $5(\Gamma + \Sigma_s) \cdot \epsilon d_t^n$.

We now end the proof of the theorem 1.5 by establishing the lemma 4.5.

The functions r_ϵ, T_ϵ and L_ϵ have been introduced in the theorem 1.4. We note them shortly r, T and L. In the sequel, we will not use their ϵ-slow/fast properties. For any $\hat{x} \in X$, let $n(\hat{x})$ be the smallest integer satisfying $\log L(\hat{x}) \leq n\epsilon$ and $\log T(\hat{x}) \leq n\epsilon$. Let $n_1 \geq 1$ large enough and $r_0 > 0$ small enough such that the set

$$\widehat{H} := \{\hat{x} \in X \ , \ r(\hat{x}) \geq r_0 \ , \ n(\hat{x}) \leq n_1\}$$

satisfies $\nu(\widehat{H}) \geq 1 - \epsilon/2$. We have in particular for all $\hat{x} \in \widehat{H}$ and $n \geq n_1$ (see the theorem 1.4) :

(a_1) $\operatorname{Lip} f_{\hat{x}}^{-n} \leq e^{-n\chi_1 + 2n\epsilon}$ on $B_{x_0}(r_0)$,

(a_2) for all $z \in f_{\hat{x}}^{-n}(B_{x_0}(r_0))$, $|\frac{1}{n} \log \| \bigwedge^s d_z f^n \| - \Sigma_s| \leq 2\epsilon$.

We consider two concentric families of balls $(B_i)_{1 \leq i \leq m} \subset (B_i^\gamma)_{1 \leq i \leq m} \subset \mathbb{P}^k$ whose radii are respectively equal to r and $r + \gamma \leq r_0/2$ (with $0 < \gamma \ll r$) and satisfy the following properties :

(b_1) the balls B_i^γ are disjoint,

(b_2) $\mu(\cup_{i=1}^m B_i) \geq 1 - \epsilon/2$,

(b_3) $\mu(B_i) \geq (1-\epsilon)\mu(B_i^\gamma)$.

We may increase n_1 such that for all $n \geq n_1$ and $1 \leq i \leq m$:

(c_1) $e^{-n\lambda_1 + 2n\epsilon}(r + \gamma) < \gamma$ (in particular $e^{-n\lambda_1 + 2n\epsilon} < 1$),

(c_2) if $\widehat{B_i} := \pi^{-1}(B_i)$, then $\nu\left(\widehat{f}^{-n}(\widehat{B_i} \cap \widehat{H}) \cap \widehat{B_i}\right) \geq (1-\epsilon)\nu(\widehat{B_i} \cap \widehat{H})\nu(\widehat{B_i})$.

The last assertion is a consequence of the mixing property of ν.

Let us temporarily fix $i \in \{1, \dots, m\}$ and note $B := B_i$, $B^\gamma := B_i^\gamma$ and $\widehat{B^\gamma} := \pi^{-1}(B^\gamma)$. Observe that if $\hat{x} \in \widehat{H} \cap \widehat{B^\gamma}$, then the set $f_{\hat{x}}^{-n}(B^\gamma)$ is well defined since $\hat{x} \in \widehat{H} \subset \{r(\hat{x}) \geq r_0\}$ and the radius of B^γ satisfies $r + \gamma \leq r_0/2$.

Let us consider the collection $\mathcal{C}_n(B)$ of sets of the form $f_{\hat{x}}^{-n}(B^\gamma)$ which do intersect B, where $\hat{x} \in \widehat{H} \cap \widehat{B^\gamma}$. Note that the elements of $\mathcal{C}_n(B)$ are *disjoint* open subsets of $\mathbb{P}^k$. We will establish the following estimates for $n \geq n_1$:

$$\mathsf{Card}(\mathbf{R}_n^\epsilon \cap B^\gamma) \geq \mathsf{Card}(\mathcal{C}_n(B)) \geq d_t^n (1-\epsilon)^2 \nu(\widehat{H} \cap \widehat{B}). \tag{16}$$

Let $\mathcal{E}$ be an element of $\mathcal{C}_n(B)$: $\mathcal{E} = f_{\hat{x}}^{-n}(B^\gamma)$ intersects B and $\hat{x} \in \widehat{H} \cap \widehat{B^\gamma}$. By (a_1) and (c_1), $f_{\hat{x}}^{-n}$ contracts on $B^\gamma \subset B_{x_0}(r_0)$. Moreover $\mathcal{E}$ is contained in B^γ, because $\mathcal{E}$ intersects B and its diameter is less than γ by (c_1). We thus obtain a point $p \in \mathcal{E}$ which is fixed by $f_{\hat{x}}^{-n} : B^\gamma \to B^\gamma$. This point is n-periodic and repulsive for f, so $p \in \mathbf{R}_n$. We have also $p \in f_{\hat{x}}^{-n}(B^\gamma) \subset f_{\hat{x}}^{-n}(B_{x_0}(r_0))$ which leads to $p \in \mathbf{R}_n^\epsilon$ by (a_2). This gives the first inequality in (16).

Let us now justify the second inequality. According to (c_2) and the relation $\mu = \pi_* \nu$ we obtain :

$$(1-\epsilon)\nu(\widehat{B} \cap \widehat{H})\mu(B) \leq \mu\left[\pi\left(\widehat{f}^{-n}(\widehat{H} \cap \widehat{B})\right) \cap B\right] \leq \mu\Big(\bigcup f_{\hat{x}}^{-n}(B^\gamma)\Big),$$

where the union runs over the elements of $\mathcal{C}_n(B)$. By the jacobian property we have $\mu(f_{\hat{x}}^{-n}(B^\gamma)) = \mu(B^\gamma)/d_t^n$ and the right hand side is thus equal to $\mathsf{Card}(\mathcal{C}_n(B))\mu(B^\gamma)/d_t^n$. We then get the desired inequality by (b_3).

The estimates (16) imply finally :

$$\mathsf{Card}(\mathbf{R}_n^\epsilon) \geq \sum_{i=1}^m \mathsf{Card}(\mathbf{R}_n^\epsilon \cap B_i^\gamma) \geq \sum_{i=1}^m \mathsf{Card}(\mathcal{C}_n(B_i)) \geq d_t^n (1-\epsilon)^2 \nu(\widehat{H} \cap \cup_{i=1}^m \widehat{B_i}). \tag{17}$$

As $\nu(\widehat{H}) \geq 1 - \epsilon/2$ and $\nu(\cup_{i=1}^m \widehat{B_i}) = \mu(\cup_{i=1}^m B_i) \geq 1 - \epsilon/2$ (see (b_2)) we have $\nu(\widehat{H} \cap \cup_{i=1}^m \widehat{B_i}) \geq 1 - \epsilon$. So, by (17), $\mathsf{Card}(\mathbf{R}_n^\epsilon) \geq d_t^n (1-\epsilon)^3$, and the lemma 4.5 is proved.

5 Appendix : properties of resonant maps

We prove here the proposition 2.7 and 2.9 we stated in the subsection 2.3. A result similar to the proposition 2.7 may be found in the article [GK], lemma 1.1.

5.1 Proof of the proposition 2.7

We want to prove that for all $2 \leq j \leq k$, the component

$$\pi_j(K_{\tau(x)})(K_x) = \pi_j(K_{\tau(x)})(\pi_1(K_x), \dots, \pi_{j-1}(K_x))$$

is made of j-resonant monomials (the dependance of the $j-1$ first components in the right hand side is a consequence of (2), subsection 2.3). For simplicity, we may assume that $k_j = 1$ for all $1 \leq j \leq k$, therefore $\lambda_j = \Lambda_j$. As we are concerned with a degree property, we do not compute the coefficients of polynomials (we set them equal to 1). Recall that $P_\alpha = z_1^{\alpha_1} \cdots z_k^{\alpha_k}$. By writing $\pi_j(K_{\tau(x)}) = \sum_{\beta \in \mathfrak{R}_j} P_\beta$, it suffices to consider the following polynomial, where $\beta \in \mathfrak{R}_j$:

$$P_\beta(\pi_1(K_x), \ldots, \pi_{j-1}(K_x)) = (\pi_1(K_x))^{\beta_1} \ldots (\pi_{j-1}(K_x))^{\beta_{j-1}}. \tag{18}$$

For all $1 \leq l \leq j-1$, we write $\pi_l(K_x) = \sum_{i=0}^{\mathsf{Card}(\mathfrak{R}_l)} P_{\alpha(l,i)}$, where $\alpha(l,0) = (0, \ldots, 1, \ldots, 0)$ (so that $P_{\alpha(l,0)} = z_l$) and $\{\alpha(l,i)\,,\, 1 \leq i \leq \mathsf{Card}(\mathfrak{R}_l)\} = \mathfrak{R}_l$. Observe that we have for any $1 \leq l \leq j-1$ and $0 \leq i \leq \mathsf{Card}(\mathfrak{R}_l)$:

$$\sum_{p=1}^{j-1} \alpha_p(l,i)\Lambda_p = \sum_{p=1}^{l} \alpha_p(l,i)\Lambda_p = \Lambda_l. \tag{19}$$

We now expand (18) and prove that we get a sum of j-resonant monomials. By the Newton's formula, the monomials that appear in the expansion of (18) have the form :

$$P_\gamma = \prod_{i=0}^{\mathsf{Card}(\mathfrak{R}_1)} P_{\alpha(1,i)}^{\beta_1(i)} \ \ldots \ \prod_{i=0}^{\mathsf{Card}(\mathfrak{R}_{j-1})} P_{\alpha(j-1,i)}^{\beta_{j-1}(i)}\ , \tag{20}$$

where $\sum_{i=0}^{\mathsf{Card}(\mathfrak{R}_l)} \beta_l(i) = \beta_l$ for all $1 \leq l \leq j-1$. We have to prove that $\gamma \in \mathfrak{R}_j$, that is : $\sum_{p=1}^{j-1} \gamma_p \Lambda_p = \Lambda_j$. First observe that $|\gamma| \geq |\beta| \geq 2$ and that $P_\gamma = z_1^{\gamma_1} \ldots z_{j-1}^{\gamma_{j-1}}$ (indeed there are no $z_j, \ldots, z_k$ in the right hand side of (18)). By (20) we have for all $1 \leq p \leq j-1$:

$$\gamma_p = \sum_{i=0}^{\mathsf{Card}(\mathfrak{R}_1)} \alpha_p(1,i)\beta_1(i) + \ldots + \sum_{i=0}^{\mathsf{Card}(\mathfrak{R}_{j-1})} \alpha_p(j-1,i)\beta_{j-1}(i) = \sum_{l=1}^{j-1} \sum_{i=0}^{\mathsf{Card}(\mathfrak{R}_l)} \alpha_p(l,i)\beta_l(i).$$

Thus the sum $\sum_{p=1}^{j-1} \gamma_p \Lambda_p$ is equal to :

$$\sum_{p=1}^{j-1} \Lambda_p \left(\sum_{l=1}^{j-1} \sum_{i=0}^{\mathsf{Card}(\mathfrak{R}_l)} \alpha_p(l,i)\beta_l(i) \right) = \sum_{l=1}^{j-1} \left(\sum_{i=0}^{\mathsf{Card}(\mathfrak{R}_l)} \beta_l(i) \left(\sum_{p=1}^{j-1} \alpha_p(l,i)\Lambda_p \right) \right). \tag{21}$$

By using successively (19), $\sum_{i=0}^{\mathsf{Card}(\mathfrak{R}_l)} \beta_l(i) = \beta_l$ and $\beta \in \mathfrak{R}_j$, the right hand of (21) is equal to

$$\sum_{l=1}^{j-1} \left(\sum_{i=0}^{\mathsf{Card}(\mathfrak{R}_l)} \beta_l(i)\, \Lambda_l \right) = \sum_{l=1}^{j-1} \beta_l \Lambda_l = \Lambda_j.$$

We have therefore $\gamma \in \mathfrak{R}_j$, which completes the proof of the proposition.

5.2 Proof of the proposition 2.9

Let $\mathcal{R}$ be a resonant bundle map. By the proposition 2.7, the degree of the iterates of $\mathcal{R}$ is bounded by $\tilde{q}$. We define the function $\| \mathcal{R} \| := \max\{|\mathcal{R}^{(1)}|, \ldots, |\mathcal{R}^{(\tilde{q})}|\}$: for all $x \in X$, $\| \mathcal{R} \| (x)$ is the maximum of the coefficients of the polynomial map R_x (see subsection 2.1 for the definition of $|\mathcal{R}^{(j)}|$). We show in the next proposition that for all $1 \leq j \leq l$, $\| \pi_j(\mathcal{R}^n) \|$ is close to $\| \pi_j(\mathcal{A}^n) \|$. This fact will be useful for proving the proposition 2.9.

Proposition 5.1 *Let $\mathcal{R} : E(\rho_\epsilon) \to E(\rho_\epsilon)$ be a resonant bundle map where ρ_ϵ is a ϵ-slow function. There exist $\theta \geq 1$ (depending only on $\tilde{q}$) and a $\theta\epsilon$-fast function $M_{\theta\epsilon} : X \to [1, +\infty[$ such that*

$$\forall j \in \{1, \ldots, l\} \ , \ \forall n \geq 1 \ , \ e^{n(\Lambda_j - \epsilon)} \leq \| \pi_j(\mathcal{R}^n) \| \leq M_{\theta\epsilon} e^{n(\Lambda_j + \theta\epsilon)}.$$

Proof : The estimate from below follows from $\| \pi_j(\mathcal{R}^n) \| \geq |\pi_j(\mathcal{A}^n)| \geq e^{n(\Lambda_j - \epsilon)}$. We prove the estimate from above. We may assume that $k_j = 1$ and $\lambda_j = \Lambda_j$ for all $1 \leq j \leq k$. Let also $\Delta := \mathsf{Card}(\mathfrak{R}) + 1$. Let $1 \leq s \leq k$ and define the assertion (i_s) : there exists a $(\tilde{q} + \ldots + \tilde{q}^{s-1})\epsilon$-fast function $M_s : X \to [1, +\infty[$ such that

$$\forall j \in \{1, \ldots, s\} \ , \ \forall n \geq 1 \ , \ \| \pi_j(\mathcal{R}^n) \| \leq M_s e^{n(\Lambda_j + \tilde{q}^{2s}\epsilon)}.$$

Let $\psi_{\tilde{q}\epsilon}$ be a $\tilde{q}\epsilon$-fast function such that $\psi_{\tilde{q}\epsilon} \geq \| \mathcal{R} \|$ (cf lemma 2.2). We proceed by induction on s. The assertion (i_1) is satisfied with $M_1 := 1$. Indeed, $\pi_1(\mathcal{R}^n) = \pi_1(\mathcal{A}^n)$ because $\mathfrak{R}_1$ is empty (see subsection 2.3). Assume that (i_{s-1}) is satisfied for $1 \leq s-1 \leq k-1$. We define

$$M_s := \frac{\psi_{\tilde{q}\epsilon} M_{s-1}^{\tilde{q}} \Delta^{\tilde{q}+1} e^{-\Lambda_s}}{1 - e^{-(\tilde{q}^{2s}-1)\epsilon}}.$$

Observe that M_s is $(\tilde{q} + \ldots + \tilde{q}^{s-1})\epsilon$-fast. As $M_s \geq M_{s-1}$, the assertion (i_s) is proved if we establish the following assertions for all $n \geq 1$:

$$(ii_n) : \| \pi_s(\mathcal{R}^n) \| \leq M_s e^{n(\Lambda_s + \tilde{q}^{2s}\epsilon)}.$$

The assertion (ii_1) is fulfilled because :

$$\| \pi_s(\mathcal{R}) \| \leq \| \mathcal{R} \| \leq \psi_{\tilde{q}\epsilon} \leq \psi_{\tilde{q}\epsilon} \frac{M_{s-1}^{\tilde{q}} \Delta^{\tilde{q}+1} e^{\tilde{q}^{2s}\epsilon}}{1 - e^{-(\tilde{q}^{2s}-1)\epsilon}} = M_s e^{\Lambda_s + \tilde{q}^{2s}\epsilon}.$$

Let us assume that (ii_n) is true and establish (ii_{n+1}). We have

$$\pi_s(R_x^{n+1}) = \pi_s(R_{\tau^n(x)})(\pi_1(R_x^n), \ldots, \pi_s(R_x^n)). \tag{22}$$

By (2) (subsection 2.3), the polynomial $\pi_s(R_{\tau^n(x)})$ has the form

$$\pi_s(R_{\tau^n(x)})(z_1, \ldots, z_s) = \sigma(\tau^n(x)) z_s + \sum_{\alpha \in \mathfrak{R}_s} \sigma_\alpha(\tau^n(x)) P_\alpha(z_1, \ldots, z_{s-1}),$$

where $\sigma(\tau^n(x)), \sigma_\alpha(\tau^n(x)) \in \mathbb{C}$. So by (22), $\pi_s(R_x^{n+1})$ is the sum of I_n and J_n :

$$I_n := \sigma(\tau^n(x))\pi_s(R_x^n),$$

$$J_n := \sum_{\alpha\in\mathfrak{R}_s} \sigma_\alpha(\tau^n(x))P_\alpha(\pi_1(R_x^n),\ldots,\pi_{s-1}(R_x^n)). \tag{23}$$

To simplify the exposition, we note in the sequel $u := e^{\Lambda_s+\tilde{q}^{2s}\epsilon}$ and M_s for $M_s(x)$, $\psi_{\tilde{q}\epsilon}$ for $\psi_{\tilde{q}\epsilon}(x)$, etc. We obtain by (ii_n) and $|\sigma(\tau^n(x))| \le e^{\Lambda_s+\epsilon}$:

$$\| I_n \| \le e^{\Lambda_s+\epsilon} M_s u^n = \left[M_s e^{-(\tilde{q}^{2s}-1)\epsilon}\right] u^{n+1}.$$

The end of the proof consists in verifying the following estimate, by using (i_{s-1}) :

$$\| J_n \| \le \left[\Delta^{\tilde{q}+1}\psi_{\tilde{q}\epsilon} M_{s-1}^{\tilde{q}} e^{-\Lambda_s}\right] u^{n+1}. \tag{24}$$

Indeed the two preceding lines imply (ii_{n+1}) by the very definition of M_s :

$$\left\| \pi_s(\mathcal{R}^{n+1}) \right\| \le \| I_n \| + \| J_n \| \le \left[M_s e^{-(\tilde{q}^{2s}-1)\epsilon} + \Delta^{\tilde{q}+1}\psi_{\tilde{q}\epsilon} M_{s-1}^{\tilde{q}} e^{-\Lambda_s}\right] u^{n+1} = M_s u^{n+1}.$$

We now prove (24). By the proposition 2.7, J_n is a linear combination of resonant monomials. Recall also that $\| J_n \|$ is the maximum of the coefficients of J_n in the basis $(P_\beta)_{\beta\in\mathfrak{R}_s}$. Fix a resonant degree $\beta \in \mathfrak{R}_s$. Let $\alpha \in \mathfrak{R}_s$ and expand $P_\alpha(\pi_1(R_x^n),\ldots,\pi_{s-1}(R_x^n))$. Observe that for all $1 \le j \le s-1$, $\pi_j(R_x^n)$ is a sum of at most $\Delta = \mathsf{Card}(\mathfrak{R}) + 1$ monomials, because $\mathcal{R}$ is resonant. So the degree $\beta \in \mathfrak{R}_s$ appears in the preceding expansion at most $\Delta^{\alpha_1+\ldots+\alpha_{s-1}}$ times. It implies that the P_β-coefficient $\sigma_\beta(\alpha)$ of $\sigma_\alpha(\tau^n(x))P_\alpha(\pi_1(R_x^n),\ldots,\pi_{s-1}(R_x^n))$ satisfies by (i_{s-1}) and $|\sigma_\alpha(\tau^n(x))| \le \psi_{\tilde{q}\epsilon}(x)e^{n\epsilon}$:

$$\begin{aligned} |\sigma_\beta(\alpha)| &\le |\sigma_\alpha(\tau^n(x))|\Delta^{\alpha_1+\ldots+\alpha_{s-1}} \| \pi_1(R_x^n) \|^{\alpha_1} \ldots \| \pi_{s-1}(R_x^n) \|^{\alpha_{s-1}} \\ &\le \psi_{\tilde{q}\epsilon}(x)e^{n\epsilon}[\Delta M_{s-1}e^{n\tilde{q}^{2(s-1)}\epsilon}]^{\alpha_1+\cdots+\alpha_{s-1}} e^{n(\alpha_1\Lambda_1+\cdots+\alpha_{s-1}\Lambda_{s-1})}. \end{aligned}$$

Observe that $\alpha \in \mathfrak{R}_s$ implies $\alpha_1+\ldots+\alpha_{s-1} \le \tilde{q}$ and $\alpha_1\Lambda_1+\ldots+\alpha_{s-1}\Lambda_{s-1} = \Lambda_s$. We deduce (use $1+\tilde{q}\tilde{q}^{2(s-1)} \le \tilde{q}^2\tilde{q}^{2(s-1)} = \tilde{q}^{2s}$) :

$$|\sigma_\beta(\alpha)| \le [\psi_{\tilde{q}\epsilon}\Delta^{\tilde{q}} M_{s-1}^{\tilde{q}} e^{-\Lambda_s}]e^{(n+1)\tilde{q}^{2s}\epsilon} e^{(n+1)\Lambda_s} = [\psi_{\tilde{q}\epsilon}\Delta^{\tilde{q}} M_{s-1}^{\tilde{q}} e^{-\Lambda_s}]u^{n+1}.$$

We now use (23) to obtain $\| J_n \| \le \sum_{\alpha\in\mathfrak{R}_s} |\sigma_\beta(\alpha)| \le \Delta[\psi_{\tilde{q}\epsilon}\Delta^{\tilde{q}} M_{s-1}^{\tilde{q}} e^{-\Lambda_s}]u^{n+1}$, this completes the proof of (24). We let finally $\theta = \max\{\tilde{q}+\ldots+\tilde{q}^{k-1}, \tilde{q}^{2k}\}$ to get the two assertions : the function $M_{\theta\epsilon} := M_k$ is $\theta\epsilon$-fast and $\| \pi_j(\mathcal{R}^n) \| \le M_{\theta\epsilon}e^{n(\Lambda_j+\theta\epsilon)}$. □

We now prove the proposition 2.9. Recall that

$$(\lambda_1,\ldots,\lambda_k) = (\Lambda_1,\ldots,\Lambda_1,\cdots,\Lambda_j,\ldots,\Lambda_j,\cdots,\Lambda_l,\ldots,\Lambda_l),$$

where Λ_j appears k_j times. We note $(e_i)_i$ the canonical orthonormal basis of $\mathbb{C}^k$. If $A : \mathbb{C}^k \to \mathbb{C}^k$ is linear, we define $\bigwedge^s A : \bigwedge^s \mathbb{C}^k \to \bigwedge^s \mathbb{C}^k$ as the linear extension of the map L satisfying $L(e_{i_1}\wedge\cdots\wedge e_{i_s}) := A(e_{i_1})\wedge\cdots A(e_{i_s})$ for any $1 \le i_1 < \cdots < i_s \le k$.

Proposition 2.9 *Let $\mathcal{R} : E(\rho_\epsilon) \to E(\rho_\epsilon)$ be a resonant bundle map where ρ_ϵ is a ϵ-slow function. Let $\eta := k\theta$ (θ depends on $\tilde{q}$, see proposition 5.1). There exists a $\eta\epsilon$-fast function $H_{\eta\epsilon} : X \to [1, +\infty[$ such that for all $w \in E_x(\rho_\epsilon)$, $s \in \{1, \ldots, k\}$ and $n \geq 1$: $\left| \frac{1}{n} \log |\bigwedge^s d_w R_x^n| - (\lambda_1 + \ldots + \lambda_s) \right| \leq \frac{1}{n} \log H_{\eta\epsilon}(x) + \eta\epsilon$.*

Proof : We first give the proof in the case $k_j = 1$ for all $1 \leq j \leq k$ (it implies $\lambda_j = \Lambda_j$). Let $x \in X$ and $w \in E_x(\rho_\epsilon)$. By the remark 2.8 (subsection 2.3), the matrix of $d_w R_x^n$ in the canonical basis $(e_i)_{1 \leq i \leq k}$ is lower triangular, so we have :

$$d_w R_x^n(e_i) = \zeta_n(i) e_i + \sum_{j=i+1}^{k} \omega_n(i,j) e_j \quad \text{and} \quad d_w R_x^n(e_k) = \zeta_n(k) e_k.$$

We fix $1 \leq i \leq k$ and $i+1 \leq j \leq k$. We give bounds for $\zeta_n(i)$ and $\omega_n(i,j)$. We have

$$e^{n(\Lambda_i - \epsilon)} \leq |\zeta_n(i)| \leq e^{n(\Lambda_i + \epsilon)} \tag{25}$$

because the diagonal part of $d_w R_x^n$ is equal to A_x^n (see remark 2.8). Recall that $\pi_j(R_x^n)$ is a sum of at most $\mathsf{Card}(\mathfrak{R}_j)$ resonant monomials with degree α lower than $\tilde{q}$ (see the proposition 2.7). We deduce by the proposition 5.1 and $\rho_\epsilon \leq 1$ that :

$$|\omega_n(i,j)| \leq \sum_{\alpha \in \mathfrak{R}_j} \| \pi_j(R_x^n) \| \, |\alpha| \rho_\epsilon(x)^{|\alpha|-1} \leq \mathsf{Card}(\mathfrak{R}_j) M_{\theta\epsilon}(x) e^{n(\Lambda_j + \theta\epsilon)} \tilde{q}. \tag{26}$$

Let $M'_{\theta\epsilon}(x) := \mathsf{Card}(\mathfrak{R}) \tilde{q} M_{\theta\epsilon}(x) \geq 1$. In the sequel, we will multiply $M'_{\theta\epsilon}$ by constants depending only on k and s without mentionning it. By (25), (26) and the inequality $\Lambda_j < \Lambda_i$, we get :

$$|d_w R_x^n(e_i)| \leq M'_{\theta\epsilon}(x) e^{n(\Lambda_i + \theta\epsilon)}. \tag{27}$$

Now we focus on the coefficients of the matrix $\bigwedge^s d_w R_x^n$ in the orthonormal basis $e_{i_1} \wedge \ldots \wedge e_{i_s}$, with $1 \leq i_1 < \ldots < i_s \leq k$. We begin with the vector $\bigwedge^s d_w R_x^n(e_1 \wedge \ldots \wedge e_s)$ which is equal to :

$$\zeta_n(1) \ldots \zeta_n(s) e_1 \wedge \ldots \wedge e_s + \sum_{(i_1, \ldots, i_s) \neq (1, \ldots, s)} \omega'_n(i_1, \ldots, i_s) e_{i_1} \wedge \ldots \wedge e_{i_s}. \tag{28}$$

Observe now the following estimates for $(i_1, \ldots, i_s) \neq (1, \ldots, s)$ and $n \geq 1$. The first inequality is a consequence of (26) :

$$|\omega'_n(i_1, \ldots, i_s)| \leq M'_{\theta\epsilon}(x)^s e^{n(\Lambda_{i_1} + \ldots + \Lambda_{i_s} + s\theta\epsilon)} \leq M'_{\theta\epsilon}(x)^s e^{n(\Lambda_1 + \ldots + \Lambda_s + k\theta\epsilon)}. \tag{29}$$

We obtain with the lines (25), (28) and (29) that for any $n \geq 1$:

$$e^{n(\Lambda_1 + \ldots + \Lambda_s - s\epsilon)} \leq |\bigwedge^s d_w R_x^n(e_1 \wedge \ldots \wedge e_s)| \leq M'_{\theta\epsilon}(x)^k e^{n(\Lambda_1 + \ldots + \Lambda_s + k\theta\epsilon)}. \tag{30}$$

We now focus on $\bigwedge^s d_w R_x^n(e_{i_1} \wedge \ldots \wedge e_{i_s})$. The Hadamard's inequality implies with (27) :

$$\begin{aligned} |\bigwedge^s d_w R_x^n(e_{i_1} \wedge \ldots \wedge e_{i_s})| &\leq |d_w R_x^n(e_{i_1})| \ldots |d_w R_x^n(e_{i_s})| \\ &\leq M'_{\theta\epsilon}(x)^s e^{n(\Lambda_{i_1} + \ldots + \Lambda_{i_s} + k\theta\epsilon)} \\ &\leq M'_{\theta\epsilon}(x)^k e^{n(\Lambda_1 + \ldots + \Lambda_s + k\theta\epsilon)}. \end{aligned}$$

We finally obtain the following estimates for any $n \geq 1$, where $\eta = k\theta$ and $H_{\eta\epsilon}$ is equal to the $\eta\epsilon$-fast function $(M'_{\theta\epsilon})^k$:

$$\Big|\frac{1}{n}\log|\bigwedge^s d_w R_x^n| - (\lambda_1 + \ldots + \lambda_s)\Big| \leq \frac{1}{n}\log H_{\eta\epsilon}(x) + \eta\epsilon.$$

We sketch the proof in the general case, i.e. when $k = \sum_{i=1}^l k_j$ and $k_j \geq 1$. We may assume that the block diagonal matrix A_x^n is a lower (block) triangular matrix. Indeed, for all $1 \leq j \leq l$, there exists a matrix $V_j \in \mathbb{U}_{k_j}(\mathbb{C})$ such that $T_j^n := V_j[\pi_j(A_x^n)]V_j^{-1}$ is lower triangular. The metric property :

$$\forall (x,v) \in \mathcal{L}^j \ , \ e^{n(\Lambda_j - \epsilon)}|v| \leq |\pi_j(A_x^n)(v)| \leq e^{n(\Lambda_j+\epsilon)}|v|$$

implies that the modulus of the coefficients of T_j^n are $\leq e^{n(\Lambda_j+\epsilon)}$, and that the diagonal coefficients $(\zeta_n^{(j)}(i))_{1\leq i\leq k}$ of T_j^n satisfy :

$$e^{n(\Lambda_j-\epsilon)} \leq |\zeta_n^{(j)}(i)| \leq e^{n(\Lambda_j+\epsilon)}.$$

We thus obtain estimates analogous to (25). Let V be the block diagonal matrix $(V_1, \ldots, V_l)$. The matrix $T_x^n := V[A_x^n]V^{-1}$ is therefore block diagonal, with lower triangular blocks $(T_1^n, \ldots, T_l^n)$. The matrix $S_x^n := V[d_w R_x^n]V^{-1}$ is also lower triangular (see remark 2.8) and its coefficients outside the block diagonal matrix T_x^n satisfy estimates analogous to (26) (the $(V_j)_{1\leq j\leq l}$ are unitary transformations). We deduce as before that for any $n \geq 1$:

$$\Big|\frac{1}{n}\log|\bigwedge^s S_x^n| - (\lambda_1 + \ldots + \lambda_s)\Big| \leq \frac{1}{n}\log H_{\eta\epsilon}(x) + \eta\epsilon.$$

The conclusion follows from $|\bigwedge^s S_x^n| = |\bigwedge^s V\ [\bigwedge^s d_w R_x^n]\ \bigwedge^s V^{-1}| = |\bigwedge^s d_w R_x^n|$. □

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