Dynamics of fibered endomorphisms of $\mathbb{P}^k$

Christophe Dupont and Johan Taflin*

September 12, 2019

Abstract

We study the structure and the Lyapunov exponents of the equilibrium measure of endomorphisms of $\mathbb{P}^k$ preserving a fibration. We extend the decomposition of the equilibrium measure obtained by Jonsson for polynomial skew products of $\mathbb{C}^2$. We also show that the sum of the sectional exponents satisfies a Bedford-Jonsson formula when the fibration is linear, and that this function is plurisubharmonic on families of fibered endomorphisms. In particular, the sectional part of the bifurcation current is a closed positive current on the parameter space.

Key words : Equilibrium measure, Lyapunov exponents, Bifurcation current.

MSC 2010 : 32H50, 37F10.

1 Introduction

Let f be a holomorphic endomorphism of $\mathbb{P}^k$ of degree $d \geq 2$ which preserves a rational fibration parametrized by a projective space i.e. there exist a dominant rational map $\pi \colon \mathbb{P}^k \dashrightarrow \mathbb{P}^r$ and a holomorphic map $\theta \colon \mathbb{P}^r \to \mathbb{P}^r$ such that

$$\pi \circ f = \theta \circ \pi. \tag{1.1}$$

The generic fiber of π has dimension $q := k - r$. Another way to express (1.1) is that f permutes the fibers of π and this permutation is given by θ. We are interested in the relationships between the dynamics of f and the one of θ.

This type of maps has been recently used to exhibit interesting dynamical phenomena in $\mathbb{P}^2$ (see [Duj16], [ABD$^+$16], [BT17], [Duj17], [Taf17]). All these examples, except [BT17], come from polynomial skew products of $\mathbb{C}^2$, whose dynamical properties have been studied by Jonsson in [Jon99]. It is therefore interesting for future examples to extend the results of [Jon99] to a broader framework. Our initial motivation was to study the particular case where π is the standard linear fibration defined by $\pi[y : z] = [y]$ with $y := (y_0, \dots, y_r) \in \mathbb{C}^{r+1}$ and $z = (z_0, \dots, z_{q-1}) \in \mathbb{C}^q$. However, some of the techniques can be extended to a more general setting. In what follows, we choose the setting of each result in order to avoid unnecessary technical details. We refer to the end of this introduction for the possible scope of the techniques, in particular when $k = 2$ thanks to the works of Dabija-Jonsson [DJ08, DJ10] and Favre-Peirera [FP11, FP15] on endomorphisms of $\mathbb{P}^2$ preserving a fibration, a foliation or a web.

*Research partially supported by ANR project Fatou ANR-17-CE40-0002-01.

Green currents and Lyapunov exponents – The maps f, π and θ are given by homogeneous polynomials and one shows that f and θ have the same algebraic degree (this can be seen, for instance, by comparing the cardinality of preimages of generic points by $\pi \circ f$ and $\theta \circ \pi$). Both maps have a Green current, T_f and T_θ respectively, which are positive closed $(1,1)$-currents with continuous local potentials. Their self-intersections are well-defined and their supports define dynamically meaningful filtrations

$$\mathscr{J}_i(f) := \mathrm{supp}(T_f^i) \quad \text{and} \quad \mathscr{J}_j(\theta) := \mathrm{supp}(T_\theta^j),$$

for $i \in \{1, \ldots, k\}$ and $j \in \{1, \ldots, r\}$. The *equilibrium measures* of f and θ are defined by $\mu_f := T_f^k$ and $\mu_\theta := T_\theta^r$. Since f and θ are semi-conjugated by π, a natural question is whether there exists a relation between T_f^i and the pull back of T_θ by π. Our first result gives such a relationship if $i > q$. More precisely, we will see in Section 3 how to define $\pi^* T_\theta$ and if S denotes the result normalized by its mass,

$$S := \frac{\pi^* T_\theta}{\|\pi^* T_\theta\|},$$

then we have the following result.

Theorem 1.1. *Let $f\colon \mathbb{P}^k \to \mathbb{P}^k$ and $\theta\colon \mathbb{P}^r \to \mathbb{P}^r$ be two endomorphisms of degree $d \geq 2$. Assume there exists a dominant rational map $\pi\colon \mathbb{P}^k \dashrightarrow \mathbb{P}^r$ whose indeterminacy set $I(\pi)$ is disjoint from $\mathscr{J}_q(f)$ and such that $\theta \circ \pi = \pi \circ f$. Then for $j \in \{1, \ldots, r\}$, the current S^j is well-defined, satisfies $S^j \neq T_f^j$ and $T_f^{q+j} = T_f^q \wedge S^j$. In particular, $\mu_f = T_f^q \wedge S^r$ and $\pi_* \mu_f = \mu_\theta$.*

Let us emphasize that the proof only relies on the properties of the currents T_f and T_θ and is coordinate free. Using the classification obtained in [DJ08], one can check easily that the assumption $\mathscr{J}_q(f) \cap I(\pi) = \varnothing$ is always satisfied when $k = 2$, in which case $q = 1$, see Section 2 for details. This is also the case for the standard linear fibration in any dimension (see Lemma 6.1). In general, we do not know any example where this assumption does not hold.

The main point in Theorem 1.1 is the formula $\mu_f = T_f^q \wedge S^r$ which can be seen as a generalization of the decomposition of μ_f obtained by Jonsson [Jon99] for polynomial skew products of $\mathbb{C}^2$. Indeed, for μ_θ-almost every $a \in \mathbb{P}^r$ the fiber $L_a := \overline{\pi^{-1}(a)}$ has dimension q and we can define the probability measure

$$\mu_a := \frac{T_f^q \wedge [L_a]}{\|\pi^* T_\theta\|^r}.$$

Corollary 1.2. *Let $\phi\colon \mathbb{P}^k \to \mathbb{R}$ be a continuous function. Under the assumptions of Theorem 1.1 we have*

$$\int_{\mathbb{P}^k} \phi(x) d\mu_f(x) = \int_{\mathbb{P}^r} \left(\int_{L_a} \phi(x) d\mu_a(x) \right) d\mu_\theta(a).$$

The other results in this paper can also be seen as consequences of the formula $\mu_f = T_f^q \wedge S^r$ and the main technical difficulties come from the fact that the currents S and $[L_a]$ are singular. As a direct consequence of Theorem 1.1, we obtain in the following result that if μ_θ is absolutely continuous with respect to Lebesgue measure (i.e. θ is a Lattès mapping of $\mathbb{P}^r$, see [BD05]) then μ_f is absolutely continuous with respect to the trace measure $\sigma_{T_f^q} := T_f^q \wedge \omega_{\mathbb{P}^k}^r$. Here, $\omega_{\mathbb{P}^k}$ (resp. $\omega_{\mathbb{P}^r}$) is the Fubini-Study form on $\mathbb{P}^k$ (resp. $\mathbb{P}^r$) normalized such that $\omega_{\mathbb{P}^k}^k$ (resp. $\omega_{\mathbb{P}^r}^r$) is a probability measure.

Corollary 1.3. *Under the assumptions of Theorem 1.1, if $\mu_\theta \ll \omega_{\mathbb{P}^r}^r$ then $\mu_f \ll \sigma_{T_f^q}$.*

Let us note that when $k = 2$, the property $\mu_f \ll \sigma_{T_f}$ implies that the smallest exponent of μ_f is minimal, equal to $\frac{1}{2}\log d$, see [Duj12, Theorem 3.6]. Hence we obtain:

Corollary 1.4. *Let $f = [P(x, y) : Q(x, y) : R(x, y, z)]$ be a holomorphic endomorphism of $\mathbb{P}^2$ of degree $d \geq 2$. If $\theta = [P(x, y) : Q(x, y)]$ is a Lattès mapping of $\mathbb{P}^1$, then f is semi-extremal: the Lyapunov exponents of the equilibrium measure μ_f satisfy $\lambda_1 > \lambda_2 = \frac{1}{2}\log d$.*

This corollary applies in particular for Desboves mappings of $\mathbb{P}^2$, their Lyapunov exponents are thus $\lambda_1 > \lambda_2 = \frac{1}{2}\log d$, with $d = 4$, see Section 2. The following Theorem, combined with Corollary 1.2, provides an other proof of Corollary 1.4 and allow to generalize it to fibered endomorphisms satisfying $\mu_\theta \ll \omega_{\mathbb{P}^r}^r$. It is a consequence of $\pi_*\mu_f = \mu_\theta$ and holds for more general smooth dynamical systems.

Theorem 1.5. *Under the assumptions of Theorem 1.1, if Λ is a Lyapunov exponent of multiplicity m for μ_θ then Λ is a Lyapunov exponent of multiplicity at least m of μ_f.*

Standard linear fibration – Now we restrict ourselves to the cases where π is the standard linear fibration. Theorem 1.7 below is already new when $k = 2$ and $r = 1$. The indeterminacy set $I(\pi)$ of π corresponds to $\{y = 0\} \simeq \mathbb{P}^{q-1}$ and each fiber $L_a = \overline{\pi^{-1}(a)}$ is a linear projective space $\mathbb{P}^q$ in which $I(\pi)$ can be identified with the hyperplane at infinity, i.e. $L_a \setminus I(\pi) \simeq \mathbb{C}^q$. If f preserves the fibration defined by π then f acts on each periodic fiber as a regular polynomial endomorphism of $\mathbb{C}^q$. This class of maps has been studied by Bedford-Jonsson in [BJ00]. In particular, they obtained a formula for the sum of the Lyapunov exponents of the equilibrium measure. More precisely, let R be a regular polynomial endomorphism of $\mathbb{C}^q$ of degree d i.e. R extends to an endomorphism of $\mathbb{P}^q$ of degree d. We denote by T_R, Crit_R and G_R respectively the Green current, the critical set and the Green function in $\mathbb{C}^q$ of R. The restriction of R to the hyperplane at infinity $\mathbb{P}^q \setminus \mathbb{C}^q \simeq \mathbb{P}^{q-1}$ is an endomorphism of $\mathbb{P}^{q-1}$ and we denote by Λ_0 the sum of the Lyapunov exponents of its equilibrium measure.

Theorem 1.6 (Bedford-Jonsson [BJ00])**.** *Let R be a regular polynomial endomorphism of $\mathbb{C}^q$ of degree d. The sum Λ_R of the Lyapunov exponents of its equilibrium measure satisfies*

$$\Lambda_R = \log d + \Lambda_0 + \langle T_R^{q-1} \wedge [\mathrm{Crit}_R], G_R \rangle.$$

We give a generalization of this formula in the fibered setting. To this aim, we introduce some notation. If $\pi \circ f = \theta \circ \pi$ then we denote by Λ_f (resp. Λ_θ) the sum of the Lyapunov exponents of μ_f (resp. μ_θ). Theorem 1.5 implies that

$$\Lambda_\sigma := \Lambda_f - \Lambda_\theta \tag{1.2}$$

is the sum of the Lyapunov exponents of μ_f in the direction of the fibers (the Oseledec subspaces of these exponents are tangent to the fibers). The indeterminacy set $I(\pi) \simeq \mathbb{P}^{q-1}$ is invariant by f thus $f_{I(\pi)}$ can be seen as an endomorphism of $\mathbb{P}^{q-1}$ and we denote by Λ_0 the sum of the Lyapunov exponents of this restriction.

Observe now that the algebraic set Crit_f is not irreducible because f preserves the linear fibration. Indeed, by Lemma 6.3 we can define $[C_\infty] := \pi^*[\mathrm{Crit}_\theta]$ and

$$[C_\sigma] := [\mathrm{Crit}_f] - [C_\infty].$$

We shall say that $[C_\infty]$ is the *fibered part* of Crit_f (it is foliated by the fibers of π) and that $[C_\sigma]$ is the *sectional part* of Crit_f. Finally, we define the *relative Green function* as the unique lower semicontinuous function $G\colon \mathbb{P}^k \to [0,+\infty]$ such that $dd^cG = T_f - S$ and $\min G = 0$.

Theorem 1.7. *Let f be an endomorphism of $\mathbb{P}^k$ of degree $d \geq 2$ which preserves the standard linear fibration. Then*

$$\Lambda_\sigma = \log d + \Lambda_0 + \langle T_f^{q-1} \wedge S^r \wedge [C_\sigma], G\rangle.$$

In particular, $\Lambda_\sigma \geq \frac{q+1}{2}\log d$.

The idea of the proof is to apply the formula of Bedford-Jonsson to each n-periodic fiber of f. The current S^r can indeed be seen as the limit of the average of the currents of integration on the n-periodic fibers, and the above formula follows by taking the limits. However, in order to implement that idea, we need some additional care since the currents involved are singular. Note that Astorg-Bianchi [AB18, Section 4] also make the use of n-periodic points to express the horizontal Lyapunov exponent of polynomial skew products of $\mathbb{C}^2$ (in view of the study of bifurcations, see the next paragraph).

Families of fibered endomorphisms − We now consider a family $(f_\lambda)_{\lambda\in M}$ of endomorphisms of $\mathbb{P}^k$ which preserves the standard linear fibration π i.e. there exists a family $(\theta_\lambda)_{\lambda\in M}$ of endomorphisms of $\mathbb{P}^r$ such that $\pi\circ f_\lambda = \theta_\lambda\circ\pi$. The family $(f_\lambda)_{\lambda\in M}$ induces a dynamical system $f(\lambda,x) := (\lambda, f_\lambda(x))$ on $M\times\mathbb{P}^k$ whose critical set Crit_f is the gluing of the critical sets of f_λ, $\lambda\in M$. In the same way, there exists a positive closed (i,i)-current T_f^i on $M\times\mathbb{P}^k$ whose slices are equal to $T_{f_\lambda}^i$. In [BB07] Bassanelli and Berteloot established a formula involving the currents $[\mathrm{Crit}_f]$ and T_f^k and the dd^c of $\Lambda_f\colon \lambda\mapsto\Lambda_{f_\lambda}$ where Λ_{f_λ} is the sum of the Lyapunov exponents of μ_{f_λ} (see also [Pha05]). They proved that

$$dd^c\Lambda_f = p_*(T_f^k \wedge [\mathrm{Crit}_f]),$$

where $p\colon M\times\mathbb{P}^k \to M$ is the projection. This current is called the *bifurcation current* $T_{\mathrm{Bif}}(f)$ and its support coincides with several bifurcation phenomena in the family $(f_\lambda)_{\lambda\in M}$, see the article by Berteloot-Bianchi-Dupont [BBD18].

Similar objects can be defined for the family $(\theta_\lambda)_{\lambda\in M}$ and again, it is natural to connect them with the ones defined for $(f_\lambda)_{\lambda\in M}$. Since this family preserves the fibration, as above the set Crit_f is not irreducible and we have $[\mathrm{Crit}_f] = [C_\infty] + [C_\sigma]$ where $[C_\infty] := \Pi^*[\mathrm{Crit}_\theta]$ with $\Pi(\lambda,x) = (\lambda,\pi(x))$. This decomposition induces a decomposition of $T_{\mathrm{Bif}}(f)$ in a *fibered part* and a *sectional part*. The result below states that the fibered part of $T_{\mathrm{Bif}}(f)$ coincides with $T_{\mathrm{Bif}}(\theta)$.

Theorem 1.8. *Let M be a complex manifold and consider two holomorphic families $(f_\lambda)_{\lambda\in M}$ and $(\theta_\lambda)_{\lambda\in M}$ of endomorphisms of $\mathbb{P}^k$ and $\mathbb{P}^r$ respectively such that $\pi\circ f_\lambda = \theta_\lambda\circ\pi$ where π is the standard linear fibration. The $(1,1)$-current*

$$T_{\mathrm{Bif},\sigma}(f) := T_{\mathrm{Bif}}(f) - T_{\mathrm{Bif}}(\theta)$$

is positive. Moreover, if $S^r := \Pi^*T_\theta^r$ then

$$T_{\mathrm{Bif}}(\theta) = p_*(T_f^q \wedge S^r \wedge [C_\infty]), \quad T_{\mathrm{Bif},\sigma}(f) = p_*(T_f^q \wedge S^r \wedge [C_\sigma]).$$

A different way of seeing this result is the following. By Theorem 1.5, we know that $\Lambda_\sigma = \Lambda_f - \Lambda_\theta$ is the sum of the Lyapunov exponents of μ_f which are not in the Lyapunov spectrum of μ_θ, see Equation (1.2). Theorem 1.8 yields that Λ_σ is a plurisubharmonic function on M and gives a formula of $dd^c\Lambda_\sigma$ in terms of currents on $M \times \mathbb{P}^k$,

$$dd^c\Lambda_\sigma = p_*(T_f^q \wedge S^r \wedge [C_\sigma]).$$

As remarked above, Astorg-Bianchi initiated in [AB18] the study of bifurcations for skew products of $\mathbb{C}^2$, they proved in that setting that Λ_σ is plurisubharmonic. Following an idea coming from [AB18], for each $n \geq 1$ we can consider the bifurcation current associated to the dynamics of the family $(f_\lambda)_{\lambda\in M}$ on the n-periodic fibers. To be more precise, if $\lambda \in M$ and $a \in \mathbb{P}^r$ are such that $\theta_\lambda^n(a) = a$ then we denote by $\Lambda(f^n_{\lambda|L_a})$ the sum of Lyapunov exponents of $f^n_{\lambda|L_a}$ seen as a polynomial endomorphism of $L_a \simeq \mathbb{P}^q$. Then we define

$$\Lambda_{\sigma,n}(\lambda) := \frac{1}{nd^{rn}} \sum_{\theta_\lambda^n(a)=a} \Lambda(f^n_{\lambda|L_a}) \quad \text{and} \quad T_{\mathrm{Bif},n}(f) := dd^c\Lambda_{\sigma,n}.$$

It is easy to see that if all the cycles of the family $(\theta_\lambda)_{\lambda\in M}$ can be followed holomorphically on M then $\Lambda_{\sigma,n}$ is plurisubharmonic. Actually, this holds in general and we can express the current $T_{\mathrm{Bif},\sigma}(f)$ in terms $T_{\mathrm{Bif},n}(f)$.

Corollary 1.9. *Let $(f_\lambda)_{\lambda\in M}$, $(\theta_\lambda)_{\lambda\in M}$ and π be as in Theorem 1.8. For each $n \geq 1$ the function $\Lambda_{\sigma,n}$ is plurisubharmonic and*

$$T_{\mathrm{Bif},\sigma}(f) = \lim_{n\to\infty} T_{\mathrm{Bif},n}(f).$$

Moreover, if $[\mathrm{Per}_{\theta,n}]$ denotes the current of integration on $\{(\lambda, a) \in M \times \mathbb{P}^r \mid \theta_\lambda^n(a) = a\}$ by taking into account multiplicities, then

$$T_{\mathrm{Bif},n} = \frac{p_*(T_f^q \wedge \Pi^*[\mathrm{Per}_{\theta,n}] \wedge [C_\sigma])}{d^{rn}}.$$

The first part of this result was obtained in [AB18, Corollary 4.8] in the special case of skew products of $\mathbb{C}^2$ under the hypothesis that $dd^c\Lambda_\theta = 0$. And, as observed by Astorg-Bianchi, a consequence of Corollary 1.9 is that if the dynamics on $(f_\lambda)_{\lambda\in M}$ bifurcates on one periodic fiber then, asymptotically when $n \to \infty$, it bifurcates on a positive proportion of the n-periodic fibers.

Final remarks and outline of the paper – To conclude this introduction, let us explain in which setting results similar to Theorem 1.1 and Theorem 1.5 could be obtained. First, observe that the assumption that the base space is $\mathbb{P}^r$ is unnecessary as long as $\dim(I(\pi)) \leq q-1$. Indeed, if π is a dominant meromorphic map between $\mathbb{P}^k$ and a compact complex manifold X of dimension r with $\dim(I(\pi)) \leq q-1$, $(q = k-r)$, then the restriction of π to a generic linear subspace of dimension r in $\mathbb{P}^k$ gives a surjective holomorphic map from $\mathbb{P}^r$ to X. Then by results in [DHP08, Section 2 & 3], X is projective and then by [Laz84], X is isomorphic to $\mathbb{P}^r$. Observe that this argument uses the smoothness of the base. The case with a singular base might appear naturally but goes beyond the scope of this paper.

Another natural setting is the following. Assume that f is an endomorphism of $\mathbb{P}^k$ which preserves a family $(L_a)_{a\in X}$ of algebraic sets of dimension q and of degree α parametrized by a complex manifold X of dimension r, i.e. there exists an endomorphism θ of X such that $f(L_a) = L_{\theta(a)}$. It is natural to expect that under some assumptions on the family $(L_a)_{a\in X}$ and if θ possesses an equilibrium measure μ_θ, the measure μ_f can be written as $\mu_f = T_f^q \wedge S^r$ where

$$S^r := \int_X \frac{[L_a]}{\alpha} d\mu_\theta(a).$$

Indeed, it is easy to check, using the classifications in [DJ10] and [FP15] and the proof of Theorem 1.1, that this is the case when $k = 2$ and the family $(L_a)_{a\in X}$ defines a web with algebraic leaves. However, in some of these examples $S = T_f$. This holds for (ii)-(iii)-(iv) in [DJ10, Theorem A] (see Theorem 2.1 below) and for (ii) in [FP15, Theorem E]. Otherwise, $S \neq T_f$ in these theorems.

Finally, let us mention that results of Dinh-Nguyên-Truong [DNT12, DNT15] suggest that one might expect some of the results above (as $\pi_*\mu_f = \mu_\theta$ and Theorem 1.5) to extend to the case of dominant meromorphic self-maps of compact Kähler manifolds with large (or dominant) topological degree which preserve meromorphic fibrations.

The paper is organized as follows. Section 2 is devoted to geometric examples of fibered endomorphisms of $\mathbb{P}^2$. In Section 3, we give some technical results on the pull-back and the intersection of currents. In Section 4 and Section 5, we prove Theorem 1.1, Corollary 1.3 and Theorem 1.5 respectively. In Section 6, we restrict ourselves to the cases where π is the standard linear fibration and we establish our generalized Bedford-Jonsson formula in that context. Section 7 is devoted to bifurcations of such maps.

Acknowledgements – We thank Charles Favre for useful comments on the preliminary version of this paper. We also thank the referee for his cautious reading and remarks which allow us to improve the exposition.

2 Examples

2.1 Fibered endomorphisms of $\mathbb{P}^2$

Let us specify the classification of fibered endomorphisms of $\mathbb{P}^2$ established in [DJ08].

Theorem 2.1. *Let f be a holomorphic endomorphism of $\mathbb{P}^2$ of degree $d \geq 2$, with an irreducible invariant pencil of curves $\pi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$. Then, in suitable homogeneous coordinates, one of the following case occurs:*

1. *π is a pencil of lines $\pi[x : y : z] = [x : y]$. Then $f = [P(x, y) : Q(x, y) : R(x, y, z)]$, where P, Q, R are homogeneous polynomials of degree d.*
2. *π is a binomial pencil $\pi[x : y : z] = [x^h y^{k-h} : z^k]$, where $0 < h < k$ and h, k are relatively prime. Then f is equal to $[x^d : y^d : S(x, y, z)]$ where $S(x, y, z) = z^{d-kl} \prod_{i=1}^{l} (z^k + c_i x^h y^{k-h})$, $0 \leq l \leq d/k$ and $c_i \in \mathbb{C}^*$.*
3. *π is the binomial pencil $\pi[x : y : z] = [xy : z^2]$. Then f is equal to $[y^d : x^d : T(x, y, z)]$ where $T(x, y, z) = z^{d-2l} \prod_{i=1}^{l} (z^2 + c_i xy)$, $0 \leq l \leq d/2$ and $c_i \in \mathbb{C}^*$.*

In the first case, we have $I(\pi) = \{[0 : 0 : 1]\}$ and $\theta[X : Y] = [P(X, Y) : Q(X, Y)]$. In the two other cases $I(\pi) = \{[1 : 0 : 0], [0 : 1 : 0]\}$ and $\theta[X : Y] = [X^d : Y^{d-kl} \prod_{i=1}^{l} (Y + c_i X)^k]$, which is the polynomial mapping $z^{d-kl} \prod_{i=1}^{l} (z + c_i)^k$. In particular, the indeterminacy set $I(\pi)$ always consists in superattracting periodic points of period 1 or 2, hence does not intersect the support of T_f. This property will be proved more generally in Lemma 6.1.

Now let us also recall the classification of endomorphisms of $\mathbb{P}^2$ with an invariant irreducible web, proved in [DJ10].

Theorem 2.2. *Let f be a holomorphic endomorphism of $\mathbb{P}^2$ of degree $d \geq 2$. Let C be an irreducible curve of the dual space $\check{\mathbb{P}}^2$. Assume that every line of $\mathbb{P}^2$ belonging to C is mapped to another such line (the web C is thus invariant by f). Then C is either (i) a line, (ii) a smooth conic, (iii) a smooth cubic or (iv) a nodal cubic.*

In particular, a fibration is preserved only when the irreducible web induced by C is a pencil of line. Beyond our framework of fibered endomorphisms, Favre-Peirera classified endomorphisms with an invariant foliation or transcendental web, see [FP11, FP15]. In higher dimensions, the classification of fibered endomorphisms of $\mathbb{P}^k$ remains an open problem. We expect that the assumptions of Theorem 1.1 cover a lot of these endomorphisms.

2.2 Semi-extremal endomorphisms

An endomorphism of $\mathbb{P}^2$ of degree d is called *extremal* if the Lyapunov exponents of its equilibrium measure μ_f satisfy $\lambda_1 = \lambda_2 = \frac{1}{2}\log d$. These endomorphisms have been characterized by the four equivalent properties: $\mu \ll \omega_{\mathbb{P}^2}^2$, $\dim_H \mu_f = 4$, T_f is a positive smooth $(1,1)$-form on some open set of $\mathbb{P}^2$, and f is a Lattès mapping, see [BD05, BL01, DD04].

The *semi-extremal* endomorphisms are defined by $\lambda_1 > \lambda_2 = \frac{1}{2}\log d$. A natural question is to find examples and characterizations of these endomorphisms. We already know that they satisfy $\dim_H \mu = 2 + \frac{\log d}{\lambda_1}$, see [Dup11]. Corollary 1.4 provides an interesting class of semi-extremal endomorphisms. In particular, the family of Desboves mappings

$$f_c : [z : w : t] \mapsto \left[-z(z^3 + 2w^3) : w(2z^3 + w^3) : t(w^3 - z^3 + c(z^3 + w^3 + t^3))\right] \ , \ c \in \mathbb{C}^*,$$

belongs to this class, since $[z : w] \to [-z(z^3 + 2w^3) : w(2z^3 + w^3)]$ is a Lattès map on $\mathbb{P}^1$. Theorem 1.7 specifies that their largest exponent satisfies $\lambda_1 \geq \log d$. A natural question is to determine whether the lower bound $\lambda_1 \geq \log d$ holds for every semi-extremal endomorphism. Bonifant, Dabija and Milnor studied Desboves mappings in [BD02] and [BDM07, Section 4], with a focus on the study of the invariant smooth cubic curve $C := \{z^3+w^3+t^3 = 0\}$ as an attractor. This family is also very interesting in bifurcation theory: Bianchi-Taflin [BT17] proved that its bifurcation locus coincides with $\mathbb{C}^*$. This locus has non-empty interior, this property can not occur for bifurcations of rational maps on $\mathbb{P}^1$.

3 Basics on pluripotential theory

In Section 6 and Section 7, we will intersect currents supported by fibers of a rational map $\pi\colon \mathbb{P}^k \dashrightarrow \mathbb{P}^r$ and singular currents or functions. Moreover, it will be crucial that these intersections depend continuously on the fiber. It seems complicated to obtain these statements for an arbitrary dominant rational map π. The aim of the first part of this section is to give two results in that direction when π is the standard linear fibration. In a second part, we explain how to define the pull-back $\pi^*\tau$ of a positive closed $(1,1)$-current τ on $\mathbb{P}^r$ by a rational map and how to define its self-intersections $(\pi^*\tau)^j$ in the setting of Theorem 1.1.

3.1 Continuous families of currents

Let $\pi\colon \mathbb{P}^k \dashrightarrow \mathbb{P}^r$ be the standard linear fibration defined by $\pi[y : z] = [y]$ where $y := (y_0, \ldots, y_r) \in \mathbb{C}^{r+1}$ and $z = (z_0, \ldots, z_{q-1}) \in \mathbb{C}^q$. We recall that the indeterminacy set $I(\pi)$ of π corresponds to $\{y = 0\} \simeq \mathbb{P}^{q-1}$ and each fiber $L_a := \overline{\pi^{-1}(a)}$ is a projective space in which $I(\pi)$ can be identified with the hyperplane at infinity, i.e. $L_a \setminus I(\pi) \simeq \mathbb{C}^q$.

In the following two results, we consider an integer $0 \leq l \leq k$ and a family $(R_a)_{a\in\mathbb{P}^r}$ of positive closed $(k-l,k-l)$-currents in $\mathbb{P}^k$ such that $a \mapsto R_a$ is continuous. We also consider an open set $U \subset \mathbb{P}^k$ and an upper semicontinuous function $v\colon U \to [-\infty, 0]$ such that $dd^c v = T_1 - T_2$, where T_1 and T_2 are two positive closed $(1,1)$-currents where T_2 has continuous local potentials.

Lemma 3.1. *Assume there exist two analytic subsets $X, Y \subset \mathbb{P}^k$ such that v is continuous on $U\backslash X$ and such that for all $a \in \mathbb{P}^r$ we have $\mathrm{supp}(R_a) \subset L_a \cap Y$ and $\dim(L_a \cap X \cap Y) \leq l-1$. Then, for all $a \in \mathbb{P}^r$ the current vR_a is well-defined on U and depends continuously on a.*

Proof. The facts that vR_a is well-defined and that its mass is locally uniformly bounded with respect to a follow easily from the Oka inequality obtained by Fornæss-Sibony [FS95]. In order to give some details, we freely use the terminology coming from [FS95]. Let $a \in \mathbb{P}^r$ and $x \in U$. Since $\dim(L_a \cap X \cap Y) \leq l-1$, there exists an $(k-l,l)$ Hartogs figure H disjoint from $L_a \cap X \cap Y$ such that its hull $\widehat{H}$ is a neighborhood of x in U. By continuity of $a \mapsto L_a$, there exists a neighborhood V of a such that $L_{a'} \cap X \cap Y \cap H = \varnothing$ for all $a' \in V$. Since $v \leq 0$ and $dd^c v = T_1 - T_2$ where T_2 has continuous local potential, up to a continuous function v is equal to a plurisubharmonic function on $\widehat{H}$ and [FS95, Proposition 3.1] implies that $vR_{a'}$ is well-defined for all $a' \in V$. Moreover, again possibly by exchanging v by $v + \phi$ with ϕ continuous, we can assume that $vR_{a'} \leq 0$ and $dd^c(vR_{a'}) \geq 0$ on $\widehat{H}$. Hence, we can apply the Oka inequality [FS95, Theorem 2.4]. If K is a compact set contained in the interior of $\widehat{H}$ then there exists a constant $C > 0$ such that for all $a' \in V$

$$\|vR_{a'}\|_K \leq C\|vR_{a'}\|_H.$$

Since $a \mapsto R_a$ is continuous and v is continuous on $\cup_{a'\in V}\mathrm{supp}(R_{a'}) \cap H$, we obtain that the mass of $vR_{a'}$ is uniformly bounded on K for $a' \in V$ and we conclude using the compactness of $\mathbb{P}^r$.

To prove the continuity of $a \mapsto vR_a$, let $(a_n)_{n\geq 1}$ be a sequence in $\mathbb{P}^r$ converging to a_0. Since vR_a has locally uniformly bounded mass, we can assume that vR_{a_n} converges to a current R'. We must have

$$\mathrm{supp}R' \subset \limsup_{n\to\infty}(\mathrm{supp}(R_{a_n})) \subset \limsup_{n\to\infty}(L_{a_n} \cap Y) \subset L_a \cap Y.$$

On the other hand, by continuity of $a \mapsto R_a$ and since v is continuous on $U \setminus X$, we have that $R' = vR_{a_0}$ outside $L_a \cap Y \cap X$ which has dimension smaller than or equal to $l-1$. Hence, the support theorem of Bassanelli [Bas94] for currents T such that T and $dd^c T$ have order 0 implies that $R' = vR_{a_0}$ on U. □

Lemma 3.2. *Let $(\nu_n)_{n\geq 1}$ a sequence of probabilities in $\mathbb{P}^r$ which converges to ν. Let us define $R := \int R_a d\nu$ and $R_n := \int R_a d\nu_n$. If vR_a is well defined on U and $a \mapsto vR_a$ is continuous then vR_n and vR are well-defined on U and satisfy $vR_n = \int vR_a d\nu_n$, $vR = \int vR_a d\nu$ and $\lim_{n\to\infty} vR_n = vR$.*

Proof. Let ϕ be a (l,l) smooth form with compact support in U. We can assume that the support of ϕ is contained in a small ball $B \subset U$ on which $v = u_1 - u_2$ where u_1, u_2 are plurisubharmonic on B and u_2 is continuous. Hence, there exists a decreasing sequence $(u_{1,j})_{j\geq 1}$ of continuous plurisubharmonic functions on B converging pointwise to u_1. Define $v_j := u_{1,j} - u_2$. Since vR_a is well defined then $\langle v_j R_a, \phi\rangle$ decreases to $\langle vR_a, \phi\rangle$ by [FS95, Corollary 3.3]. In particular, if we define $\psi_j(a) := \langle v_j R_a, \phi\rangle$ and $\psi(a) := \langle vR_a, \phi\rangle$ then ψ_j decreases pointwise to ψ which is a continuous function since $a \mapsto vR_a$ is continuous. Hence, by Dini's theorem ψ_j converges uniformly to ψ.

On the other hand, by monotone convergence theorem $\langle v_j R, \phi\rangle$ decreases to $\langle vR, \phi\rangle$ which is potentially equal to $-\infty$. But, by definition of R we have

$$\lim_{j\to\infty}\langle v_j R, \phi\rangle = \lim_{j\to\infty}\int_{\mathbb{P}^r}\langle v_j R_a, \phi\rangle d\nu(a) = \lim_{j\to\infty}\int_{\mathbb{P}^r}\psi_j d\nu = \int_{\mathbb{P}^r}\psi d\nu = \int_{\mathbb{P}^r}\langle vR_a, \phi\rangle d\nu(a),$$

i.e. $vR = \int vR_a d\nu$. The same holds for R_n, and then $\lim_{n\to\infty} vR_n = vR$ since $a \mapsto vR_a$ is continuous. □

3.2 Pull-back of $(1,1)$-currents by rational maps

In this subsection, $\pi\colon \mathbb{P}^k \dashrightarrow \mathbb{P}^r$ is a dominant rational map with $\dim(I(\pi)) \leq q-1$.

As π is not supposed to be a submersion on $\mathbb{P}^k \setminus I(\pi)$, the definition of the pull-back operator π^* on currents requires some work. However, we will only consider currents given by wedge products of positive closed $(1,1)$-currents with continuous local potentials, which greatly simplifies the problem. If τ is a positive closed $(1,1)$-current on $\mathbb{P}^r$ which is equal locally to $dd^c u$ then $\pi^*_{|\mathbb{P}^k\setminus I(\pi)}\tau$ can be defined locally on $\mathbb{P}^k \setminus I(\pi)$ as $dd^c u\circ\pi$. Méo [Méo96] proved that $\tau \mapsto \pi^*_{|\mathbb{P}^k\setminus I(\pi)}\tau$ is continuous. Moreover, since $I(\pi)$ has codimension at least 2, the trivial extension of $\pi^*_{|\mathbb{P}^k\setminus I(\pi)}\tau$ to $\mathbb{P}^k$ is again a positive closed $(1,1)$-current that we denote by $\pi^*\tau$. We summarize in the following proposition the properties about pull-back we shall need in the sequel. Recall that if R_1 and R_2 are two positive closed currents on $\mathbb{P}^k$ of bidegree $(1,1)$ and (j,j) respectively then the wedge product $R_1\wedge R_2$ is well-defined if the local potentials of R_1 are integrable with respect to $R_2\wedge\omega_{\mathbb{P}^k}^{k-j}$ (see e.g. [BT82]).

Proposition 3.3. *Let $\pi\colon \mathbb{P}^k \dashrightarrow \mathbb{P}^r$ be a dominant rational map whose indeterminacy set has a dimension smaller than or equal to $q-1$. If τ is a positive closed $(1,1)$-current of mass 1 on $\mathbb{P}^r$ then $\|\pi^*\tau\|$ is equal to the algebraic degree $\deg(\pi)$ of π. If τ has continuous local potentials then the self-intersections $(\pi^*\tau)^j$ are well-defined for $j\in\{1,\ldots,r\}$. The currents $(\pi^*\tau)^j$ coincide with the trivial extension of the standard pull-back of τ^j if τ is smooth. Moreover, if (u_n) is a sequence of continuous functions which converges uniformly to 0 then the currents $\tau_n := \tau + dd^c u_n$ satisfy $\lim_{n\to\infty}(\pi^*\tau_n)^j = (\pi^*\tau)^j$.*

Proof. Let τ be a positive closed $(1,1)$-current of mass 1 on $\mathbb{P}^r$. As we have said, in [Méo96] Méo proved that $\pi^*_{|\mathbb{P}^k\setminus I(\pi)}\tau$ depends continuously on τ. On the other hand, since $I(\pi)$ has codimension at least 2, the trivial extension, denoted by $\pi^*\tau$, is a positive closed $(1,1)$-current on $\mathbb{P}^k$. Moreover, as in the proof of Lemma 3.1, the Oka inequality obtained in [FS95] implies that the mass of $\pi^*\tau$ is bounded independently of τ. Hence, $\tau\mapsto\pi^*\tau$ is also continuous. Indeed, if $(\tau_n)_{n\geq1}$ is a sequence of positive closed $(1,1)$-currents converging to τ then $(\pi^*\tau_n)_{n\geq1}$ has uniformly bounded mass on $\mathbb{P}^k$ and using the continuity of $\pi^*_{|\mathbb{P}^k\setminus I(\pi)}$ each limit value R has to be equal to $\pi^*_{|\mathbb{P}^k\setminus I(\pi)}\tau$ on $\mathbb{P}^k\setminus I(\pi)$. Finally, $R = \pi^*\tau$ since $I(\pi)$ has codimension at least 2. To see that $\|\pi^*\tau\|$ is in fact independent of τ, observe that if $\tau = \omega_{\mathbb{P}^r} + dd^c u$ where u is a continuous function, then $u\circ\pi$ is in $L^1(\mathbb{P}^k)$ and $\langle dd^c(u\circ\pi), \omega_{\mathbb{P}^k}^{k-1}\rangle = 0$. Moreover, $(\pi^*\tau) - (\pi^*\omega_{\mathbb{P}^r}) = dd^c(u\circ\pi)$ so $\|\pi^*\tau\| = \|\pi^*\omega_{\mathbb{P}^r}\|$. The general case follows by continuity since smooth forms are dense in the space of positive closed $(1,1)$-currents on $\mathbb{P}^r$. Since $\|\pi^*\tau\|$ is independent of τ, we obtain that it is equal to $\deg(\pi)$ by taking an hyperplane in $\mathbb{P}^r$.

We now assume as in the statement that $\dim(I(\pi))\leq q-1$. Let R be a positive closed (j,j)-current on $\mathbb{P}^k$ with $j\in\{1,\ldots,r\}$. If τ has continuous local potentials then $\pi^*\tau$ has continuous local potentials except on $I(\pi)$, i.e. the set of points where these local potentials are unbounded is contained in $I(\pi)$. Hence, using the assumption on $\dim(I(\pi))$ we can deduce from [FS95] that these local potentials are integrable with respect to $R\wedge\omega_{\mathbb{P}^k}^{k-j}$ and thus $(\pi^*\tau)\wedge R$ is well-defined. In particular, $(\pi^*\tau)^j$ is well-defined for $j\in\{1,\dots,r+1\}$. The fact that these currents coincide with the trivial extension of $\pi^*_{|\mathbb{P}^k\setminus I(\pi)}(\tau^j)$ if τ is smooth and $j\in\{1,\dots,r\}$ follows exactly as above. Observe however that for $j=r+1$, $\pi^*(\tau^{r+1})$ vanishes whereas $(\pi^*\tau)^{r+1}$ has mass $\deg(\pi)^{r+1}$ and thus differs from 0. This implies that the support of $(\pi^*\tau)^{r+1}$ is contained in $I(\pi)$ and thus $\dim(I(\pi))\geq q-1$ i.e. $\dim(I(\pi))=q-1$.

We prove the last assertion by induction. The case $j=1$ follows from the first part of this proof. Assume the assertion is true for $j-1$ with $j\in\{2,\dots,r\}$. Observe that since (u_n) converges uniformly to 0, $\mathbb{P}^k\setminus I(\pi)$ is covered by open sets Ω where we can write $\pi^*\tau=dd^cv$ and $\pi^*\tau_n=dd^cv_n$ where (v_n) is a sequence of continuous functions converging uniformly to v. Hence, if ϕ is a smooth form with compact support on Ω then

$$\begin{aligned}\langle(\pi^*\tau_n)^j-(\pi^*\tau)^j,\phi\rangle&=\langle dd^c(v_n(\pi^*\tau_n)^{j-1}-v(\pi^*\tau)^{j-1}),\phi\rangle\\&=\langle v((\pi^*\tau_n)^{j-1}-(\pi^*\tau)^{j-1}),dd^c\phi\rangle+\langle(v_n-v)(\pi^*\tau_n)^{j-1},dd^c\phi\rangle.\end{aligned}$$

The inductive hypothesis implies that the first term in this sum converges to 0. The second term also converges to 0 since (v_n-v) converges uniformly to 0. Hence, any limit value of $(\pi^*\tau_n)^j$ has to be equal to $(\pi^*\tau)^j$ on $\mathbb{P}^k\setminus I(\pi)$ and this equality extends to $\mathbb{P}^k$ since $\dim(I(\pi))\leq q-1$. □

Remark 3.4. *The degree* $\deg(\pi)$ *of* π *is not necessary equal to* 1. *If* f *preserves the fibration defined by* π *then the fibration defined by* $\pi\circ f$ *is still preserved by* f *and if* f *has degree* d *then* $\deg(\pi\circ f)=d\deg(\pi)$. *Another example in our setting is the binomial pencil of* $\mathbb{P}^2$ *considered in [DJ08] and [FP11] where the degree is an arbitrary integer, see Section 2.*

Remark 3.5. *We have shown in the preceding proof that* $\dim(I(\pi))\geq q-1$. *This fact also follows from the definition of* $I(\pi)$ *as the common zeros of the* $r+1$ *homogeneous polynomials defining* π.

4 Structure of the Green currents

This section is devoted to the proofs of Theorem 1.1, Corollary 1.2 and Corollary 1.3.

Let f be an endomorphism of $\mathbb{P}^k$ of degree $d\geq 2$. Recall that the *Green current* T_f of f can be defined as

$$T_f=\lim_{n\to\infty}\frac{1}{d^n}f^{n*}\omega_{\mathbb{P}^k}.$$

We refer to [DS10] for a detailed study of this current. In what follows, we will use that T_f has Hölder local potentials and if $l\in\{1,\dots,k\}$ then its self-intersection T_f^l satisfies $T_f^l=\lim_{n\to\infty}d^{-ln}f^{n*}\omega_{\mathbb{P}^k}^l$.

Proof of Theorem 1.1. First observe that since $I(\pi)\cap\mathscr{J}_q(f)=\varnothing$, a cohomological arguments implies that the dimension of $I(\pi)$ is at most $q-1$ and π satisfies the assumption of Proposition 3.3. Using this proposition, we define

$$R:=\deg(\pi)^{-1}\pi^*\omega_{\mathbb{P}^r}\quad\text{and}\quad S:=\deg(\pi)^{-1}\pi^*T_\theta$$

which are two positive closed $(1,1)$-currents of mass 1. We also deduce from $I(\pi)\cap\mathscr{J}_q(f)=\varnothing$ that there exists a neighborhood U of $I(\pi)$ such that $\mathscr{J}_q(f)\cap U=\varnothing$. Since R is a smooth form on $\mathbb{P}^k\setminus I(\pi)$, there is a constant $C>0$ such that $R\leq C\omega_{\mathbb{P}^k}$ on $\mathbb{P}^k\setminus U$ and thus

$T_f^q \wedge R^j \leq C^j T_f^q \wedge \omega_{\mathbb{P}^k}^j$ on $\mathbb{P}^k$ for $1 \leq j \leq r$. Applying the operator $d^{-n(q+j)} f^{n*}$ to this inequality gives

$$T_f^q \wedge \left(\frac{1}{d^{nj}} f^{n*} R^j\right) \leq C^j T_f^q \wedge \left(\frac{1}{d^{nj}} f^{n*} \omega_{\mathbb{P}^k}^j\right).$$

The equidistribution results for f and the fact that T_f has continuous local potentials (see [DS10]) imply that the right-hand side converges to $C^j T_f^{q+j}$. On the other hand, as $\pi \circ f^n = \theta^n \circ \pi$ we have by Proposition 3.3

$$T_f^q \wedge \left(\frac{1}{d^{nj}} f^{n*} R^j\right) = T_f^q \wedge \left(\frac{1}{\deg(\pi)^j d^{nj}} f^{n*} \pi^* \omega_{\mathbb{P}^r}^j\right) = T_f^q \wedge \pi^* \left(\frac{1}{\deg(\pi)^j d^{nj}} \theta^{n*} \omega_{\mathbb{P}^r}^j\right).$$

Recall that $d^{-n} \theta^{n*} \omega_{\mathbb{P}^r} = T_\theta + dd^c u_n$ where u_n are continuous functions converging uniformly to 0. Hence, by Proposition 3.3 the sequence above converges to $T_f^q \wedge S^j$ and thus $T_f^q \wedge S^j \leq C^j T_f^{q+j}$. Moreover, $T_f^q \wedge S^j$ is invariant by $d^{-(q+j)} f^*$ and T_f^{q+j} is extremal in the cone of such currents (see [Sib99] when $q+j=1$ and [DS09] for the general case) so $T_f^q \wedge S^j = T_f^{q+j}$.

The proof of $S^j \neq T_f^j$ for $j \in \{1, \ldots, r\}$ simply comes from the fact (observed in the proof of Proposition 3.3) that S^{r+1} is supported in $I(\pi)$ which has dimension at most $q-1$. On the other hand, since T_f has Hölder local potentials, it follows from [Sib99] that $T_f^j \wedge S^{r+1-j}$ gives no mass to analytic sets of dimension $q-2+j \geq q-1$.

Finally, in order to prove the last assertion, observe that since $I(\pi) \cap \mathscr{J}_q(f) = \varnothing$, the current $T_f^{q+j} = T_f^q \wedge S^j$ satisfies

$$\pi_*(T_f^q \wedge S^j) = \deg(\pi)^{-j} \pi_*(T_f^q \wedge \pi^* T_\theta^j) = \deg(\pi)^{-j} (\pi_* T_f^q) \wedge T_\theta^j. \tag{4.1}$$

The current $\pi_* T_f^q$ is positive and closed of bidegree $(0,0)$ on $\mathbb{P}^r$ thus it is a positive multiple of $[\mathbb{P}^r]$. Since π_* preserves the mass of measures, the equation (4.1) with $j = r$ implies $\pi_* T_f^q = \deg(\pi)^r [\mathbb{P}^r]$ and thus $\pi_*(T_f^{q+j}) = \deg(\pi)^{r-j} T_\theta^j$. In particular, $\pi_* \mu_f = \mu_\theta$. □

Proof of Corollary 1.2. We shall use the formula $\mu_f = T_f^q \wedge S^r$. The proof would have been straightforward if π were a submersion on $\mathbb{P}^k \setminus I(\pi)$ and the current T_f^q were smooth. However, since T_f has continuous local potentials, we can use regularization as follows.

Let $\phi \colon \mathbb{P}^k \to \mathbb{R}$ be a continuous function. The critical set Crit_π of π is by definition the union of $I(\pi)$ with the set of points in $\mathbb{P}^k \setminus I(\pi)$ where the differential of π has rank strictly less than r. This set is algebraic so the measure μ_f gives no mass to it. Hence, if $\chi_n \colon \mathbb{P}^k \to [0,1]$ are smooth functions with compact support in $\mathbb{P}^k \setminus \mathrm{Crit}_\pi$ which converge locally uniformly to 1 on $\mathbb{P}^k \setminus \mathrm{Crit}_\pi$ then $\langle \mu_f, \phi \rangle = \lim_{n\to\infty} \langle \mu_f, \chi_n \phi \rangle$. Moreover, for μ_θ-almost all $a \in \mathbb{P}^r$ the algebraic set $L_a := \overline{\pi^{-1}_{|\mathbb{P}^k \setminus I(\pi)}(a)}$ has dimension q and $L_a \cap \mathrm{Crit}_\pi$ has dimension $q-1$. In particular, the current $[L_a]$ has mass $\deg(\pi)^r$ and coincides with the trivial extension of $\pi^*_{|\mathbb{P}^k \setminus \mathrm{Crit}_\pi} \delta_a$. This implies that if Ψ is a smooth (q,q)-form on $\mathbb{P}^k$ then the function $\pi_*(\chi_n \phi \Psi)$ on $\mathbb{P}^r$ is equal μ_θ-everywhere to $a \mapsto \langle \Psi \wedge [L_a], \chi_n \phi \rangle$.

On the other hand, the current T_f has continuous local potentials so there exists a continuous function g such that $T_f = \omega_{\mathbb{P}^k} + dd^c g$. Let $(g_l)_{l \geq 1}$ be a sequence of smooth functions converging uniformly to g and define $T_l := \omega_{\mathbb{P}^k} + dd^c g_l$. The uniform convergence implies that if R is a positive closed (r,r)-current then $T_l^q \wedge R$ converges to $T_f^q \wedge R$. Hence,

$\mu_f = T_f^q \wedge S^r$ implies

$$\begin{aligned}\langle \mu_f, \phi\rangle &= \lim_{n\to\infty}\langle \mu_f, \chi_n\phi\rangle = \lim_{n\to\infty}\lim_{l\to\infty}\langle T_l^q \wedge S^r, \chi_n\phi\rangle = \lim_{n\to\infty}\lim_{l\to\infty}\langle \mu_\theta, \pi_*(\chi_n\phi T_l^q)\rangle/\deg(\pi)^r \\ &= \lim_{n\to\infty}\lim_{l\to\infty}\int_{\mathbb{P}^r}\langle T_l^q \wedge \frac{[L_a]}{\deg(\pi)^r}, \chi_n\phi\rangle d\mu_\theta(a) = \lim_{n\to\infty}\int_{\mathbb{P}^r}\langle T_f^q \wedge \frac{[L_a]}{\deg(\pi)^r}, \chi_n\phi\rangle d\mu_\theta(a) \\ &= \int_{\mathbb{P}^r}\langle T_f^q \wedge \frac{[L_a]}{\deg(\pi)^r}, \phi\rangle d\mu_\theta(a),\end{aligned}$$

where the last equality comes from the fact that for μ_θ-almost all a, $L_a \cap \mathrm{Crit}_\pi$ has dimension $q-1$ and $T_f^q \wedge [L_a]$ gives no mass to such sets. □

By the Radon-Nikodym theorem, the following result implies Corollary 1.3.

Corollary 4.1. *Under the assumptions of Theorem 1.1, if there exists a positive $\omega_{\mathbb{P}^r}^r$-integrable function h such that $\mu_\theta = h\omega_{\mathbb{P}^r}^r$ then there exists $A > 0$ such that $\mu_f \leq A\,(h\circ\pi)\,\sigma_{T_f^q}$. In particular, $\mu_f \ll \sigma_{T_f^q}$.*

Proof. Let $R := \deg(\pi)^{-1}\pi^*\omega_{\mathbb{P}^r}$ as in the proof of Theorem 1.1. The same theorem gives that $\mu_f = \deg(\pi)^{-r}T_f^q \wedge (\pi^*\mu_\theta) = \frac{h\circ\pi}{\deg(\pi)^r}T_f^q \wedge R^r$. On the other hand, we have seen in the proof of Theorem 1.1 that $T_f^q \wedge R^r \leq C^r\sigma_{T_f^q}$ thus $\mu_f \leq \left(\frac{C}{\deg(\pi)}\right)^r (h\circ\pi)\,\sigma_{T_f^q}$. □

5 Lyapunov exponents

This section is dedicated to the proof of Theorem 1.5. The main ingredients are the formula $\pi_*\mu_f = \mu_\theta$, the fact that these measures put no mass on proper analytic sets and the local uniform convergence in Oseledec Theorem. We refer the reader to [BP13, Chapter 5–6] for the details on Oseledec theorem we shall need.

Proof of Theorem 1.5. First, let us set some notations. Let Crit_θ be the critical set of θ. We denote by $\widehat{\mathbb{P}^r} := \{(y_n)_{n\in\mathbb{Z}} \in \mathbb{P}^r \mid \theta(y_n) = y_{n+1}\}$ the natural extension and by $\widehat{\theta}$ the left-shift on $\widehat{\mathbb{P}^r}$. The projection $\mathrm{proj}\colon \widehat{\mathbb{P}^r} \to \mathbb{P}^r$ defined by $\mathrm{proj}((y_n)_{n\in\mathbb{Z}}) = y_0$ satisfies $\theta\circ\mathrm{proj} = \mathrm{proj}\circ\widehat{\theta}$. The measure μ_θ has a unique lift $\widehat{\mu}_\theta$ which is invariant by $\widehat{\theta}$ and such that $\mathrm{proj}_*\widehat{\mu}_\theta = \mu_\theta$. In what follows, if $\widehat{y}$ is in $\widehat{\mathbb{P}^r}$ we will write y_0 instead of $\mathrm{proj}(\widehat{y})$. The measure $\widehat{\mu}_\theta$ inherits several properties from μ_θ. Since μ_θ is ergodic and integrates the quasi-plurisubharmonic functions, it follows that $\widehat{\mu}_\theta$ is ergodic, gives no mass to $\cup_{p\in\mathbb{Z}}\hat{\theta}^p(\mathrm{proj}^{-1}(\mathrm{Crit}_\theta))$ and integrates the functions $\log\|D\theta^{\pm 1}\|$. In particular, for $\widehat{\mu}_\theta$-almost all $\widehat{y} = (y_n)_{n\in\mathbb{Z}}$ the differentials

$$D_{\widehat{y}}\theta^{-n} := (D_{y_{-n}}\theta^n)^{-1}, \quad D_{\widehat{y}}\theta^n := D_{y_0}\theta^n$$

are well defined for $n \geq 1$. Moreover, by Oseledec theorem, there exist distincts numbers $\Lambda_1, \ldots, \Lambda_s \in \mathbb{R}$ and a $\widehat{\theta}$-invariant set $Y \subset \widehat{\mathbb{P}^r}$, included in $(\mathbb{P}^r \setminus \mathrm{Crit}_\theta)^{\mathbb{Z}}$, such that $\widehat{\mu}_\theta(Y) = 1$ and for each $\widehat{y} \in Y$ the tangent space $T_{y_0}\mathbb{P}^r$ admits a splitting $T_{y_0}\mathbb{P}^r = \bigoplus_{i=1}^s V_i(\widehat{y})$ which satisfies

$$D_{y_0}\theta(V_i(\widehat{y})) = V_i(\widehat{\theta}(\widehat{y})) \quad \text{and} \quad \lim_{n\to\pm\infty}\frac{1}{n}\log\|D_{\widehat{y}}\theta^n(u)\| = \Lambda_i$$

uniformly for $u \in V_i(\widehat{y})$ with $\|u\| = 1$. By definition, the multiplicity of Λ_i is the dimension m_i of $V_i(\widehat{y})$. Moreover, the subspaces $V_i(\widehat{y})$ can be characterized by $V_i(\widehat{y}) \setminus \{0\} = \{u \in T_{y_0}\mathbb{P}^r \setminus \{0\} \mid \lim_{n\to\pm\infty} n^{-1}\log\|D_{\widehat{y}}\theta^n(u)\| = \Lambda_i\}$.

The natural extension $\widehat{\mathbb{P}^k}$, the map $\widehat{f}$ and the measure $\widehat{\mu}_f$ are defined in the same way with respect to f and since $\pi\circ f = \theta\circ\pi$, the map π lifts to a map $\widehat{\pi}\colon \widehat{\mathbb{P}^k} \to \widehat{\mathbb{P}^r}$ such that $\widehat{\pi}\circ\widehat{f}=\widehat{\theta}\circ\widehat{\pi}$. The uniqueness of the lift $\widehat{\mu}_\theta$ of μ_θ and the fact that $\pi_*\mu_f=\mu_\theta$ imply that $\widehat{\pi}_*\widehat{\mu}_f=\widehat{\mu}_\theta$. In particular, the set $\widehat{\pi}^{-1}(Y)$ has full $\widehat{\mu}_f$-measure. The measure $\widehat{\mu}_f$ also admits a Oseledec decomposition on a $\widehat{f}$-invariant set Z of full $\widehat{\mu}_f$-measure. And since the measure μ_f gives no mass to proper analytic sets, if Crit_π denotes the critical set of π (i.e. the union of $I(\pi)$ with the set of points in $\mathbb{P}^k\setminus I(\pi)$ where the differential of π has rank strictly less than r) then the $\widehat{f}$-invariant set

$$X:=Z\cap\widehat{\pi}^{-1}(Y)\cap(\mathbb{P}^k\setminus(\mathrm{Crit}_f\cup\mathrm{Crit}_\pi))^{\mathbb{Z}}$$

also has full $\widehat{\mu}_f$-measure.

Now, fix $i\in\{1,\dots,s\}$ and for $\widehat{x}\in X$ consider the subspace of $T_{x_0}\mathbb{P}^k$ defined by

$$W_i(\widehat{x})=(D_{x_0}\pi)^{-1}(V_i(\widehat{\pi}(\widehat{x}))).$$

As X is disjoint from the critical sets of f and π, these subspaces have dimension m_i+q and define a Df-invariant distribution i.e. $D_{x_0}f(W_i(\widehat{x}))=W_i(\widehat{f}(\widehat{x}))$. Therefore, Oseledec theorem applies to the measurable cocycle defined by the action of Df on $W_i(\widehat{x})$ and induces $\widehat{\mu}_f$-almost everywhere a decomposition $W_i(\widehat{x})=\bigoplus_{j=1}^l F_{ij}(\widehat{x})$ for some integer $l\geq 1$. Since this cocycle is a sub-cocycle of the standard one, each $F_{ij}(\widehat{x})$ is associated to a Lyapunov exponent λ_j of μ_f and satisfies

$$F_{ij}(\widehat{x})\setminus\{0\}=\{v\in W_i(\widehat{x})\setminus\{0\}\mid \lim_{n\to\pm\infty}n^{-1}\log\|D_{\widehat{x}}f^n(v)\|=\lambda_j\}.$$

In particular, $\dim(F_{ij}(\widehat{x}))$ is bounded by the multiplicity of λ_j. Now we show that if $F_{ij}(\widehat{x})$ is not contained in $\ker D_{x_0}\pi$ then $\lambda_j=\Lambda_i$, that property will be sufficient to conclude. So let $\widehat{x}\in X$ and $j\in\{1,\dots,l\}$ such that $F_{ij}(\widehat{x})\not\subset\ker D_{x_0}\pi$. In particular, there exists $v\in F_{ij}(\widehat{x})\setminus\ker D_{x_0}\pi$ and so

$$\begin{aligned}\Lambda_i=\lim_{n\to+\infty}\frac{1}{n}\log\|D_{\widehat{x}}(\theta^n\circ\pi)(v)\|&=\lim_{n\to+\infty}\frac{1}{n}\log\|D_{\widehat{x}}(\pi\circ f^n)(v)\|\\&\leq\lim_{n\to+\infty}\frac{1}{n}\log\|D_{\widehat{x}}f^n(v)\|=\lambda_j.\end{aligned}$$

Here, the inequality comes from $I(\pi)\cap\mathrm{supp}(\mu_f)=\varnothing$ and thus, there exists $C>0$ such that $\|D_x\pi(v)\|\leq C\|v\|$ for all $x\in\mathrm{supp}(\mu_f)$ and $v\in T_x\mathbb{P}^k$.

For the converse inequality, observe that since the subspaces $W_i(\widehat{x})$ and $F_{ij}(\widehat{x})$ depends measurably on $\widehat{x}$, there exist a constant $c>0$ and $B\subset X$ with $\widehat{\mu}_f(B)>0$ such that for each $\widehat{x}\in B$ there is a subspace $E(\widehat{x})\subset F_{ij}(\widehat{x})$ of positive dimension satisfying $E(\hat{x})\cap\ker D_{x_0}\pi=\{0\}$ and $\|D_{x_0}\pi(v)\|\geq c\|v\|$ for all $v\in E(\widehat{x})$. By Poincaré recurrence theorem, for $\widehat{\mu}_f$-almost all $\widehat{x}\in X$ there exists an increasing sequence $(k_n)_{n\geq1}$ of integers such that $\widehat{f}^{k_n}(\widehat{x})\in B$ for all $n\geq 1$. Therefore, if $v_n\in F_{ij}(\widehat{x})$ is such that $\|v_n\|=1$ and $D_{\widehat{x}}f^{k_n}(v_n)\in E(\widehat{f}^{k_n}(\widehat{x}))$ then

$$\begin{aligned}\lambda_j=\lim_{n\to+\infty}\frac{1}{k_n}\log\|D_{\widehat{x}}f^{k_n}(v_n)\|&\leq\lim_{n\to+\infty}\frac{1}{k_n}\log\|D_{\widehat{x}}(\pi\circ f^{k_n})(v_n)\|\\&=\lim_{n\to+\infty}\frac{1}{k_n}\log\|D_{\widehat{x}}(\theta^{k_n}\circ\pi)(v_n)\|\leq\Lambda_i.\end{aligned}$$

Here, the first equality comes from the fact that the convergence in Oseledec theorem is uniform on the unit sphere of $F_{ij}(\widehat{x})$. The last inequality uses a similar argument as well as the uniform bound $\|D_x\pi(v)\|\leq C\|v\|$ for $x\in\mathrm{supp}(\mu_f)$ and the fact that $v_n\notin\ker D_{x_0}\pi$ since $D_{\widehat{x}}f^{k_n}(v_n)\in E(\widehat{f}^{k_n}(\widehat{x}))$.

We have shown that if $F_{ij}(\widehat{x})\not\subset\ker D_{x_0}\pi$ then $\lambda_j=\Lambda_i$. Thus there is a unique $j\in\{1,\dots,l\}$ with this property. As $\dim(W_i(\widehat{x}))=m_i+q$ and $\dim(\ker D_{x_0}\pi)=q$, the dimension of $F_{ij}(\widehat{x})$ has to be at least m_i. □

6 Generalized Bedford-Jonsson's formula

In this section, we assume that the fibration π is the standard linear fibration $\pi\colon \mathbb{P}^k \dashrightarrow \mathbb{P}^r$ defined by $\pi[y:z] = [y]$ where $y := (y_0, \ldots, y_r) \in \mathbb{C}^{r+1}$ and $z = (z_0, \ldots, z_{q-1}) \in \mathbb{C}^q$. We first analyse some basic properties of maps preserving such a fibration and then we give a proof of Theorem 1.7.

As we have said in the introduction, the indeterminacy set of π corresponds to $\{y = 0\} \simeq \mathbb{P}^{q-1}$ and each fiber $L_a := \overline{\pi^{-1}(a)}$ is a projective space $\mathbb{P}^q$ in which $I(\pi)$ can be identified with the hyperplane at infinity, i.e. $L_a \setminus I(\pi) \simeq \mathbb{C}^q$. If $f\colon \mathbb{P}^k \to \mathbb{P}^k$ preserves such a fibration then it lifts to a polynomial endomorphism F of $\mathbb{C}^{k+1}$ of the form

$$F(y, z) = (\Theta(y), R(y, z)),$$

where $y \in \mathbb{C}^{r+1}$, $z \in \mathbb{C}^q$ and Θ, R are homogeneous polynomials. The map Θ is a lift of the endomorphism θ of $\mathbb{P}^r$ such that $\theta \circ \pi = \pi \circ f$. The inequality $\|\Theta(y)\|_\infty \leq \|F(y, z)\|_\infty$ implies that the functions

$$G_\Theta(y) := \lim_{n\to\infty} \frac{1}{d^n} \log \|\Theta^n(y)\|_\infty \quad \text{and} \quad G_F(y, z) := \lim_{n\to\infty} \frac{1}{d^n} \log \|F^n(y, z)\|_\infty,$$

satisfy $G_\Theta(y) \leq G_F(y, z)$. The difference of these functions descends to $\mathbb{P}^k$ and we define it as the *relative Green function* of f, $G[y:z] := G_F(y, z) - G_\Theta(y)$. It is the unique lower semicontinuous function such that

$$dd^c G = T_f - S \quad \text{and} \quad \min G = 0.$$

Here, T_f is the Green current of f and $S := \pi^* T_\theta$. It is easy to check that G is invariant (i.e. $d^{-1} G \circ f = G$), continuous on $\mathbb{P}^k \setminus I(\pi)$ and $\{G = +\infty\} = I(\pi)$. As we will see in the proof of the next lemma, $I(\pi)$ is an attracting set for f and G encodes the speed of convergence toward it. In particular, the assumptions of Theorem 1.1 are satisfied.

Lemma 6.1. *If f preserves the linear fibration $\pi\colon \mathbb{P}^k \dashrightarrow \mathbb{P}^r$, then $\mathscr{J}_q(f) \cap I(\pi) = \varnothing$.*

Proof. Using the notations introduced above, we will prove that if $\epsilon > 0$ is small enough then the region

$$U_\epsilon := \{[y:z] \in \mathbb{P}^k \mid \|y\|_\infty < \epsilon \|z\|_\infty\}$$

satisfies $f(\overline{U_\epsilon}) \subset \overline{U_{\epsilon/2}}$. In particular $\cap_{n\geq 1} f^n(U_\epsilon) = I(\pi)$. Let us define

$$\alpha := \max_{\|y\|_\infty = 1} \|\Theta(y)\|_\infty \quad \text{and} \quad \beta := \min_{\|z\|_\infty = 1} \|R(0, z)\|_\infty.$$

Since f is a well-defined endomorphism of $\mathbb{P}^k$, we have $\beta > 0$. Thus, if $[y:z] \in \overline{U_\epsilon}$ then

$$\|\Theta(y)\|_\infty \leq \alpha \|y\|_\infty^d \leq \alpha \epsilon^d \|z\|_\infty^d \ \text{ and } \ \|R(y, z)\|_\infty \geq \beta \|z\|_\infty^d - \gamma\epsilon \|z\|_\infty^d,$$

where γ is the sum of the moduli of the coefficients of $R(y, z) - R(0, z)$. Therefore, if $\epsilon > 0$ is small enough and $[y:z] \in \overline{U_\epsilon}$ then

$$\|\Theta(y)\|_\infty \leq \alpha \epsilon^d \|z\|_\infty \leq \epsilon(\beta - \gamma\epsilon)\|z\|_\infty^d/2 \leq \epsilon \|R(y, z)\|_\infty/2,$$

which gives $f[y:z] \in \overline{U_{\epsilon/2}}$.

On the other hand, if H is a generic linear subspace of $\mathbb{P}^k$ of dimension r then $H \cap I(\pi) = \varnothing$ thus $H \cap U_\epsilon = \varnothing$ for $\epsilon > 0$ small enough. Hence, we can regularize the current of integration $[H]$ to obtain a positive closed (q, q) smooth form $\widetilde{\omega}$ of mass 1 supported in $\mathbb{P}^k \setminus U_\epsilon$. Equidistribution results for smooth forms (see [DS10]) give $T_f^q = \lim_{n\to\infty} d^{-nq} f^{n*} \widetilde{\omega}$. The fact that $f(U_\epsilon) \subset U_\epsilon$ implies that $\operatorname{supp}(T_f^q) \cap U_\epsilon = \varnothing$ and thus $\mathscr{J}_q(f) \cap I(\pi) = \varnothing$. $\square$

Remark 6.2. *The fact that $I(\pi)$ is an attracting set of dimension $q-1$ implies by [Taf18, Proposition 1.1] that $\mathscr{J}_q(f)\cap I(\pi)=\varnothing$, the proof relies there on the growth of iterated neighborhoods of the attracting set. We use here equidistribution towards the current T_f^q, we will also use this result in the parametric setting of Lemma 7.1.*

The next result follows easily from the fact that f preserves the fibration defined by π.

Lemma 6.3. *The critical current of f admits a decomposition*

$$[\mathrm{Crit}_f]=[C_\infty]+[C_\sigma],$$

where $[C_\infty]:=\pi^*[\mathrm{Crit}_\theta]$ and $[C_\sigma]$ is the current of integration on an algebraic set, called the sectional part of Crit_f.

Proof. If $\rho\colon\mathbb{C}^{k+1}\setminus\{0\}\to\mathbb{P}^k$ is the standard projection then the critical current of f can be defined by $\rho^*[\mathrm{Crit}_f]=dd^c\log|\det DF|$. The fact that F has the form $F(y,z)=(\Theta(y),R(y,z))$ implies that $\det DF=\det D\Theta\times\det D_zR$ where D_zR denotes the $q\times q$ matrix formed by the partial derivatives of R in the $z_0,\dots,z_{q-1}$ directions. Hence,

$$[\mathrm{Crit}_f]=[C_\infty]+[C_\sigma]$$

where $\rho^*[C_\sigma]=dd^c\log|\det D_zR|$ and $\rho^*[C_\infty]=dd^c\log|\det D\Theta|$. It is easy to check that $[C_\infty]=\pi^*[\mathrm{Crit}_\theta]$. □

If a is a n-periodic point of θ then $f^n_{|L_a}$ can be identified to a regular polynomial endomorphism of $\mathbb{C}^q\simeq L_a\backslash I(\pi)$ as follows. Since $\theta^n(a)=a$, there exists $A=(a_0,\dots,a_r)\in\mathbb{C}^{r+1}$ such that $a=[a_0:\dots:a_r]$ and $\Theta^n(A)=A$. Hence, if $[y:z]\in\mathbb{P}^k$ belongs to $L_a\setminus I(\pi)$ then there exists a unique $Z\in\mathbb{C}^q$ such that $[y:z]=[A:Z]$ and $f^n_{|L_a\setminus I(\pi)}$ can be identified with $R_n(Z):=R_{\Theta^{n-1}(A)}\circ\cdots\circ R_A(Z)$ where $R_A(Z):=R(A,Z)$. With the same notations, $G[y:z]=G_F(A,Z)-G_\Theta(A)$ and, as $\Theta^n(A)=A$ implies $G_\Theta(A)=0$, we have

$$G[y:z]=\lim_{l\to\infty}\frac{1}{d^{ln}}\max(\log\|R_n^l(Z)\|_\infty,0),$$

which is exactly the Green function associated to the polynomial mapping R_n. Hence, using this identification the equilibrium measure of $f^n_{|L_a}$ is $T_f^q\wedge[L_a]$. The critical current of $f^n_{|L_a}$ is $[\mathrm{Crit}_{f^n}]\wedge[L_a]$ if a is not a critical point of θ^n (i.e. if the wedge product is well-defined). In fact, without any assumption on a, since $[\mathrm{Crit}_{f^n}]=\sum_{i=0}^{n-1}f^{i*}[\mathrm{Crit}_f]$ one can check using Lemma 6.3 that the critical current associated to the restriction of f^n to $L_a\simeq\mathbb{P}^q$ corresponds to $\left(\sum_{i=0}^{n-1}f^{i*}[C_\sigma]\right)\wedge[L_a]+d^n[I(\pi)]$. Hence, the Bedford-Jonsson formula for regular polynomial endomorphisms of $\mathbb{C}^q$ (see Theorem 1.6) yields the following result.

Lemma 6.4. *Let $a\in\mathbb{P}^r$ be such that $\theta^n(a)=a$. If Λ_0 (resp. $\Lambda(f^n_{|L_a})$) denotes the sum of the Lyapunov exponents of $f_{|I(\pi)}$ (resp. $f^n_{|L_a}$) with respect to its equilibrium measure then*

$$\Lambda(f^n_{|L_a})=n\log d+n\Lambda_0+\sum_{i=0}^{n-1}\langle T_f^{q-1}\wedge[C_\sigma]\wedge[L_{\theta^i(a)}],G\rangle.$$

Proof. Since the dynamics of R_n on the hyperplane at infinity can be identified to the one of f^n on $I(\pi)$, the discussion above and the Bedford-Jonsson formula give

$$\Lambda(f^n_{|L_a})=n\log d+n\Lambda_0+\left\langle T_f^{q-1}\wedge\left(\sum_{i=0}^{n-1}f^{i*}[C_\sigma]\right)\wedge[L_a],G\right\rangle.$$

We conclude by using

$$f_*^i(GT_f^{q-1} \wedge [L_a]) = GT_f^{q-1} \wedge \frac{f_*^i[L_a]}{d^{qi}} \quad \text{and} \quad \frac{f_*^i[L_a]}{d^{qi}} = [L_{\theta^i(a)}],$$

where the first equality follows from the invariance of T_f and G, and the second one from the invariance of the fibration. □

Observe that since π is a submersion on $\mathbb{P}^k \setminus I(\pi)$, we have

$$S^r = (\pi^* T_\theta)^r = \pi^* \mu_\theta = \int_{\mathbb{P}^r} [L_a] d\mu_\theta(a)$$

on $\mathbb{P}^k \setminus I(\pi)$. As $I(\pi)$ has dimension $q-1$, these equalities extends to $\mathbb{P}^k$. This allows us to use the continuity results obtained in Section 3 to prove the following result.

Lemma 6.5.

$$\Lambda_\sigma = \lim_{n\to\infty} \frac{1}{nd^{rn}} \sum_{\theta^n(a)=a} \Lambda(f^n_{|L_a}).$$

Proof. By Lemma 6.1, there exists a neighborhood Ω of $I(\pi)$ such that $U := \mathbb{P}^k \setminus \overline{\Omega}$ contains $\mathscr{J}_q(f)$. Since f preserves the fibration defined by π, its differential preserves the subbundle $\ker D\pi$ of the tangent bundle over $\mathbb{P}^k \setminus I(\pi)$. Hence, we can define on U the Jacobian $|\mathrm{Jac}_\sigma f|$ of Df in the direction of $\ker D\pi$ with respect to a smooth metric. The function $u := \log |\mathrm{Jac}_\sigma f|$ is bounded from above on U and is locally the sum of a potential on $[C_\sigma]$ and a smooth function. In particular, it satisfies $dd^c u = [C_\sigma] + (T_1 - T_2)$ on U, where T_1 and T_2 are two positive smooth forms. Since $\Lambda_\sigma = \Lambda_f - \Lambda_\theta$ (see Equation (1.2) after Theorem 1.6) and $\pi_*\mu_f = \mu_\theta$, we have $\Lambda_\sigma = \langle \mu_f, u\rangle$. On the other hand, u is continuous on $U \setminus C_\sigma$ and $\dim(L_a \cap C_\sigma) = q-1$. Hence, by Lemma 3.1 with $v = u$, $X = C_\sigma$ and $Y = \mathbb{P}^k$, we have that for each $a \in \mathbb{P}^r$ the current $u[L_a]$ is well defined on U and depends continuously on a. Moreover, since T_f has continuous local potentials, $a \mapsto u[L_a] \wedge T_f^q$ is also continuous. Therefore, using that $\mu_f = T_f^q \wedge S^r = \int T_f^q \wedge [L_a] d\mu_\theta$, we obtain by Lemma 3.2 that

$$\Lambda_\sigma = \int_{\mathbb{P}^r} \langle T_f^q \wedge [L_a], u\rangle d\mu_\theta = \lim_{n\to\infty} \frac{1}{d^{rn}} \sum_{\theta^n(a)=a} \langle T_f^q \wedge [L_a], u\rangle, \tag{6.1}$$

where the last equality comes from the equidistribution of periodic points of θ towards μ_θ (see [BD99]). Let us recall that the measure $T_f^q \wedge [L_a]$ corresponds to the equilibrium measure of the polynomial mapping $f^n_{|L_a}$. Therefore, if $\Lambda(f^n_{|L_a})$ denotes the sum of the Lyapunov exponents of $f^n_{|L_a}$ with respect to this measure, we have by definition of u

$$\begin{aligned}\Lambda(f^n_{|L_a}) = \langle T_f^q \wedge [L_a], \log |\mathrm{Jac} f^n_{|L_a}|\rangle = \sum_{i=0}^{n-1} \langle T_f^q \wedge [L_a], u \circ f^i\rangle = \sum_{i=0}^{n-1} \langle f_*^i(T_f^q \wedge [L_a]), u\rangle \\ = \sum_{i=0}^{n-1} \langle T_f^q \wedge [L_{\theta^i(a)}], u\rangle, \end{aligned} \tag{6.2}$$

where the last equality comes from $d^{-q} f^* T_f^q = T_f^q$ and $d^{-q} f_*[L_a] = [L_{\theta(a)}]$. Combining (6.1) and (6.2) gives $\Lambda_\sigma = \lim_{n\to\infty} \frac{1}{nd^{rn}} \sum_{\theta^n(a)=a} \Lambda(f^n_{|L_a})$. □

We can now finish the proof of Theorem 1.7.

Proof of Theorem 1.7. Lemma 6.4 and Lemma 6.5 imply

$$\begin{aligned}\Lambda_\sigma &= \lim_{n\to\infty}\frac{1}{nd^{rn}}\sum_{\theta^n(a)=a}\left(n\log d + n\Lambda_0(f) + \sum_{i=0}^{n-1}\langle T_f^{q-1}\wedge[C_\sigma]\wedge[L_{\theta^i(a)}],G\rangle\right)\\ &= \log d + \Lambda_0(f) + \lim_{n\to\infty}\frac{1}{d^{rn}}\sum_{\theta^n(a)=a}\langle T_f^{q-1}\wedge[C_\sigma]\wedge[L_a],G\rangle\\ &= \log d + \Lambda_0(f) + \langle T_f^{q-1}\wedge S^r\wedge[C_\sigma],G\rangle,\end{aligned}$$

where the last equality comes from Lemma 3.1 and Lemma 3.2 applied with $v=-G$, $U=\mathbb{P}^k$, $X=I(\pi)$ and $Y=[C_\sigma]$. To be more precise, as we have seen before $a\mapsto u[L_a]$ is continuous and $dd^cu=[C_\sigma]+(T_1-T_2)$ on U where T_1 and T_2 are smooth. Hence, $a\mapsto[C_\sigma]\wedge[L_a]$ is continuous. Since $\dim(I(\pi)\cap C_\sigma)=q-2$, Lemma 3.1 implies that $a\mapsto G[C_\sigma]\wedge[L_a]$ is continuous. Thus, the continuity of the local potentials of T_f gives that $a\mapsto G[C_\sigma]\wedge[L_a]\wedge T_f^{q-1}$ is continuous. And finally, Lemma 3.2 gives $\langle T^{q-1}\wedge S^r\wedge[C_\sigma],G\rangle = \int\langle T^{q-1}\wedge[L_a]\wedge[C_\sigma],G\rangle d\mu_\theta = \lim_{n\to\infty}\int\langle T^{q-1}\wedge[L_a]\wedge[C_\sigma],G\rangle d\mu_n$, where μ_n is the average of the Dirac masses on the n-periodic points of θ. □

7 Sectional and fiber-wise bifurcation currents

In this section, we consider a family $(f_\lambda)_{\lambda\in M}$ of endomorphisms of $\mathbb{P}^k$ which preserves the standard linear fibration i.e. there exists a family $(\theta_\lambda)_{\lambda\in M}$ of endomorphisms of $\mathbb{P}^r$ such that $\pi\circ f_\lambda=\theta_\lambda\circ\pi$, where π is defined in Section 6. We denote by $\Pi\colon M\times\mathbb{P}^k \dashrightarrow M\times\mathbb{P}^r$, $f\colon M\times\mathbb{P}^k\to M\times\mathbb{P}^k$ the maps $\Pi(\lambda,x)=(\lambda,\pi(x))$, $f(\lambda,x)=(\lambda,f_\lambda(x))$ and by $p_{\mathbb{P}^k}\colon M\times\mathbb{P}^k\to\mathbb{P}^k$, $p\colon M\times\mathbb{P}^k\to M$ the two projections.

It is possible to define a $(1,1)$-current in $M\times\mathbb{P}^k$ whose slices by p are exactly the Green currents T_{f_λ} of f_λ, we refer to [DS06] for a definition of the slices in our situation. Indeed, by [BB07] the following limit exists

$$T_f := \lim_{n\to\infty}\frac{1}{d^n}f^{n*}(p_{\mathbb{P}^k}^*\omega_{\mathbb{P}^k}),$$

and defines the Green current associated to the family $(f_\lambda)_{\lambda\in M}$. The approach of [BB07] consists in lifting (locally) the family $(f_\lambda)_{\lambda\in M}$ to a family $(F_\lambda)_{\lambda\in M}$ of polynomial endomorphisms of $\mathbb{C}^{k+1}$ and show that the potentials of the lift of $d^{-n}f^{n*}(p_{\mathbb{P}^k}^*\omega_{\mathbb{P}^k})$ converge locally uniformly to a function, called the Green function of the family $(F_\lambda)_{\lambda\in M}$, which by definition is a potential of the lift of T_f. Moreover, since this convergence is locally uniform, T_f has continuous local potentials, its self-intersections T_f^l are well-defined and satisfy $T_f^l=\lim_{n\to\infty}d^{-nl}f^{n*}(p_{\mathbb{P}^k}^*\omega_{\mathbb{P}^k}^l)$ for $l\in\{1,\ldots,k\}$. When $l=k$, this gives a (k,k)-current T_f^k on $M\times\mathbb{P}^k$ whose slices are the equilibrium measure of f_λ. A positive closed (k,k)-current with this property is called an *equilibrium current* in [Pha05]. Bassanelli-Berteloot proved that the bifurcation current $T_{\mathrm{Bif}}(f):=dd^c\Lambda_f$ satisfies

$$T_{\mathrm{Bif}}(f)=p_*(T_f^k\wedge[\mathrm{Crit}_f]).$$

Pham obtained this result in the more general setting of polynomial-like maps and proved that T_f^k can be replaced by any equilibrium current, see [Pha05, pages 8-9].

Following [AB18] where the special case of polynomial skew products has been studied, we are interested in the relationship between the bifurcation currents $T_{\mathrm{Bif}}(f)$ associated to $(f_\lambda)_{\lambda\in M}$ and $T_{\mathrm{Bif}}(\theta)$ associated to $(\theta_\lambda)_{\lambda\in M}$ when $\theta_\lambda\circ\pi=\pi\circ f_\lambda$. There are two reasons to restrict ourselves to the linear fibration. The first one is that π is a submersion on $\mathbb{P}^k \setminus I(\pi)$. It implies in particular that the critical sets $\mathrm{Crit}_{f_\lambda}$ have a decomposition into a *sectional part* and a *fibered part*, the latter being given by $\pi^{-1}(\mathrm{Crit}_{\theta_\lambda})$ (see Lemma 7.2). The second reason is that the support of T_f^q, which is the Green current of order q of the family $(f_\lambda)_{\lambda\in M}$, is disjoint from $M \times I(\pi)$. Indeed, it is easy to check that for all $\lambda \in M$ the map f_λ satisfies the condition $\mathscr{J}_q(f_\lambda) \cap I(\pi) = \varnothing$. However, for an arbitrary family it is not clear whether this condition for each parameter implies $\mathrm{supp}(T_f^q) \cap M \times I(\pi) = \varnothing$.

Lemma 7.1. *Let $(f_\lambda)_{\lambda\in M}$ and π be as above. Then $\mathrm{supp}(T_f^q) \cap M \times I(\pi) = \varnothing$ and $\Pi_*(T_f^q) = [M \times \mathbb{P}^r]$.*

Proof. We will use the same arguments as for Lemma 6.1 locally in the parameter space. Let λ_0 be a fixed parameter in M. Since the region $U_\epsilon = \{[y : z] \in \mathbb{P}^k \mid \|y\|_\infty < \epsilon \|z\|_\infty\}$ is a trapping region for f_{λ_0} for $\epsilon > 0$ small enough, then if N is a small enough neighborhood of λ_0 then $f(N \times U_\epsilon) \subset N \times U_\epsilon$. Let $\widetilde{\omega}$ denote the positive closed (q, q)-form supported in $\mathbb{P}^k \setminus U_\epsilon$ obtained in the proof of Lemma 6.1. There exists a smooth $(q-1, q-1)$-form ϕ on $\mathbb{P}^k$ such that $\widetilde{\omega} = \omega_{\mathbb{P}^k}^q + dd^c\phi$ and there exists $C > 0$ such that $-C\omega_{\mathbb{P}^k}^{q-1} \leq \phi \leq C\omega_{\mathbb{P}^k}^{q-1}$. Let $p_{\mathbb{P}^k} : N \times \mathbb{P}^k \to \mathbb{P}^k$ be the canonical projection. As we have said in the beginning of this Section,

$$\lim_{n\to\infty} \frac{1}{d^{n(q-1)}} f^{n*}(p_{\mathbb{P}^k}^* \omega_{\mathbb{P}^k}^{q-1}) = T_f^{q-1}.$$

This and the inequalities $-C\omega_{\mathbb{P}^k}^{q-1} \leq \phi \leq C\omega_{\mathbb{P}^k}^{q-1}$ imply that $\lim_{n\to\infty} \frac{1}{d^{nq}} f^{n*}(p_{\mathbb{P}^k}^* \phi) = 0$ and thus

$$\lim_{n\to\infty} \frac{1}{d^{nq}} f^{n*}(p_{\mathbb{P}^k}^* \widetilde{\omega}) = \lim_{n\to\infty} \frac{1}{d^{nq}} f^{n*}(p_{\mathbb{P}^k}^* \omega_{\mathbb{P}^k}^q) = T_f^q.$$

Therefore, since $\mathrm{supp}(p_{\mathbb{P}^k}^* \widetilde{\omega}) \subset N \times (\mathbb{P}^k \setminus U_\epsilon)$ and $f(N \times U_\epsilon) \subset N \times U_\epsilon$, we have $\mathrm{supp}(T_f^q) \cap N \times I(\pi) = \varnothing$ on $N \times \mathbb{P}^k$, as desired. This implies that the current $\Pi_*(T_f^q)$ is a well-defined positive closed $(0, 0)$-current on $M \times \mathbb{P}^r$, i.e. it is equal to $\alpha[M \times \mathbb{P}^r]$ for some $\alpha > 0$. Finally $\alpha = 1$ since the fibers of Π have degree 1. □

The next result follows exactly as Lemma 6.3.

Lemma 7.2. *Let $(f_\lambda)_{\lambda\in M}$, $(\theta_\lambda)_{\lambda\in M}$ and π be as in Theorem 1.8. Then the current of integration on the critical set Crit_f of the family $(f_\lambda)_{\lambda\in M}$ has a decomposition*

$$[\mathrm{Crit}_f] = [C_\infty] + [C_\sigma],$$

where $[C_\infty] := \Pi^*[\mathrm{Crit}_\theta]$ and $[C_\sigma]$ is the current of integration on an analytic set of $M \times \mathbb{P}^k$.

Remark 7.3. *The fact that the critical set is not irreducible for the family $(f_\lambda)_{\lambda\in M}$ implies directly that the bifurcation current $T_{\mathrm{Bif}}(f)$ admits a similar decomposition. For a general family in one variable (i.e. $k = 1$), possibly by exchanging M by a branched cover, each critical point can be followed individually thus Crit_f has as many irreducible components as there are critical points. This gives rise to the decomposition of $T_{\mathrm{Bif}}(f)$ into the currents associated to the activation of each critical point, see [DF08].*

The last ingredient to prove Theorem 1.8 is the following result about slicing.

Lemma 7.4. *Let $(f_\lambda)_{\lambda\in M}$, $(\theta_\lambda)_{\lambda\in M}$ and π be as in Theorem 1.8. If $S := \Pi^*(T_\theta)$ then $T_f^q \wedge S^r$ is an equilibrium current for $(f_\lambda)_{\lambda\in M}$.*

Proof. It follows easily from the definition that if R is a positive closed current in $M \times \mathbb{P}^k$ such that the slice R_λ is well-defined and if u is a continuous function on $\mathrm{supp}(R)$ then $(uR)_\lambda = u_{\{\lambda\}\times\mathbb{P}^k} R_\lambda$ and $(dd^c uR)_\lambda = dd^c(u_{\{\lambda\}\times\mathbb{P}^k} R_\lambda)$. This implies that the slice $(T_f^q)_\lambda$ of T_f^q is equal to $((T_f)_\lambda)^q$ which is the Green current of order q of f_λ. Moreover, Lemma 7.1 yields $\mathrm{supp}(T_f^q) \cap M \times I(\pi) = \varnothing$, therefore $S = \Pi^*(T_\theta)$ has continuous local potentials on $\mathrm{supp}(T_f^q)$. Thus the slice $(T_f^q \wedge S^r)_\lambda$ is equal to $((T_f)_\lambda)^q \wedge (\pi^*(T_\theta)_\lambda)^r$ which is equal to the equilibrium measure of f_λ by Theorem 1.1, i.e. $T_f^q \wedge S^r$ is an equilibrium current for $(f_\lambda)_{\lambda\in M}$. □

Proof of Theorem 1.8. We denote by $p\colon M \times \mathbb{P}^k \to M$ and $\widetilde{p}\colon M \times \mathbb{P}^r \to M$ the two projections. Observe that $p = \widetilde{p} \circ \Pi$ on $M \times (\mathbb{P}^k \setminus I(\pi))$. Since $T_f^q \wedge S^r$ is an equilibrium current, it follows from Pham's article [Pha05] that

$$dd^c \Lambda_f = p_*(T_f^q \wedge S^r \wedge [\mathrm{Crit}_f]) = p_*(T_f^q \wedge S^r \wedge [C_\infty]) + p_*(T_f^q \wedge S^r \wedge [C_\sigma]),$$

where the last inequality comes from the decomposition obtained in Lemma 7.2. Moreover, as $\Lambda_f = \Lambda_\theta + \Lambda_\sigma$, in order to prove that $dd^c\Lambda_\sigma = p_*(T_f^q \wedge S^r \wedge [C_\sigma])$ it is sufficient to prove that $dd^c\Lambda_\theta = p_*(T_f^q \wedge S^r \wedge [C_\infty])$. To this end, observe that by Lemma 7.1

$$p_*(T_f^q \wedge S^r \wedge [C_\infty]) = \widetilde{p}_*(\Pi_*(T_f^q \wedge \Pi^*(T_\theta^r \wedge [\mathrm{Crit}_\theta]))) = \widetilde{p}_*(T_\theta^r \wedge [\mathrm{Crit}_\theta]).$$

On the other hand, Bassanelli-Berteloot formula applied to the family $(\theta_\lambda)_{\lambda\in M}$ gives $dd^c\Lambda_\theta = \widetilde{p}_*(T_\theta^r \wedge [\mathrm{Crit}_\theta])$ which concludes the proof. □

Using this formula for $T_{\mathrm{Bif},\sigma}(f)$, we can relate this current to the bifurcations in the fibers.

Proof of Corollary 1.9. Define

$$\Lambda_{\sigma,n}(\lambda) := \frac{1}{nd^{rn}} \sum_{\theta_\lambda^n(a)=a} \Lambda(f^n_{\lambda|L_a}).$$

By Lemma 6.5 for each $\lambda \in M$ we have

$$\Lambda_\sigma(\lambda) = \lim_{n\to\infty} \Lambda_{\sigma,n}(\lambda).$$

Observe now that by the Briend-Duval's inequality the function $\Lambda_{\sigma,n}(\lambda)$ is uniformly bounded from below by $(q\log d)/2$. It is also locally uniformly bounded from above since $\Lambda_{\sigma,n}(\lambda) \leq q\log(\max_{x\in\mathbb{P}^k}(\|D_x f_\lambda\|))$. Hence, by the dominated convergence theorem, $\Lambda_{\sigma,n}$ converges to Λ_σ in L^1_{loc}, in particular, $dd^c\Lambda_{\sigma,n}$ converges to $dd^c\Lambda_\sigma$ in the sense of currents.

On the other hand, Theorem 1.8 implies that $dd^c\Lambda_\sigma = p_*(T_f^q \wedge S^r \wedge [C_\sigma])$. Let $[\mathrm{Per}_{\theta,n}]$ denote the current of integration on $\{(\lambda, a) \in M \times \mathbb{P}^r \,|\, \theta_\lambda^n(a) = a\}$ (where we take into account the multiplicities) and let us set $S_n := \Pi^*[\mathrm{Per}_{\theta,n}]/d^{rn}$. In order to prove the corollary, it is sufficient to prove that $dd^c\Lambda_{\sigma,n} = p_*(T_f^q \wedge S_n \wedge [C_\sigma])$ for every $n \geq 1$.

To this end, let $n \geq 1$ and observe that outside a ramification locus $\Sigma_n \subset M$ of codimension at least 1 the periodic points of θ_λ of period n can be followed holomorphically i.e. each $\lambda \in M \setminus \Sigma_n$ admits a neighborhood N and a family $\{\gamma_j\}_{j\in J}$ of holomorphic maps $\gamma_j\colon N \to \mathbb{P}^r$ such that $[\mathrm{Per}_{\theta,n}] = \sum_{j\in J}[\Gamma_j]$ on $N \times \mathbb{P}^r$ where Γ_j is the graph of γ_j. If $j \in J$ and $\Theta(\lambda, y) := (\lambda, \theta_\lambda(y))$ then $\Theta(\Gamma_j)$ is another graph $\Gamma_{j'}$ with $j' \in J$. Hence, if we gather the periodic points of a same cycle, we obtain a current

$$\sum_{i=0}^{n-1} p_*(T_f^q \wedge \Pi^*[\Theta^i(\Gamma_j)] \wedge [C_\sigma])$$

which is equal to the bifurcation current of the family $(f^n_{\lambda|L_{\gamma_j(\lambda)}})_{\lambda\in N}$ of endomorphisms of $L_{\gamma_j(\lambda)} \simeq \mathbb{P}^q$, i.e. equal to $dd^c\Lambda(f^n_{\lambda|L_{\gamma_j(\lambda)}})$. Since this equality is true for every $j \in J$, we get the formula $dd^c\Lambda_{\sigma,n} = p_*(T^q_f \wedge S_n \wedge [C_\sigma])$ on $M \setminus \Sigma_n$. To conclude, observe that none of these currents gives mass to Σ_n. Indeed, as we have remarked above $\Lambda_{\sigma,n}$ is uniformly bounded from below by $(q\log d)/2$, so $dd^c\Lambda_{\sigma,n}$ gives no mass to the proper analytic set Σ_n. For the second current, the analytic set $X_n := \mathrm{supp}(S_n \wedge [C_\sigma])$ has dimension $m+q-1$, where $m := \dim(M)$, and its intersections with the fibers of p have dimension $q-1$. Hence, the dimension of $X_n \cap p^{-1}(\Sigma_n)$ is strictly less than $m+q-2$ and the current $T^q_f \wedge S_n \wedge [C_\sigma]$ gives no mass to this set since T_f has continuous local potentials. This implies that $p_*(T^q_f \wedge S_n \wedge [C_\sigma])$ gives no mass to Σ_n, as desired. □

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Christophe Dupont
Univ Rennes
IRMAR, CNRS UMR 6625
F-35000 Rennes, France
christophe.dupont@univ-rennes1.fr

Johan Taflin
Université de Bourgogne Franche-Comté
IMB, CNRS UMR 5584
21078 Dijon Cedex, France
johan.taflin@u-bourgogne.fr