On the dimension of invariant measures of endomorphisms of $\mathbb{CP}^k$

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Abstract

Let f be an endomorphism of $\mathbb{CP}^k$ and ν be an f-invariant measure with positive Lyapunov exponents $(\lambda_1, \ldots, \lambda_k)$. We prove a lower bound for the pointwise dimension of ν in terms of the degree of f, the exponents of ν and the entropy of ν. In particular our result can be applied for the maximal entropy measure μ. When $k = 2$, it implies that the Hausdorff dimension of μ is estimated by $\dim_{\mathcal{H}} \mu \geq \frac{\log d}{\lambda_1} + \frac{\log d}{\lambda_2}$, which is half of the conjectured formula. Our method for proving these results consists in studying the distribution of the ν-generic inverse branches of f^n in $\mathbb{CP}^k$. Our tools are a volume growth estimate for the bounded holomorphic polydiscs in $\mathbb{CP}^k$ and a normalization theorem for the ν-generic inverse branches of f^n.

Key Words: holomorphic dynamics, dimension theory.
MSC: 37C45, 37F10

1 Introduction

Let f be a smooth map acting on a compact Riemannian manifold M and ν be an f-invariant measure on M. By Young [Y], the pointwise dimension of ν is defined by (provided the limit exists):

$$\delta(x) = \lim_{r \to 0} \frac{\log \nu(B_x(r))}{\log r},$$

where $B_x(r)$ is the ball in M of center x and radius r (take lim inf and lim sup to define the lower and upper pointwise dimensions $\underline{\delta}$ and $\bar{\delta}$). That function actually describes the geometrical behaviour of ν with respect to the metric of M: if $a \leq \underline{\delta} \leq \bar{\delta} \leq b$ hold ν-a.e., then the Hausdorff dimension of ν also satisfies $a \leq \dim_{\mathcal{H}} \nu \leq b$ [Y]. Recall that $\dim_{\mathcal{H}} \nu$ is defined as the infimum of the Hausdorff dimension of the full ν-measure borel subsets in M. In particular we have $\dim_{\mathcal{H}} \nu \leq \dim_{\mathcal{H}} \mathrm{supp}(\nu)$. We refer to the book of Pesin [P] for an introduction to dimension theory in dynamical systems.

Given a dynamical system (M, f, ν), one can expect relations between the dimension of ν, its Lyapunov exponents $\lambda_k \leq \ldots \leq \lambda_1$ and its entropy h_ν (see [Le], [P]). The situation has been completely described when f is a smooth diffeomorphism and ν is an f-invariant hyperbolic measure (i.e. with no zero exponents). Young [Y] first proved in the case of surfaces the formula $\delta = h_\nu/\lambda_1 - h_\nu/\lambda_2$ ν-a.e., where $\lambda_2 < 0 < \lambda_1$. In higher dimensions, Ledrappier-Young [LY] established that the unstable pointwise dimension of ν satisfies ν-a.e.

$$\delta^u = \frac{h_1}{\lambda_1} + \sum_{i=2}^{u} \frac{h_i - h_{i-1}}{\lambda_i}, \tag{1}$$

where $h_1 \leq \ldots \leq h_u = h_\nu$ denote the conditional entropies of ν along the unstable manifolds $\mathcal{W}^1 \subset \ldots \subset \mathcal{W}^u$ (a similar formula holds for the stable dimension δ^s). Later Barreira-Pesin-Schmeling [BPS] proved the formula $\delta = \delta^s + \delta^u$ ν-a.e. by showing a product property for the invariant hyperbolic measures.

In this article, we focus on the holomorphic endomorphisms f of $\mathbb{CP}^k$ of degree $d \geq 2$. These mappings define non invertible ramified coverings of topological degree d^k. We refer to the article of Dinh-Sibony [DS] for a survey of their dynamical properties. The question of the Hausdorff dimension for the equilibrium measure was raised by Fornaess-Sibony [FS2] (see subsection 1.1).

When $k = 1$, f defines a rational map on $\mathbb{CP}^1$, and Mañé [M] proved the formula $\delta = h_\nu/\lambda$ ν-a.e. for any ergodic measure satisfying $h_\nu > 0$. Here λ denotes the single exponent of ν, it has multiplicity 2 for the underlying real system. The proof heavily relies on the Koebe distortion theorem. The present article deals with the higher dimensional case, which is not conformal. We obtain the following result:

Theorem A: *Let f be a holomorphic endomorphism of $\mathbb{CP}^k$ of degree $d \geq 2$ and ν be an ergodic f-invariant measure with positive Lyapunov exponents $\lambda_k \leq \ldots \leq \lambda_1$. Then we have:*

$$\forall x \in \mathbb{CP}^k \ \nu\text{-a.e.} \ , \ \underline{\delta}(x) \ \geq \ \frac{\log d^{k-1}}{\lambda_1} + \frac{h_\nu - \log d^{k-1}}{\lambda_k}.$$

The proof is outlined in section 2, the method consists in studying the distribution of the ν-generic inverse branches of f^n in $\mathbb{CP}^k$. Our main tools are a volume growth estimate for holomorphic polydiscs in $\mathbb{CP}^k$ and a normalization theorem for the ν-generic inverse branches of f^n. That result provides, in some sense, a substitute for the one-dimensional Koebe distortion theorem.

1.1 Application to the equilibrium measure μ of f

The equilibrium measure is defined as the limit (in the sense of distributions) of the smooth (k, k) form $d^{-kn} f^{n*} \omega^k$, where ω^k is the standard volume form on $\mathbb{CP}^k$. Fornaess-Sibony [FS1] proved that μ is mixing and that $\log \operatorname{Jac} f \in L^1(\mu)$. Briend-Duval established that the exponents of μ are bounded below by $\log \sqrt{d}$ [BD1] and that μ is the unique measure of maximal entropy ($h_\mu = \log d^k$) [BD2]. Concerning the Hausdorff dimension of μ, Mañé's formula asserts that $\dim_{\mathcal{H}} \mu = \log d/\lambda$ when $k = 1$. Binder-DeMarco [BDeM] conjectured for $k \geq 2$:

Conjecture : *For every system* $(\mathbb{CP}^k, f, \mu)$, $\dim_{\mathcal{H}} \mu = \frac{\log d}{\lambda_1} + \cdots + \frac{\log d}{\lambda_k}$.

We note that this formula is consistent with (1) if we set $h_i = \log d^i$ for the conditional entropies of μ. Binder-DeMarco [BDeM] proved that $\dim_{\mathcal{H}} \mu \leq 2k - 2\,(\Sigma_{i=1}^k \lambda_i - k \log \sqrt{d})/\lambda_1$ in a polynomial setting by using volume estimates. Dinh-Dupont [DD] extended that estimate to meromorphic endomorphisms of $\mathbb{CP}^k$.

From theorem A we deduce the following bound. It proves half of the conjectured formula when $k = 2$.

Corollary A: *Let f be an endomorphism of $\mathbb{CP}^k$ of degree $d \geq 2$ and μ be its equilibrium measure. If $\lambda_k \leq \cdots \leq \lambda_1$ denote the Lyapunov exponents of μ, then*

$$\dim_{\mathcal{H}} \mu \geq \frac{\log d^{k-1}}{\lambda_1} + \frac{\log d}{\lambda_k}.$$

In particular, $\dim_{\mathcal{H}} \mu \geq \frac{\log d}{\lambda_1} + \frac{\log d}{\lambda_2}$ *for every system* $(\mathbb{CP}^2, f, \mu)$.

Now we can establish the conjecture for a class of non conformal systems by combining corollary A with the upper bound stated above:

Corollary B: *Let f be an endomorphism of $\mathbb{CP}^k$ of degree $d \geq 2$ and μ be its equilibrium measure. If $\lambda_k = \log \sqrt{d}$ and $\lambda_{k-1} = \ldots = \lambda_1$, then*

$$\dim_{\mathcal{H}} \mu = \frac{\log d^{k-1}}{\lambda_1} + \frac{\log d}{\lambda_k}.$$

1.2 Application to measures with large entropy

Let f be an endomorphism of $\mathbb{CP}^k$ of degree $d \geq 2$ and ν be an f-invariant ergodic measure. De Thélin [dT] proved that if $\log \mathrm{Jac}\, f \in L^1(\nu)$ and $h_\nu > \log d^{k-1}$, then the Lyapunov exponents of ν satisfy $\frac{1}{2}(h_\nu - \log d^{k-1}) \leq \lambda_k \leq \cdots \leq \lambda_1$. In [Du] we recently constructed ergodic measures satisfying $h_\nu > \log d^{k-1}$ and showed that the preceding estimate holds without assuming the integrability of $\log \mathrm{Jac}\, f$. By theorem A, we deduce the following bounds for the largest Lyapunov exponent of ν.

Corollary C: *Let f be an endomorphism of $\mathbb{CP}^k$ of degree $d \geq 2$ and ν be an f-invariant ergodic measure.*

1. *If* $\log d^{k-1} < h_\nu$, *then* $\lambda_1 \geq (1 - 1/k) \log \sqrt{d}$.

2. *If* $\log d^{k-1} < h_\nu < (1 + 1/k) \log d^{k-1}$, *then* $\lambda_1 \geq \frac{1}{2}(h_\nu - \log d^{k-1}) + \varphi(h_\nu)$, *where* $\varphi(h_\nu) > 0$.

The first point follows from the observation $\underline{\delta} \leq 2k$. For the second point, the function φ is defined as $\varphi(h_\nu) := \frac{1}{2}[(1+1/k)\log d^{k-1} - h_\nu]$. Let us observe that the latter is false for the equilibrium measure μ, its Lyapunov exponents are indeed $\lambda_1 = \ldots = \lambda_k = \frac{1}{2}(h_\mu - \log d^{k-1}) = \log\sqrt{d}$ when f is a Lattès example [BeDu].

1.3 Organization of the article

The proof of theorem A relies on theorem B, which is stated in section 2: that result describes the distribution of the ν-generic inverse branches in $\mathbb{CP}^k$. Section 3 deals with notations and the normalization theorem for the inverse branches. The proof of theorem A is detailed in section 4. We show theorem B in sections 5 and 6. In an appendix we establish the growth lemma.

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2 Statement of theorem B and outline of its proof

Let us fix f a holomorphic endomorphism of $\mathbb{CP}^k$ with degree $d \geq 2$ and ν an ergodic f-invariant measure with positive exponents $\lambda_k \leq \ldots \leq \lambda_1$. The fractional time q_n is defined as the entire part of $n\lambda_k/\lambda_1$. We denote by $f^{-n}_{y_n}$ the inverse branch of f^n mapping $y_n := f^n(y)$ to y. We set $\mathcal{E}_\rho$ as an arbitrary maximal ρ-separated subset in $\mathbb{CP}^k$. We define $\mathcal{E}_\rho(q)$ as the finite set of $p \in \mathcal{E}_\rho$ satisfying $q \in B_p(\rho)$ and denote $B_x^{\Omega_\epsilon}(r) := B_x(r) \cap \Omega_\epsilon$.

Theorem B : *Let f be an endomorphism of $\mathbb{CP}^k$ of degree $d \geq 2$ and ν be an ergodic f-invariant measure with positive Lyapunov exponents $\lambda_k \leq \ldots \leq \lambda_1$. For every $\epsilon > 0$, there exist $\Omega_\epsilon \subset \mathbb{CP}^k$ and $r_0 = r_0(\epsilon) > 0$ satisfying:*

1. *$\nu(\Omega_\epsilon) > 1 - \epsilon$.*
2. *for every $x \in \Omega_\epsilon$ and n large enough, the collection of inverse branches*

$$\mathcal{P}_n(x) := \left\{ f^{-n}_{y_n} B_p(s_n) \ , \ y \in B_x^{\Omega_\epsilon}(s_n e^{-n\lambda_k + 3n\epsilon}) \ , \ p \in \mathcal{E}_{s_n}(y_n) \right\}$$

is well defined for $s_n := r_0 e^{-8n\epsilon}$ and satisfies $\mathsf{Card}\, \mathcal{P}_n(x) \leq d^{(k-1)(n-q_n)} e^{20kn\epsilon}$.

Theorem B is used in the proof of theorem A (see section 4). We sketch below the proof of theorem B. It relies on propositions A and B. For simplicity, we shall work up to $e^{\pm n\epsilon}$ error terms (for instance we replace $e^{-n\lambda_k+3n\epsilon}$ by $e^{-n\lambda_k}$ and s_n by 1).

We define a *polydisc* as any holomorphic map $\eta : \mathbb{D}^{k-1}(r) \to \mathbb{CP}^k$. Let ω denote the Fubini-Study $(1,1)$-form on $\mathbb{CP}^k$ and define $\mathsf{Vol}\, \eta := \int_{\mathbb{D}^{k-1}(r)} \eta^* \omega^{k-1}$: this is the *volume* of η counted with multiplicity. Let $\{B_j\,,\, j \in J\}$ be a finite covering of $\mathbb{CP}^k$ which consists of open sets bounded in the affine charts. We say that η is *bounded* if its image is contained in some B_j. We shall need the

Growth lemma : *If $\eta : \mathbb{D}^{k-1}(2) \to \mathbb{CP}^k$ is bounded, then $\mathsf{Vol}\, f^m \circ \eta_{|\mathbb{D}^{k-1}} \leq d^{(k-1)m}$ for every $m \geq 1$.*

That geometric result does not depend on the measure ν: the proof relies on the existence of a Green current for every endomorphism of $\mathbb{CP}^k$ (see the appendix). That lemma allows us to establish the next proposition: let us fix $x \in \Omega_\epsilon$ and denote by $\mathcal{L}_n$ the set of polydiscs $L_n : \mathbb{D}^{k-1} \to B_x(e^{-n\lambda_k})$.

Proposition A : *For every $L_n \in \mathcal{L}_n$, we have $\mathsf{Vol}\, f^n \circ L_n \leq d^{(k-1)(n-q_n)}$.*

This estimate follows from the growth lemma taking $m = n - q_n$ and $\eta = f^{q_n} \circ L_n$. Indeed, the polydisc $f^{q_n} \circ L_n$ is bounded since $f^{q_n}(B_x(e^{-n\lambda_k})) \subset B_{x_{q_n}}(e^{-n\lambda_k} \cdot e^{q_n\lambda_1}) \simeq B_{x_{q_n}}(1)$: that comes from the fact that λ_1 is the largest exponent and $q_n\lambda_1 \simeq n\lambda_k$.

Our second tool is a normalization theorem for the inverse branches of f^n established by Berteloot-Dupont-Molino [BDM]. That theorem basically asserts that every inverse branch $P_n \in \mathcal{P}_n(x)$ looks like a parallelepiped with characteristic dimensions $e^{-n\lambda_1} \leq \ldots \leq e^{-n\lambda_k}$, it plays the role of a distortion theorem. The normalization theorem allows us to prove:

Proposition B : *There exists a finite subset $\mathcal{F}_n \subset \mathcal{L}_n$ of cardinality less than $ke^{20n\epsilon}$ such that for every $P_n \in \mathcal{P}_n(x)$, there is $L_n \in \mathcal{F}_n$ satisfying $\mathsf{Vol}\, f^n \circ L_{n|L_n^{-1}(P_n)} \geq 1$.*

We actually show that $\mathsf{Vol}\, f^n \circ L_{n|L_n^{-1}(P_n)} \geq 1$ for (almost) every polydisc $L_n \in \mathcal{L}_n$ transverse to the $e^{-n\lambda_k}$-direction of P_n. The family $\mathcal{F}_n$ then practically consists of hyperplanes parallel to the coordinates.

Finally, the upper bound $\mathsf{Card}\, \mathcal{P}_n(x) \leq d^{(k-1)(n-q_n)}$ follows using the fact that the inverse branches are pairwise disjoint (see subsection 5.2), that completes the proof of theorem B.

Let us notice that the estimates of theorems A and B can be sharpened when λ_k has multiplicity p. The same method indeed yields $\mathsf{Card}\, \mathcal{P}_n(x) \leq d^{(k-p)(n-q_n)}$ by considering the family of polydiscs $L_n : \mathbb{D}^{k-p} \to B_x(e^{-n\lambda_k})$. In particular that implies the lower bound $\dim_{\mathcal{H}} \nu \geq \frac{\log d^{k-p}}{\lambda_1} + \frac{h_\nu - \log d^{k-p}}{\lambda_k}$.

3 Generalities

3.1 The dynamical systems $(\mathbb{CP}^k, f, \nu)$

Let f be a holomorphic endomorphism of $\mathbb{CP}^k$ of degree $d \geq 2$. It is defined in homogeneous coordinates as $[P_0 : \ldots : P_k]$ where the P_i's are homogeneous polynomials of degree d without common zero (except the origin). The topological degree of f is d^k. Let $\mathcal{C}$ be the critical set of f, this is an hypersurface of degree $(d-1)(k+1)$ counted with multiplicity. Let ω be the Fubini-Study $(1,1)$-form on $\mathbb{CP}^k$ and dist the induced distance on $\mathbb{CP}^k$. We denote by $\mathrm{Jac}\, f$ the function on $\mathbb{CP}^k$ satisfying $f^*\omega^k = \mathrm{Jac}\, f \cdot \omega^k$. This is a bounded non-negative C^∞ function which vanishes on the critical set $\mathcal{C}$.

Let ν be an f-invariant ergodic measure, h_ν its entropy and $\lambda_k \leq \ldots \leq \lambda_1$ its Lyapunov exponents. We assume that those exponents are positive. In particular, the classical formula $\int_{\mathbb{CP}^k} \log \mathrm{Jac}\, f \, d\nu = 2(\lambda_1 + \ldots + \lambda_k)$ yields:

Lemma 3.1 *If the exponents of ν are positive, then* $\log \mathrm{Jac}\, f \in L^1(\nu)$ *and* $\nu(\mathcal{C}) = 0$.

We shall assume that $\lambda_k < \ldots < \lambda_1$. In particular that enables us to simplify the statements concerning the normal forms (see the next subsections). Our method easily extends when multiplicities occur.

We endow $\mathbb{C}^k$ with $|z| = \max_{1\leq i\leq k} |z_i|$. For any polynomial mapping $Q : \mathbb{C}^k \to \mathbb{C}^l$, we set $\| Q \|$ as the maximum of the modulus of its coefficients. We also denote by $(c_i)_{1\leq i\leq k}$ the canonical basis of $\mathbb{C}^k$ and by $(\pi_i)_{1\leq i\leq k}$ the projections to the axis.

3.2 Normal forms associated with the Lyapunov exponents

For every $\alpha = (\alpha_1, \ldots, \alpha_k) \in \mathbb{N}^k$, we set $|\alpha| := \alpha_1 + \ldots + \alpha_k$ and $Q_\alpha := z_1^{\alpha_1} \ldots z_k^{\alpha_k}$. Given $1 \leq i \leq k-1$, the set of i-resonant degrees is defined by:

$$\mathfrak{R}_i := \left\{ \alpha \in \mathbb{N}^k \ , \ |\alpha| \geq 2 \ , \ \alpha_1 = \ldots = \alpha_i = 0 \ \text{ and } \ \lambda_i = \alpha_{i+1}\lambda_{i+1} + \ldots + \alpha_k \lambda_k \right\}.$$

We set $I := \{1 \leq i \leq k-1 \, , \, 2\lambda_k \leq \lambda_i\}$. Observe that $\mathfrak{R}_i$ is empty if $i \notin I$. Note also that $|\alpha| \leq \theta := \lambda_1/\lambda_k$ for every $\alpha \in \mathfrak{R}_i$, hence $\mathfrak{R} := \cup_{i=1}^{k-1} \mathfrak{R}_i$ has finite cardinality. We denote $\Delta := \mathsf{Card}\, \mathfrak{R}$.

We say that a polynomial map $N : \mathbb{C}^k \to \mathbb{C}^k$ is *normal* if $N = (N_1, \ldots, N_{k-1}, 0)$ where $N_i = \sum_{\alpha \in \mathfrak{R}_i} c_i^\alpha Q_\alpha$ for some $c_i^\alpha \in \mathbb{C}$. A map $R : \mathbb{C}^k \to \mathbb{C}^k$ is *resonant* if $R = A + N$, where $A = (a_1, \ldots, a_k)$ is a linear diagonal map satisfying $e^{-\lambda_i - \epsilon} \leq |a_i| \leq e^{-\lambda_i + \epsilon}$ and N is a normal map.

Every resonant map $R = A + N$ is invertible, and $R^{-1} = A^{-1} + N'$ for some normal map N'. Moreover, if $R_i = A_i + N_i$ $(i = 1, 2)$ are resonant maps, we have $R_1 \circ R_2 = A_1 \circ A_2 + N''$ for some normal map N''. These are classical stability properties (see e.g. [GK], section 1.1 and [BDM], section 5).

3.3 Natural extension and normalization theorem

Let $\mathsf{O} := \{\hat{x} := (x_n)_{n\in\mathbb{Z}} \, , \, x_{n+1} = f(x_n)\}$ be the set of orbits, $\hat{\pi} : \mathsf{O} \to \mathbb{CP}^k$ the projection $\hat{x} \mapsto x_0$, and $s : \mathsf{O} \to \mathsf{O}$ the left shift. We also set $\tau := s^{-1}$. Note that

$\hat{\pi} \circ s = f \circ \hat{\pi}$ on O. For every $n \geq 0$, we denote $\hat{x}_n := s^n(\hat{x})$. We say that a function $\phi_\epsilon : \mathsf{O} \to \mathbb{R}^+$ is *ϵ-slow* (resp. *ϵ-fast*) if $\phi_\epsilon(\mathsf{O}) \subset]0,1]$ (resp. $[1,+\infty[$) and satisfies $\phi_\epsilon(\hat{x})e^{-\epsilon} \leq \phi_\epsilon(s(\hat{x})) \leq \phi_\epsilon(\hat{x})e^{\epsilon}$ for every $\hat{x} \in \mathsf{O}$.

We denote by $\hat{\nu}$ the s-invariant measure on O satisfying $\hat{\nu}(\hat{\pi}^{-1}(A)) = \nu(A)$ for every borel set $A \subset \mathbb{CP}^k$ (see [CFS], section 10.4). We shall work with the s-invariant set $X := \{\hat{x} = (x_n)_{n\in\mathbb{Z}} \, , \, x_n \notin \mathcal{C}\}$. It satisfies $\hat{\nu}(X) = 1$ since $\nu(\mathcal{C}) = 0$ (see lemma 3.1). For every $\hat{x} \in X$, we denote by $f_{\hat{x}}^{-n}$ the inverse branch of f^n sending x_0 to x_{-n}. Hence $f_{\hat{x}_n}^{-n}$ is the inverse branch of f^n sending $x_n = f^n(x)$ to x.

Definition 3.2 *$\mathcal{R} = (R_{\hat{x}})_{\hat{x}\in X}$ is a* resonant cocycle *if every $R_{\hat{x}}$ is a resonant map.*

Given a resonant cocycle $\mathcal{R}$, we set $R_{\hat{x}} := (a_1(\hat{x}), \dots, a_k(\hat{x})) + (N_1(\hat{x}), \dots, N_{k-1}(\hat{x}), 0)$, where $e^{-\lambda_i - \epsilon} \leq |a_i(\hat{x})| \leq e^{-\lambda_i + \epsilon}$. For every $n \geq 1$, we define $R_{\hat{x}}^n := R_{\tau^{n-1}(\hat{x})} \circ \dots \circ R_{\hat{x}}$ and $R_{\hat{x}}^{-n} := (R_{\hat{x}}^n)^{-1}$. Using the stability properties, we obtain:

$$\forall n \in \mathbb{Z} \; , \; R_{\hat{x}}^n = (a_{1,n}(\hat{x}), \dots, a_{k,n}(\hat{x})) + (N_{1,n}(\hat{x}), \dots, N_{k-1,n}(\hat{x}), 0) \, , \tag{2}$$

where $e^{-n\lambda_i - |n|\epsilon} \leq |a_{i,n}(\hat{x})| \leq e^{-n\lambda_i + |n|\epsilon}$ and $N_{i,n}(\hat{x}) := \sum_{\alpha \in \mathfrak{R}_i} c_{i,n}^\alpha(\hat{x}) Q_\alpha$.

Definition 3.3 *Let M_ϵ be an ϵ-fast function on X. A resonant cocycle $\mathcal{R}$ is M_ϵ-*adapted *if $\| N_{i,n}(\hat{x}) \| = \max_{\alpha\in\mathfrak{R}_i} |c_{i,n}^\alpha(\hat{x})| \leq M_\epsilon(\hat{x}) e^{-n\lambda_i + |n|\epsilon}$ for every $n \in \mathbb{Z}$.*

Definition 3.4 *Let r_ϵ and β_ϵ be respectively an ϵ-slow and an ϵ-fast function on X. $\mathcal{S} = (S_{\hat{x}})_{\hat{x}\in X}$ is a $(r_\epsilon, \beta_\epsilon)$-*coordinate *if for any $\hat{x} \in X$, $S_{\hat{x}} : B_{x_0}(r_\epsilon(\hat{x})) \to \mathbb{C}^k$ is an injective holomorphic map satisfying $S_{\hat{x}}(x_0) = 0$ and*

$$\forall (p, p') \in B_{x_0}(r_\epsilon(\hat{x})) \; , \; \mathit{dist}(p,p') \leq |S_{\hat{x}}(p) - S_{\hat{x}}(p')| \leq \beta_\epsilon(\hat{x}) \, \mathit{dist}(p,p').$$

The normalization theorem is stated as follows [BDM].

Theorem 3.5 *For every $\epsilon > 0$, there exist a $(r_\epsilon, \beta_\epsilon)$-coordinate $\mathcal{S}$ and an M_ϵ-adapted resonant cocycle $\mathcal{R}$ such that the following diagram commutes for $\hat{\nu}$-almost every $\hat{x} \in X$ and every $n \geq 1$:*

$$\begin{tikzcd} B_{x_0}(r_\epsilon(\hat{x})) \arrow[r, "f_{\hat{x}}^{-n}"] \arrow[d, "S_{\hat{x}}"'] & f_{\hat{x}}^{-n}\left(B_{x_0}(r_\epsilon(\hat{x}))\right) \arrow[d, "S_{\tau^n(\hat{x})}"] \\ \mathbb{C}^k \arrow[r, "R_{\hat{x}}^n"] & \mathbb{C}^k \end{tikzcd}$$

Note that the existence of r_ϵ requires the ν-integrability of $\log \| (d_x f)^{-1} \|$ (see [BDM], lemma 4.1). Here this is a consequence of lemma 3.1.

3.4 Some estimates

We denote $z := (\tilde{z}, z_k) \in \mathbb{D}^{k-1} \times \mathbb{D}$ and $\tilde{\pi}(z) := \tilde{z}$. We recall that $\Delta = \mathsf{Card}\, \mathfrak{R}$ and that $|\alpha| \le \theta = \lambda_1/\lambda_k$ for every $\alpha \in \mathfrak{R}$.

Lemma 3.6 *Let $\mathcal{R}$ be an M_ϵ-adapted resonant cocycle and $M'_\epsilon := \max\{\Delta+1, \theta, \theta(\theta-1)\}M_\epsilon$. Then for every $\hat{x} \in X$, $r \le 1$ and $z \in \mathbb{D}^k(r)$, we have:*

1. $\mathbb{D}^k\left(M'_\epsilon(\hat{x})^{-1} e^{-n\lambda_1 - n\epsilon} \cdot r\right) \subset R^n_{\hat{x}}\left(\mathbb{D}^k(r)\right) \subset \mathbb{D}^k\left(M'_\epsilon(\hat{x}) e^{-n\lambda_k + n\epsilon} \cdot r\right)$,
2. $\| \tilde{\pi} \circ d_z R^n_{\hat{x}} \| \le M'_\epsilon(\hat{x}) e^{-n\lambda_{k-1} + n\epsilon}$,
3. $e^{-n\lambda_k - n\epsilon} \le \left| \pi_k \left(\frac{\partial R^n_{\hat{x}}}{\partial z_k}(z) \right) \right|$ *and* $\left| \frac{\partial R^n_{\hat{x}}}{\partial z_k}(z) \right| \le \max\{ M'_\epsilon(\hat{x}) e^{-n\lambda_{k-1} + n\epsilon} \, , \, e^{-n\lambda_k + n\epsilon} \}$,
4. $\left| \frac{\partial^2 R^n_{\hat{x}}}{\partial z_k^2}(z) \right| \le M'_\epsilon(\hat{x}) e^{-2n\lambda_k + n\epsilon}$.

Proof: Let $\hat{x} \in X$, $r \le 1$ and $z \in \mathbb{D}^k(r)$. Using the M_ϵ-adapted property and (2), we get for every $1 \le i \le k$:

$$|\pi_i(R^n_{\hat{x}}(z))| \le |a_{n,i}(\hat{x})||z| + \Delta \parallel N_{i,n}(\hat{x}) \parallel |z|^\theta \le (\Delta+1) M_\epsilon(\hat{x}) e^{-n\lambda_i + n\epsilon} |z|.$$

We deduce $|R^n_{\hat{x}}(z)| < M'_\epsilon(\hat{x}) e^{-n\lambda_k + n\epsilon} r$ for every $z \in \mathbb{D}^k(r)$. Similarly, for every $w \in \mathbb{D}^k(r)$ and $1 \le i \le k$, we have:

$$|\pi_i(R^{-n}_{\hat{x}}(w))| \le (\Delta+1) M_\epsilon(\hat{x}) e^{n\lambda_i + n\epsilon} |w| \le M'_\epsilon(\hat{x}) e^{n\lambda_1 + n\epsilon} |w|.$$

Hence $|R^{-n}_{\hat{x}}(w)| < r$ for every $w \in \mathbb{D}^k\left(M'_\epsilon(\hat{x})^{-1} e^{-n\lambda_1 - n\epsilon} r\right)$. That proves the point 1. For the point 2, observe that for every $1 \le i \le k-1$ and $z \in \mathbb{D}^k(r)$:

$$\parallel \pi_i \circ d_z R^n_{\hat{x}} \parallel \le \max\{ \, |a_{n,i}(\hat{x})| \, , \, \theta \parallel N_{i,n}(\hat{x}) \parallel r^{\theta-1} \, \} \le M'_\epsilon(\hat{x}) e^{-n\lambda_{k-1} + n\epsilon}.$$

The point 3 now follows from the point 2 and the observation (see (2)) :

$$\left| \pi_k \left(\frac{\partial R^n_{\hat{x}}}{\partial z_k}(z) \right) \right| = \parallel \pi_k \circ d_z R^n_{\hat{x}} \parallel = |a_{k,n}(\hat{x})| \simeq e^{-n\lambda_k \pm n\epsilon}.$$

For the point 4, let us distinguish whether or not $I = \{2\lambda_k \le \lambda_i\}$ is empty. If I is empty, there are no resonant degree, hence $R^n_{\hat{x}}$ is a linear mapping and $\frac{\partial^2 R^n_{\hat{x}}}{\partial z_k^2} = 0$. If I is not empty ($\theta = \lambda_1/\lambda_k \ge 2$ in that case), we have for every $1 \le i \le \max I$:

$$\left| \pi_i \left(\frac{\partial^2 R^n_{\hat{x}}}{\partial z_k^2}(z) \right) \right| \le \theta(\theta-1) \parallel N_{i,n}(\hat{x}) \parallel r^{\theta-2} \le M'_\epsilon(\hat{x}) e^{-n\lambda_i + n\epsilon} \le M'_\epsilon(\hat{x}) e^{-2n\lambda_k + n\epsilon},$$

and $\pi_i(\frac{\partial^2 R^n_{\hat{x}}}{\partial z_k^2}) = 0$ for every $\max I + 1 \le i \le k$. □

4 Proof of theorem A

In this section we establish theorem A assuming theorem B. Our aim is to prove:

$$\forall x \in \mathbb{CP}^k \ \nu\text{-a.e.}\ , \ \liminf_{r\to 0} \frac{\log \nu(B_x(r))}{\log r} \ \geq \ \frac{\log d^{k-1}}{\lambda_1} + \frac{h_\nu - \log d^{k-1}}{\lambda_k}. \tag{3}$$

Let $\epsilon > 0$ and Ω_ϵ, r_0 be given by theorem B. We have $\nu(\Omega_\epsilon) > 1 - \epsilon$, and for every $x \in \Omega_\epsilon$ the cardinality of

$$\mathcal{P}_n(x) = \{ \, f^{-n}_{\hat{y}_n}(B_p(s_n)) \, , \, y \in B_x^{\Omega_\epsilon}(\rho_n) \, , \, p \in \mathcal{E}_{s_n}(y_n) \, \}$$

is less than $d^{(k-1)(n-q_n)} e^{20kn\epsilon}$. Here we set $\rho_n := s_n e^{-n\lambda_k + 3n\epsilon}$. We shall use Brin-Katok's theorem. Let $B_n(x,\xi) := \{z \in \mathbb{CP}^k \, , \, \mathsf{dist}(f^q(x), f^q(z)) < \xi \, , \, 0 \leq q \leq n\}$ be the n-dynamical ball centered at x with radius ξ.

Theorem [BK] *For ν-a.e. $x \in \mathbb{CP}^k$, we have*

$$\sup_{\xi > 0} \liminf_{n\to+\infty} -\frac{1}{n} \log \nu(B_n(x,\xi)) = h_\nu.$$

In particular, for ν-a.e. $x \in \mathbb{CP}^k$, there exist $\xi_\epsilon(x) > 0$ and $m_\epsilon(x) \geq 1$ such that:

$$\forall \xi \leq \xi_\epsilon(x) \, , \, \forall n \geq m_\epsilon(x) \, , \, \nu(B_n(x,\xi)) \leq e^{-n(h_\nu - \epsilon)}.$$

We may decrease r_0 and choose $m_0 \geq 1$ large enough so that $\Gamma_\epsilon := \{\xi_\epsilon \geq r_0 \, , \, m_\epsilon \leq m_0\}$ satisfies $\nu(\Gamma_\epsilon) > 1 - \epsilon$. We have:

$$\forall x \in \Gamma_\epsilon \, , \, \forall n \geq m_0 \, , \, \nu(B_n(x, r_0)) \leq e^{-n(h_\nu - \epsilon)}. \tag{4}$$

We let $\Lambda_\epsilon := \Gamma_\epsilon \cap \Omega_\epsilon$ (it satisfies $\nu(\Lambda_\epsilon) > 1 - 2\epsilon$) and define:

$$\mathcal{Q}_n(x) := \{ \, f^{-n}_{\hat{y}_n}(B_p(s_n)) \, , \, y \in B_x^{\Lambda_\epsilon}(\rho_n) \, , \, p \in \mathcal{E}_{s_n}(y_n) \, \} \ \subset \ \mathcal{P}_n(x).$$

Lemma 4.1 *For every $Q \in \mathcal{Q}_n(x)$ we have $\nu(Q) \leq e^{-n(h_\nu - \epsilon)}$.*

The proof needs the definition of Ω_ϵ and is postponed to subsection 5.1. Let $\Lambda'_\epsilon \subset \Lambda_\epsilon$ be the subset of points satisfying $\nu(B_x^{\Lambda_\epsilon}(r))/\nu(B_x(r)) \to 1$ when $r \to 0$. The Borel density lemma asserts that $\nu(\Lambda'_\epsilon) = \nu(\Lambda_\epsilon)$.

Lemma 4.2 *For every $x \in \Lambda'_\epsilon$, there exists $p(x) \geq 1$ such that:*

$$\forall n \geq p(x) \ , \ \nu\left(B_x(\rho_n)\right) \leq 2 \, \mathit{Card} \, \mathcal{P}_n(x) \cdot e^{-n(h_\nu - \epsilon)}.$$

PROOF: Let $x \in \Lambda'_\epsilon$ and $p(x) \geq 1$ so that $\nu(B_x(\rho_n)) \leq 2\nu(B_x^{\Lambda_\epsilon}(\rho_n))$ for $n \geq p(x)$. The fact that $\mathcal{Q}_n(x)$ is a covering of $B_x^{\Lambda_\epsilon}(\rho_n)$ combined with lemma 4.1 implies:

$$\nu(B_x^{\Lambda_\epsilon}(\rho_n)) \leq \sum_{Q \in \mathcal{Q}_n(x)} \nu(Q) \leq \mathsf{Card}\ \mathcal{Q}_n(x) \cdot e^{-n(h_\nu - \epsilon)}.$$

We conclude using $\mathsf{Card}\ \mathcal{Q}_n(x) \leq \mathsf{Card}\ \mathcal{P}_n(x)$. □

Lemma 4.3 *For every $x \in \Lambda'_\epsilon$, we have:*

$$\liminf_{r \to 0} \frac{\log \nu(B_x(r))}{\log r} \geq \left(\frac{\log d^{k-1}}{\lambda_1} + \frac{h_\nu - \log d^{k-1}}{\lambda_k} - \frac{21k\epsilon}{\lambda_k} \right) \frac{\lambda_k}{\lambda_k + 5\epsilon}.$$

PROOF: Lemma 4.2 yields for $n \geq p(x)$:

$$\log \nu(B_x(\rho_n)) \leq \log \mathsf{Card}\ \mathcal{P}_n(x) - n(h_\nu - \epsilon) + \log 2.$$

We use theorem B to obtain for $n \geq p(x)$:

$$\log \nu(B_x(\rho_n)) \leq (n - q_n) \log d^{k-1} - n(h_\nu - \epsilon) + 20kn\epsilon + \log 2.$$

Using $\rho_n = r_0 e^{-n\lambda_k - 5n\epsilon}$ and $q_n \geq n\lambda_k/\lambda_1 - 1$, we obtain for n large:

$$\frac{\log \nu(B_x(\rho_n))}{\log \rho_n} \geq \frac{n\lambda_k/\lambda_1 \cdot \log d^{k-1} + n(h_\nu - \log d^{k-1}) - 21kn\epsilon - \log 2}{n\lambda_k + 5n\epsilon - \log r_0}.$$

The aimed estimate follows letting $n \to \infty$. □

Finally, lemma 4.3 yields (3) as follows. Let $\Lambda' := \cap_{p \geq 1} \cup_{q \geq p} \Lambda'_{1/q}$. We have $\nu(\Lambda') = 1$ since $\nu(\Lambda'_{1/q}) > 1 - 2/q$ for every $q \geq 1$. Now for every $x \in \Lambda'$ there exists a subsequence $(q_j(x))_{j \geq 1}$ such that $x \in \Lambda'_{1/q_j(x)}$. We deduce (3) from lemma 4.3 setting $\epsilon = 1/q_j(x)$ and letting $j \to \infty$. That completes the proof of theorem A.

5 Proof of theorem B

5.1 Definition of Ω_ϵ and r_0

Let $\epsilon > 0$ and r_ϵ, β_ϵ, M'_ϵ be the ϵ-slow and ϵ-fast functions provided by theorem 3.5 and lemma 3.6. Let us choose $r_0 \leq 1$ small and $\beta_0, M'_0 \geq 1$ large such that the set

$$\widehat{\Omega}_\epsilon := \{\, \hat{x} \in X\,,\ r_\epsilon(\hat{x}) \geq r_0\,,\ \beta_\epsilon(\hat{x}) \leq \beta_0\,,\ M'_\epsilon(\hat{x}) \leq M'_0 \,\}$$

satisfies $\hat{\nu}(\widehat{\Omega}_\epsilon) > 1 - \epsilon$. We define $\Omega_\epsilon := \hat{\pi}(\widehat{\Omega}_\epsilon)$. Observe that $\nu(\Omega_\epsilon) = \hat{\nu}(\hat{\pi}^{-1}(\Omega_\epsilon)) \geq \hat{\nu}(\widehat{\Omega}_\epsilon) > 1 - \epsilon$. We fix once and for all a section of the restriction $\hat{\pi} : \widehat{\Omega}_\epsilon \to \Omega_\epsilon$. That is to say that we associate to every $x \in \Omega_\epsilon$ an element of $\widehat{\Omega}_\epsilon \cap \hat{\pi}^{-1}\{x\}$, that we denote $\hat{x}$.

We set $r_n := r_0 e^{-n\epsilon}$, $\beta_n := \beta_0 e^{n\epsilon}$, $M'_n := M'_0 e^{n\epsilon}$. We shall also need:

$$s_n := r_0 e^{-8n\epsilon} \quad , \quad \rho_n := r_0 e^{-n\lambda_k - 5n\epsilon} \quad , \quad \tau_n := \beta_0(1 + 2\beta_0 M'_0)\rho_n.$$

In the sequel, the estimates and inclusions will be written for n large only depending on $\epsilon, r_0, \beta_0, M'_0$ and $(\lambda_i)_{1 \le i \le k}$.

Lemma 5.1 *For every $x \in \Omega_\epsilon$, the maps $f^{-n}_{\hat{x}_n}$, $S_{\hat{x}_n}$ and $R^n_{\hat{x}_n}$ satisfy:*

1. *$f^{-n}_{\hat{x}_n}$ and $S_{\hat{x}_n}$ are well defined on $B_{x_n}(r_n)$.*
2. *$\mathsf{dist}(p, p') \le |S_{\hat{x}_n}(p) - S_{\hat{x}_n}(p')| \le \beta_n \, \mathsf{dist}(p, p')$ for every $(p, p') \in B_{x_n}(r_n)$.*
3. *$\mathbb{D}^k_{S_{\hat{x}_n}(p)}(r) \subset S_{\hat{x}_n}(B_p(r)) \subset \mathbb{D}^k_{S_{\hat{x}_n}(p)}(\beta_n r)$ for every $B_p(r) \subset B_{x_n}(r_n)$.*
4. *$\mathbb{D}^k\left(M'^{\,-1}_n e^{-q\lambda_1 - q\epsilon} \cdot r\right) \subset R^q_{\hat{x}_n}\left(\mathbb{D}^k(r)\right) \subset \mathbb{D}^k\left(M'_n e^{-q\lambda_k + q\epsilon} \cdot r\right)$ for every $r \le 1$ and $0 \le q \le n$.*

Proof: The fact that $\hat{x} \in \widehat{\Omega}_\epsilon$ and the ϵ-slow, ϵ-fast properties of $r_\epsilon, \beta_\epsilon$ yield $r_\epsilon(\hat{x}_n) \ge r_\epsilon(\hat{x})e^{-n\epsilon} \ge r_n$ and $\beta_\epsilon(\hat{x}_n) \le \beta_\epsilon(\hat{x})e^{n\epsilon} \le \beta_n$. All the items then follow from theorem 3.5, definition 3.4 and lemma 3.6(1). □

Now we can give the

Proof of lemma 4.1: Let $y \in \Lambda_\epsilon$ and $p \in \mathcal{E}_{s_n}(y_n)$ such that $Q = f^{-n}_{\hat{y}_n}(B_p(s_n))$. Since $B_p(s_n) \subset B_{y_n}(2s_n)$, it suffices to prove that $f^{-n}_{\hat{y}_n}(B_{y_n}(2s_n)) \subset B_n(y, r_0)$ (see (4)). We verify for that purpose that $\mathsf{dist}(f^{-q}_{\hat{y}_n}(z), f^{-q}_{\hat{y}_n}(y_n)) \le r_0$ for every $z \in B_{y_n}(2s_n)$ and $0 \le q \le n$. Using the identity $f^{-q}_{\hat{y}_n} = S^{-1}_{\hat{y}_{n-q}} \circ R^q_{\hat{y}_n} \circ S_{\hat{y}_n}$ and lemma 5.1(3,4), we get:

$$\forall z \in B_{y_n}(2s_n) \ , \ \mathsf{dist}(f^{-q}_{\hat{y}_n}(z), f^{-q}_{\hat{y}_n}(y_n)) \le 2s_n\beta_n M'_n e^{-q\lambda_k + q\epsilon} \le 2r_0 M'_0 \beta_0 e^{-5n\epsilon} \le r_0.$$

That completes the proof of lemma 4.1. □

Let us deal with the biholomorphism $\psi_{x,y} := S_{\hat{x}} \circ S^{-1}_{\hat{y}}$ when y is close to $x \in \Omega_\epsilon$.

Lemma 5.2 *There exist $R \le 1$ and $\gamma > 0$ such that for every $x \in \Omega_\epsilon$ and $y \in B^{\Omega_\epsilon}_x(r_0/2)$:*

1. *$\psi_{x,y} : \mathbb{D}^k(R) \to \mathbb{D}^k(\beta_0)$ is well defined.*
2. *$\frac{1}{\beta_0}|z - z'| \le |\psi_{x,y}(z) - \psi_{x,y}(z')| \le \beta_0 |z - z'|$ for every $(z, z') \in \mathbb{D}^k(R)$.*
3. *$\| \, d_z\psi_{x,y} - d_{z'}\psi_{x,y} \, \| \le \gamma |z - z'|$ for every $(z, z') \in \mathbb{D}^k(R)$.*

The point 2 actually implies $|d_0\psi_{x,y}(c_k)| \geq 1/\beta_0$. We therefore have $B_x^{\Omega_\epsilon}(r_0/2) = \cup_{i=1}^k W_x^i$, where $W_x^i := \{\,|\,\pi_i\,(d_0\psi_{x,y}(c_k))\,| \geq 1/\beta_0\}$. We fix for every $x \in \Omega_\epsilon$ a partition $B_x^{\Omega_\epsilon}(r_0/2) = \cup_{i=1}^k Y_x^i$, where $Y_x^i \subset W_x^i$. We complete lemma 5.2 as follows:

Lemma 5.3 $|\pi_i\,(d_z\psi_{x,y}(c_k))\,| \geq 1/(2\beta_0)$ *for every* $y \in Y_x^i$ *and* $z \in \mathbb{D}^k(R)$.

Proof of lemmas 5.2 and 5.3: Let $R' = r_0/2$, $\gamma = \beta_0/R'^2$ and $R = 1/(2\beta_0\gamma) < R'$. We prove 1, 2 on $\mathbb{D}^k(R')$ and 3, lemma 5.3 on $\mathbb{D}^k(R)$. Lemma 5.1(3) yields for every $w \in \{x, y\}$ (take $p = w$ and $n = 0$ in that lemma):

$$\forall r \leq r_0\ ,\ \mathbb{D}^k(r) \subset S_{\hat{w}}(B_w(r)) \subset \mathbb{D}^k(\beta_0 r). \tag{5}$$

Let $z \in \mathbb{D}^k(R')$. The left inclusion in (5) with $w = y$ and $r = R'$ yields $S_{\hat{y}}^{-1}(z) \in B_y(R')$. Since $B_y(R') \subset B_x(r_0)$, the right inclusion in (5) with $w = x$ gives $\psi_{x,y}(z) = S_{\hat{x}} \circ S_{\hat{y}}^{-1}(z) \in \mathbb{D}^k(r_0\beta_0) \subset \mathbb{D}^k(\beta_0)$. That proves the point 1. The point 2 then comes from lemma 5.1(2) and the point 3 from Cauchy's estimates: we indeed have $\|\,\psi_{x,y}\,\|_{C^2,\mathbb{D}^k(R')} \leq \beta_0/R'^2 = \gamma$ from point 1. Now let us deal with lemma 5.3. For every $z \in \mathbb{D}^k(R)$, the point 3 implies $\|\,d_z\psi_{x,y} - d_0\psi_{x,y}\,\| \leq \gamma R = 1/(2\beta_0)$. The desired estimate then follows from $|\pi_i(d_0\psi_{x,y}(c_k))| \geq 1/\beta_0$. □

5.2 The upper bound on Card $\mathcal{P}_n(x)$

Let $x \in \Omega_\epsilon$. Recall that $s_n = r_0 e^{-8n\epsilon}$, $\rho_n = s_n e^{-n\lambda_k + 3n\epsilon}$ and

$$\mathcal{P}_n(x) = \left\{ f_{\hat{y}_n}^{-n}(B_p(s_n))\,,\, y \in B_x^{\Omega_\epsilon}(\rho_n)\,,\, p \in \mathcal{E}_{s_n}(y_n) \right\},$$

where $\mathcal{E}_{s_n}$ is a fixed s_n-separated set in $\mathbb{CP}^k$. We want to prove

$$\mathsf{Card}\ \mathcal{P}_n(x) \leq d^{(k-1)(n-q_n)} \cdot e^{20kn\epsilon}, \tag{6}$$

where q_n denotes the entire part of $n\lambda_k/\lambda_1$. We first verify that $\mathcal{P}_n(x)$ is well defined, it therefore induces a covering of $B_x^{\Omega_\epsilon}(\rho_n)$:

Lemma 5.4 *For every* $y \in \Omega_\epsilon$ *and* $p \in \mathcal{E}_{s_n}(y_n)$*:*

1. $f_{\hat{y}_n}^{-n}$ *and* $S_{\hat{y}_n}$ *are well defined on* $B_p(s_n)$*,*

2. $S_{\hat{y}_n}(B_p(s_n)) \subset \mathbb{D}^k(2s_n\beta_n)$*.*

Proof: Observe that $B_p(s_n) \subset B_{y_n}(2s_n) \subset B_{y_n}(r_n)$ by definition of $\mathcal{E}_{s_n}(y_n)$. The items then follows from lemma 5.1(1,3). □

Now we localize the collection $\mathcal{P}_n(x)$. We recall that $\tau_n = \beta_0(1 + 2\beta_0 M_0')\rho_n$.

Lemma 5.5 *For every* $x \in \Omega_\epsilon$ *and* $P_n \in \mathcal{P}_n(x)$*, we have* $P_n \subset S_{\hat{x}}^{-1}(\mathbb{D}^k(\tau_n))$*.*

Proof: Let $P_n \in \mathcal{P}_n(x)$: there exist $y \in B_x^{\Omega_\epsilon}(\rho_n)$ and $p \in \mathcal{E}_{s_n}(y)$ satisfying $P_n = f_{\hat{y}_n}^{-n}(B_p(s_n))$. Our aim is to prove that $S_{\hat{x}}(P_n) \subset \mathbb{D}^k(\tau_n)$. We shall use $S_{\hat{x}}(P_n) = \psi_{x,y} \circ R^n_{\hat{y}_n} \circ S_{\hat{y}_n}(P)$, where $P := B_p(s_n)$. This comes from $f_{\hat{y}_n}^{-n} = S_{\hat{y}}^{-1} \circ R^n_{\hat{y}_n} \circ S_{\hat{y}_n}$ (see theorem 3.5) and $\psi_{x,y} = S_{\hat{x}} \circ S_{\hat{y}}^{-1}$. Lemmas 5.4(2) and 5.1(4) yield successively $S_{\hat{y}_n}(P) \subset \mathbb{D}^k(2s_n\beta_n)$ and $R^n_{\hat{y}_n} \circ S_{\hat{y}_n}(P) \subset \mathbb{D}^k(2s_n\beta_n M'_n e^{-n\lambda_k+n\epsilon})$, which is included in $\mathbb{D}^k(R)$. Then lemma 5.2(1,2) implies:

$$\psi_{x,y} \circ R^n_{\hat{y}_n} \circ S_{\hat{y}_n}(P) \subset \psi_{x,y}(0) + \mathbb{D}^k(2s_n\beta_n M'_n e^{-n\lambda_k+n\epsilon}\beta_0). \tag{7}$$

But $\psi_{x,y}(0) = S_{\hat{x}}(y) \in \mathbb{D}^k(\rho_n\beta_0)$ from $y \in B_x(\rho_n)$ and lemma 5.1(3). The right hand side of (7) is therefore included in $\mathbb{D}^k(\rho_n\beta_0 + 2s_n M'_n\beta_n e^{-n\lambda_k+n\epsilon}\beta_0)$, which is $\mathbb{D}^k(\tau_n)$. That proves $S_{\hat{x}}(P_n) \subset \mathbb{D}^k(\tau_n)$. □

Now let us restate propositions A and B of section 2. We parametrize the family $\mathcal{L}_n$ of polydiscs by $(i,\alpha) \in \{1,\dots,k\} \times \mathbb{D}(\tau_n)$. More precisely, let $\mathsf{L}_n^{i,\alpha} : \mathbb{D}^{k-1} \to \mathbb{D}^k(\tau_n)$ be defined as $\mathsf{L}_n^{i,\alpha}(v_1,\dots,v_{k-1}) = (v_1\tau_n,\dots,\alpha,\dots,v_{k-1}\tau_n)$, where α stands at the i-th coordinate. Pulling back $\mathsf{L}_n^{i,\alpha}$ by $S_{\hat{x}}$, we set $L_n^{i,\alpha} := S_{\hat{x}}^{-1} \circ \mathsf{L}_n^{i,\alpha}$. By lemma 5.1(3), that polydisc satisfies $L_n^{i,\alpha} : \mathbb{D}^{k-1} \to B_x(\tau_n)$.

Proposition A now take the following form.

Proposition A: *For every $(i,\alpha) \in \{1,\dots,k\} \times \mathbb{D}(\tau_n)$, $\mathsf{Vol}\, f^n \circ L_n^{i,\alpha} \le d^{(k-1)(n-q_n)}$.*

Before dealing with proposition B, let us introduce the collection

$$\mathcal{P}'_n(x) := \left\{ f_{\hat{y}_n}^{-n}(B_p(s_n/2))\,,\, y \in B_x^{\Omega_\epsilon}(\rho_n)\,,\, p \in \mathcal{E}_{s_n}(y_n) \right\}.$$

It satisfies $\mathsf{Card}\, \mathcal{P}'_n(x) = \mathsf{Card}\, \mathcal{P}_n(x)$ and its sets are pairwise disjoint. Given $P_n \in \mathcal{P}'_n(x)$, for simplicity we denote $\mathsf{Vol}\ f^n\,(L_n^{i,\alpha} \cap P_n)$ for the volume of $f^n \circ L_n^{i,\alpha}$ restricted to $(L_n^{i,\alpha})^{-1}(P_n)$. Observe that it has multiplicity 1 since f^n is injective on P_n. Proposition B is restated as follows.

Proposition B: *There exists a subset $\Lambda_n \subset \mathbb{D}(\tau_n)$ satisfying: $\mathsf{Card}\, \Lambda_n \le e^{20n\epsilon}$ and for every $P_n \in \mathcal{P}'_n(x)$, there is $(i,\alpha)(P_n) \in \{1,\dots,k\} \times \Lambda_n$ such that:*

$$\mathsf{Vol}\ f^n \left(L_n^{(i,\alpha)(P_n)} \cap P_n\right) \ge (s_n)^{k-1}.$$

Let us see how we deduce (6), thus completing the proof of theorem B. Since the sets of $\mathcal{P}'_n(x)$ are pairwise disjoint, we have:

$$\sum_{P_n \in \mathcal{P}'_n(x)} \mathsf{Vol}\ f^n \left(L_n^{(i,\alpha)(P_n)} \cap P_n\right) \le \sum_{i=1}^{k} \sum_{\alpha\in\Lambda_n} \mathsf{Vol}\ f^n \circ L_n^{i,\alpha}.$$

That implies $\mathsf{Card}\, \mathcal{P}_n(x) \cdot (s_n)^{k-1} \le k\, \mathsf{Card}\, \Lambda_n \cdot d^{(k-1)(n-q_n)}$. Then (6) follows using $s_n = r_0 e^{-8n\epsilon}$ and $\mathsf{Card}\, \Lambda_n \le e^{20n\epsilon}$.

6 Proof of propositions A and B (stated in §5.2)

6.1 Proof of proposition A

We denote by $\bar{L}_n^{i,\alpha}$ the extension of $L_n^{i,\alpha}$ to the polydisc $\mathbb{D}^{k-1}(2)$, it satisfies $\bar{L}_n^{i,\alpha} \subset B_x(2\tau_n)$. We set $\sigma_{q_n} := f^{q_n} \circ L_n^{i,\alpha}$ and $\bar{\sigma}_{q_n} := f^{q_n} \circ \bar{L}_n^{i,\alpha}$. According to section 2, proposition A is a consequence of the growth lemma combined with the following lemma:

Lemma 6.1 *The polydisc $\bar{\sigma}_{q_n}$ is bounded.*

Proof: We have to show that $\bar{\sigma}_{q_n}$ is included in some B_j. With no loss of generality, we can assume that every ball of radius r_0 in $\mathbb{CP}^k$ is contained in some B_j. For simplicity we denote $q := q_n$. Observe that it suffices to prove

$$\mathbb{D}^k(2\beta_0\tau_n) \subset R^q_{\hat{x}_q} \circ S_{\hat{x}_q}(B_{x_q}(r_q)). \tag{8}$$

Indeed, that inclusion implies using $R^q_{\hat{x}_q} \circ S_{\hat{x}_q} = S_{\hat{x}} \circ f^{-q}_{\hat{x}_q}$ and $r_q \leq r_0$:

$$f^q \circ S_{\hat{x}}^{-1}(\mathbb{D}^k(2\beta_0\tau_n)) \subset B_{x_q}(r_0).$$

The conclusion then follows from (see lemma 5.1(3) for the last inclusion):

$$\bar{\sigma}_q = f^q \circ \bar{L}_n^{i,\alpha} \subset f^q(B_x(2\tau_n)) \subset f^q \circ S_{\hat{x}}^{-1}(\mathbb{D}^k(2\beta_0\tau_n)).$$

Thus it remains to show (8). Lemma 5.1(4,3) yields:

$$\mathbb{D}^k(M_q'^{-1}e^{-q\lambda_1-q\epsilon} \cdot r_q) \subset R^q_{\hat{x}_q}(\mathbb{D}^k(r_q)) \subset R^q_{\hat{x}_q}(S_{\hat{x}_q}(B_{x_q}(r_q))).$$

Using $q\lambda_1 \leq n\lambda_k$ (which implies $q \leq n$), we obtain:

$$M_q'^{-1}e^{-q\lambda_1-q\epsilon} \cdot r_q = r_0 M_0'^{-1}e^{-q\lambda_1-3q\epsilon} \geq e^{-n\lambda_k-4n\epsilon} \geq 2\beta_0^2(1+2\beta_0M_0')r_0e^{-n\lambda_k-5n\epsilon},$$

which is equal to $2\beta_0\tau_n$. □

6.2 Proof of proposition B

We set $\eta_n := s_n e^{-n\lambda_k-4n\epsilon}/4$ and define Λ_n as a maximal η_n-separated set in $\mathbb{D}(\tau_n e^{n\epsilon})$. We have $\mathsf{Card}\ \Lambda_n \leq (\tau_n e^{n\epsilon})^2/\eta_n^2 \leq e^{20n\epsilon}$. Let us fix $P_n \in \mathcal{P}'_n(x)$ for the remainder of the section and show:

$$\exists\, (i,\alpha)(P_n) \in \{1,\dots,k\} \times \Lambda_n\ ,\ \mathsf{Vol}\ f^n\left(L_n^{(i,\alpha)(P_n)} \cap P_n\right) \geq (s_n)^{k-1}. \tag{9}$$

Let also $y \in B_x^{\Omega_\epsilon}(\rho_n)$ and $p \in \mathcal{E}_{s_n}(y)$ such that $P_n = f^{-n}_{\hat{y}_n}(P) = f^{-n}_{\hat{y}_n}(B_p(s_n/2))$.

6.2.1 Definition of $(i,\alpha)(P_n)$

We define $1 \le i(P_n) \le k$ to be the unique element satisfying $y \in Y_x^{i(P_n)}$ (see subsection 5.1). For simplicity we denote $j := i(P_n)$. We now define $\alpha(P_n) \in \Lambda_n$. Since $S_{\hat{x}}(P_n) = S_{\hat{x}} \circ f_{\hat{y}_n}^{-n}(P) \subset \mathbb{D}^k(\tau_n)$ (lemma 5.5), then $\mathsf{p}_n := S_{\hat{x}} \circ f_{\hat{y}_n}^{-n}(p)$ lies in $\mathbb{D}^k(\tau_n)$. In particular we have $\pi_j(\mathsf{p}_n) \in \mathbb{D}(\tau_n)$ and $\mathbb{D}_{\pi_j(\mathsf{p}_n)}(\eta_n) \subset \mathbb{D}(\tau_n e^{n\epsilon})$. In order to find some $\alpha(P_n) \in \Lambda_n$ satisfying (9), we shall prove:

$$\forall \alpha \in \mathbb{D}_{\pi_j(\mathsf{p}_n)}(\eta_n) \ , \ \mathsf{Vol}\ f^n\left(L_n^{j,\alpha} \cap P_n\right) \ge (s_n)^{k-1}. \tag{10}$$

Then we take for $\alpha(P_n)$ any element in $\Lambda_n \cap \mathbb{D}_{\pi_j(\mathsf{p}_n)}(\eta_n)$: that set is not empty since Λ_n is a maximal η_n-separated set in $\mathbb{D}(\tau_n e^{n\epsilon})$. That shows theorem B.

We deduce (10) from the following claim. It relies on a precise geometrical description of the inverse branches, due to the normalization theorem. We set $Q := B_p(s_n/4)$, $Q_n := f_{\hat{y}_n}^{-n}(Q)$ and identify the polydisc $L_n^{j,\alpha}$ with its source $\mathbb{D}^{k-1}$.

Claim : *For every $\alpha \in \mathbb{D}_{\pi_j(\mathsf{p}_n)}(\eta_n)$,*

(a) $L_n^{j,\alpha}$ intersects Q_n,

(b) the slice $P_n \cap L_n^{j,\alpha}$ is a domain in $\mathbb{D}^{k-1}$ with boundary in ∂P_n.

Let us see how we infer (10). Let $a \in Q_n \cap L_n^{j,\alpha}$. Since $f^n(a) \in Q$, we have $Q' := B_{f^n(a)}(s_n/4) \subset P = B_p(s_n/2)$. Hence $\Sigma := f^n(P_n \cap L_n^{j,\alpha})$ satisfies $\Sigma \subset P$ and $\partial\Sigma \subset \partial P$ (the map $f^n : P_n \to P$ is a biholomorphism). Therefore $\Sigma \cap Q'$ is an immersed polydisc containing $f^n(a)$ (the center of Q') with boundary in $\partial Q'$. The Lelong inequality [L] then implies $\mathsf{Vol}\ (\Sigma \cap Q') \ge (s_n)^{k-1}$ up to a multiplicative constant. That gives (10) and completes the proof of theorem B.

6.2.2 Proof of the claim

Let us denote $\psi := \psi_{x,y}$. For every $s \le s_n$, we set $\eta := se^{-n\lambda_k - 4n\epsilon}$, $\mathsf{A} := S_{\hat{y}_n}(B_p(s))$ and $\mathsf{A}_n := \psi_{x,y} \circ R_{\hat{y}_n}^n(\mathsf{A})$. For simplicity we assume that $\mathsf{A} = \mathbb{D}_{\mathsf{p}}^k(s)$, where $\mathsf{p} := S_{\hat{y}_n}(p)$ (see lemma 5.1(3)). For any $\tilde{u} = (u_1, \ldots, u_{k-1}) \in \mathbb{D}^{k-1}$, we define $\mathsf{v}_{\tilde{u}} : \mathbb{D} \to \mathsf{A}$ by $\mathsf{v}_{\tilde{u}}(t) := \mathsf{p} + s(\tilde{u}, t)$. The claim is a consequence of the next proposition applied with $s = s_n/2$ (for the item (b)) and $s_n/4$ (for the item (a)).

Proposition 6.2 *For every $\tilde{u} \in \mathbb{D}^{k-1}$,*

1. *$\psi_{x,y} \circ R_{\hat{y}_n}^n(\mathsf{v}_{\tilde{u}})$ is a graph over the j-axis,*
2. *its π_j-projection $\mathsf{w}_{\tilde{u}}^n := \pi_j \circ \psi_{x,y} \circ R_{\hat{y}_n}^n(\mathsf{v}_{\tilde{u}})$ contains the disc $\mathbb{D}_{\pi_j(\mathsf{p}_n)}(\eta)$.*

We need the following lemma for proving proposition 6.2.

Lemma 6.3 *For every $\tilde{u} \in \mathbb{D}^{k-1}$, $\mathsf{w}_{\tilde{u}}^n : \mathbb{D} \to \mathbb{C}$ satisfies :*

1. $\mathsf{w}_{\tilde u}^n(0) \in \mathbb{D}_{\pi_j(\mathsf{p}_n)}(se^{-n\lambda_{k-1}+3n\epsilon})$,

2. $\forall t \in \mathbb{D}$, $|\mathsf{w}_{\tilde u}^{n\prime}(t) - \mathsf{w}_{\tilde u}^{n\prime}(0)| \le se^{-2n\lambda_k+3n\epsilon}$,

3. $|\mathsf{w}_{\tilde u}^{n\prime}(0)| \ge se^{-n\lambda_k-2n\epsilon}$.

Proof of proposition 6.2: For the point 1, it suffices to verify that $\mathsf{w}_{\tilde u}^n = \pi_j \circ \psi_{x,y} \circ R_{\hat y_n}^n(\mathsf{v}_{\tilde u})$ is injective. Let $\varphi := (\mathsf{w}_{\tilde u}^n - \mathsf{w}_{\tilde u}^n(0))/\mathsf{w}_{\tilde u}^{n\prime}(0) - \mathrm{Id}$. We get from lemma 6.3(2,3):

$$\forall t \in \mathbb{D} \ , \ |\varphi'(t)| = \frac{|\mathsf{w}_{\tilde u}^{n\prime}(t) - \mathsf{w}_{\tilde u}^{n\prime}(0)|}{|\mathsf{w}_{\tilde u}^{n\prime}(0)|} \le \frac{e^{-2n\lambda_k+3n\epsilon}}{e^{-n\lambda_k-2n\epsilon}} = e^{-n\lambda_k+5n\epsilon}.$$

This implies $\mathrm{Lip}\,(\varphi) \le 1/2$, hence $\mathrm{Id} + \varphi$ and $\mathsf{w}_{\tilde u}^n$ are injective on $\mathbb{D}$. Let us prove the point 2. Since $\mathrm{Lip}\,(\varphi) \le 1/2$ and $\varphi(0) = 0$, we have $|(\mathrm{Id}+\varphi)(t)| \ge |t| - |\varphi(t)| \ge |t|/2$. That yields $|\mathsf{w}_{\tilde u}^n(t) - \mathsf{w}_{\tilde u}^n(0)| \ge |\mathsf{w}_{\tilde u}^{n\prime}(0)|/2$ for every $t \in \mathbb{S}^1$. Then lemma 6.3(3) implies:

$$\forall t \in \mathbb{S}^1 \ , \ |\mathsf{w}_{\tilde u}^n(t) - \mathsf{w}_{\tilde u}^n(0)| \ge se^{-n\lambda_k-3n\epsilon},$$

which yields $\mathbb{D}_{\mathsf{w}_{\tilde u}^n(0)}\left(se^{-n\lambda_k-3n\epsilon}\right) \subset \mathsf{w}_{\tilde u}^n$ by Jordan's theorem. We deduce from lemma 6.3(1) that:

$$\forall \tilde u \in \mathbb{D}^{k-1} \ , \ \mathbb{D}_{\pi_j(\mathsf{p}_n)}\left(se^{-n\lambda_k-3n\epsilon} - se^{-n\lambda_{k-1}+3n\epsilon}\right) \subset \mathbb{D}_{\mathsf{w}_{\tilde u}^n(0)}\left(se^{-n\lambda_k-3n\epsilon}\right).$$

Finally, the left hand side contains $\mathbb{D}_{\pi_j(\mathsf{p}_n)}(\eta) = \mathbb{D}_{\pi_j(\mathsf{p}_n)}(se^{-n\lambda_k-4n\epsilon})$. □

6.2.3 Proof of lemma 6.3

We shall use the algebraic properties of resonant maps (namely lemma 3.6). For every $(\tilde u, t) \in \mathbb{D}^{k-1} \times \mathbb{D}$, we denote $\mathsf{A}(\tilde u, t) := \mathsf{v}_{\tilde u}(t) = \mathsf{p} + s(\tilde u, t)$ and $z := \mathsf{A}(\tilde u, t)$. We also denote:

$$\mathsf{v}_{\tilde u}^n(t) := R_{\hat y_n}^n \circ \mathsf{v}_{\tilde u}(t) \quad \text{and} \quad \mathsf{h}^n(\tilde u) := R_{\hat y_n}^n \circ \mathsf{A}(\tilde u, 0).$$

We have therefore $\mathsf{p}_n = \psi \circ R_{\hat y_n}^n(\mathsf{p}) = \psi \circ \mathsf{h}^n(0)$. Observe that $\mathsf{A} \subset \mathbb{D}^k$ (lemma 5.4(2)) implies $\mathsf{v}_{\tilde u}^n \subset \mathbb{D}^k(M_n' e^{-n\lambda_k+n\epsilon}) \subset \mathbb{D}^k(R)$ (lemma 5.1(4)). One also obtains from the very definition of resonant maps (see (2), subsection 3.3):

$$\mathsf{v}_{\tilde u}^{n\prime}(t) = s \cdot \frac{\partial R_{\hat y_n}^n}{\partial z_k}(z) \quad , \quad \mathsf{v}_{\tilde u}^{n\prime\prime}(t) = s^2 \cdot \frac{\partial^2 R_{\hat y_n}^n}{\partial z_k^2}(z) \quad \text{and} \quad \pi_k \circ \mathsf{h}^n \equiv a_{k,n}(\hat y_n) \cdot \pi_k(\mathsf{p}). \tag{11}$$

We deduce from the last observation:

$$\| \, d_{\tilde u}\mathsf{h}^n \, \| = \| \, \tilde\pi \circ d_{\tilde u}\mathsf{h}^n \, \| = \left\| \, \tilde\pi \circ d_z R_{\hat y_n}^n \circ d_{(\tilde u,0)}\mathsf{A} \, \right\| = s \left\| \, \tilde\pi \circ d_z R_{\hat y_n}^n \, \right\|. \tag{12}$$

Finally let us recall that $\mathsf{w}_{\tilde u}^n = \pi_j \circ \psi \circ \mathsf{v}_{\tilde u}^n$.

1 - $\mathsf{w}_{\tilde u}^n(0) \in \mathbb{D}_{\pi_j(\mathsf{p}_n)}(se^{-n\lambda_{k-1}+3n\epsilon})$.

We have $\mathsf{w}^n_{\tilde{u}}(0) \in \pi_j \circ \psi \circ \mathsf{h}^n(\mathbb{D}^{k-1})$ and $\pi_j(\mathsf{p}_n) = \pi_j \circ \psi \circ \mathsf{h}^n(0)$. Moreover (12) yields for every $\tilde{u} \in \mathbb{D}^{k-1}$:

$$\| d_{\tilde{u}}(\pi_j \circ \psi \circ \mathsf{h}^n) \| \leq \left\| d_{\mathsf{h}^n(\tilde{u})}\psi \right\| \, \| d_{\tilde{u}}\mathsf{h}^n \| = s \left\| d_{\mathsf{h}^n(\tilde{u})}\psi \right\| \, \left\| \tilde{\pi} \circ d_z R^n_{\hat{y}_n} \right\|$$

which is less than $s\beta_0 M'_n e^{-n\lambda_{k-1}+n\epsilon} \leq s e^{-n\lambda_{k-1}+3n\epsilon}$ (lemmas 3.6(2) and 5.2(2)). That proves the point 1.

2 - $\forall t \in \mathbb{D}$, $|\mathsf{w}^{n\prime}_{\tilde{u}}(t) - \mathsf{w}^{n\prime}_{\tilde{u}}(0)| \leq s e^{-2n\lambda_k+3n\epsilon}$.
Since $\mathsf{w}^n_{\tilde{u}} = \pi_j \circ \psi \circ \mathsf{v}^n_{\tilde{u}}$, it suffices to verify that $\phi^n_{\tilde{u}} := (\psi \circ \mathsf{v}^n_{\tilde{u}})' - (\psi \circ \mathsf{v}^n_{\tilde{u}})'(0)$ satisfies $|\phi^n_{\tilde{u}}| \leq s e^{-2n\lambda_k+3n\epsilon}$. Let us write for every $t \in \mathbb{D}$:

$$\phi^n_{\tilde{u}}(t) = \left(d_{\mathsf{v}^n_{\tilde{u}}(t)}\psi - d_{\mathsf{v}^n_{\tilde{u}}(0)}\psi\right)(\mathsf{v}^{n\prime}_{\tilde{u}}(t)) + \left(d_{\mathsf{v}^n_{\tilde{u}}(0)}\psi\right)\left(\mathsf{v}^{n\prime}_{\tilde{u}}(t) - \mathsf{v}^{n\prime}_{\tilde{u}}(0)\right).$$

Using lemma 5.2(2,3), we obtain for every $t \in \mathbb{D}$:

$$|\phi^n_{\tilde{u}}(t)| \leq \gamma |\mathsf{v}^n_{\tilde{u}}(t) - \mathsf{v}^n_{\tilde{u}}(0)||\mathsf{v}^{n\prime}_{\tilde{u}}(t)| + \beta_0 |\mathsf{v}^{n\prime}_{\tilde{u}}(t) - \mathsf{v}^{n\prime}_{\tilde{u}}(0)| \leq \gamma |\mathsf{v}^{n\prime}_{\tilde{u}}|^2_{\infty,\mathbb{D}} + \beta_0 |\mathsf{v}^{n\prime\prime}_{\tilde{u}}|_{\infty,\mathbb{D}}.$$

We deduce using (11) and lemma 3.6(3,4):

$$|\phi^n_{\tilde{u}}(t)| \leq \gamma s^2 \max\{M'_n e^{-n\lambda_{k-1}+n\epsilon} \, , \, e^{-n\lambda_k+n\epsilon}\}^2 + \beta_0 s^2 M'_n e^{-2n\lambda_k+n\epsilon} \leq s e^{-2n\lambda_k+3n\epsilon}.$$

That proves the point 2.

3 - $|\mathsf{w}^{n\prime}_{\tilde{u}}(0)| = |\left(\pi_j \circ d_{\mathsf{v}^n_{\tilde{u}}(0)}\psi\right)(\mathsf{v}^{n\prime}_{\tilde{u}}(0))| \geq s e^{-n\lambda_k-2n\epsilon}$.
The line (11) and lemma 3.6(2,3) yield for $\mathsf{v}^{n\prime}_{\tilde{u}}(0) \in \mathbb{C}^k$:

$$|\tilde{\pi}(\mathsf{v}^{n\prime}_{\tilde{u}}(0))| \leq s M'_n e^{-n\lambda_{k-1}+n\epsilon} \quad \text{and} \quad |\pi_k(\mathsf{v}^{n\prime}_{\tilde{u}})(0)| \geq s e^{-n\lambda_k-n\epsilon}.$$

Now lemmas 5.2(2) and 5.3 imply (use $y \in Y^j_x$ for the second inequality):

$$\forall 1 \leq i \leq k-1 \, , \, |(\pi_j \circ d_{\mathsf{v}^n_{\tilde{u}}(0)}\psi)(c_i)| \leq \beta_0 \quad \text{and} \quad |(\pi_j \circ d_{\mathsf{v}^n_{\tilde{u}}(0)}\psi)(c_k)| \geq 1/(2\beta_0).$$

We deduce $|\mathsf{w}^{n\prime}_{\tilde{u}}(0)| \geq s\left((2\beta_0)^{-1}e^{-n\lambda_k-n\epsilon} - \beta_0 M'_n e^{-n\lambda_{k-1}+n\epsilon}\right) \geq s e^{-n\lambda_k-2n\epsilon}$, completing the proof of lemma 6.3.

7 Appendix

Let f be a holomorphic endomorphism of $\mathbb{CP}^k$ with degree $d \geq 2$. Let ω be the Fubini-Study $(1,1)$ form on $\mathbb{CP}^k$. For every holomorphic polydisc $\eta : \mathbb{D}^l(r) \to \mathbb{CP}^k$, we define $\mathsf{Vol}\, f^m \circ \eta := \int_{\mathbb{D}^l(r)} \eta^* f^{m*} \omega^l$. We recall that $\{B_j \, , \, j \in J\}$ is a finite covering of $\mathbb{CP}^k$ which consists of open sets bounded in the affine charts. We say that η is bounded if the image of that polydisc is contained in some B_j.

Growth lemma : *Let $1 \leq l \leq k$ and $\eta : \mathbb{D}^l(2) \to \mathbb{CP}^k$ be a bounded polydisc. Then*

$$\forall m \geq 1 \, , \, \mathsf{Vol}\, f^m \circ \eta_{|\mathbb{D}^l} \leq d^{lm}.$$

The proof relies on the Green current of f, which is the closed positive $(1,1)$ current on $\mathbb{CP}^k$ defined by $T = \lim_{n\to\infty} \frac{1}{d^n} f^{n*}\omega$. That current satisfies $f^*T = dT$ and $T = \omega - dd^c\varphi$ for some continuous function $\varphi : \mathbb{CP}^k \to \mathbb{R}$. Iterating that identity, we obtain:

$$\forall m \geq 1 \ , \ f^{m*}\omega = d^m T + dd^c(\varphi \circ f^m). \tag{13}$$

We refer to the article of Dinh-Sibony ([DS], section 1.2) for more details about the Green current. In order to prove the lemma, we shall use an induction concerning the mass of $T^i \wedge f^{m*}\omega^j$. Note that a similar induction was employed by Dinh to estimate the local entropy outside the support of the current T^i (see [D], theorem 2.1). In the sequel ω_0 stands for the $(1,1)$ form $dd^c|\,.\,|^2$ which induces the standard metric on $\mathbb{C}^k$.

Proof of the growth lemma: It follows from Cauchy's estimates that the family of bounded polydiscs $\mathbb{D}^l(2) \to \mathbb{CP}^k$ has bounded derivatives on $\mathbb{D}^l(3/2)$, say by 1. We deduce that for any such polydisc η and any positive current S on $\mathbb{CP}^k$ of bidegree (s,s) (with $s \leq l$):

$$\forall \rho \leq 3/2 \ , \ 0 \leq \int_{\mathbb{D}^l(\rho)} \eta^* S \wedge \eta^*\omega^{l-s} \leq \int_{\mathbb{D}^l(\rho)} \eta^* S \wedge \omega_0^{l-s}. \tag{14}$$

Let us fix $\eta : \mathbb{D}^l(2) \to \mathbb{CP}^k$ and denote by $\| S \|_\rho := \int_{\mathbb{D}^l(\rho)} \eta^* S \wedge \omega_0^{l-s}$. We shall prove for any $1 \leq q \leq l$ and $0 \leq r \leq q$:

$$(H_{q,r}) \ : \ \exists c_{q,r} \geq 1 \ , \ \exists \rho_{q,r} \in]1, 3/2[\ , \ \forall m \geq 0 \ , \ \left\| T^{q-r} \wedge f^{m*}\omega^r \right\|_{\rho_{q,r}} \leq c_{q,r}\, d^{mr}.$$

The lemma then follows by taking $S = f^{m*}\omega^l$ and $s = l$, and by using (14) and $(H_{l,l})$.

Let us establish $(H_q) :=$ "$(H_{q,r})$ holds for any $0 \leq r \leq q$" by induction on q. Observe that $(H_{q,0})$ obviously holds for any $1 \leq q \leq l$. Hence it suffices to verify $(H_{1,1})$ to end the proof of (H_1). Let $1 < \rho_{1,1} < \tau_{1,1} < \rho_{1,0} < 2$ and χ be a cut-off function with support in $\mathbb{D}^l(\tau_{1,1})$ such that $\chi \equiv 1$ on $\mathbb{D}^l(\rho_{1,1})$. We deduce from (13):

$$\| f^{m*}\omega \|_{\rho_{1,1}} = d^m \| T \|_{\rho_{1,1}} + \int_{\mathbb{D}^l(\rho_{1,1})} dd^c(\varphi \circ f^m \circ \eta) \wedge \omega_0^{l-1} =: d^m \ \| T \|_{\rho_{1,1}} + A_m.$$

On one hand $\| T \|_{\rho_{1,1}} \leq c_{1,0}$ from $(H_{0,0})$. On the other hand Stokes' theorem implies up to some multiplicative constant:

$$A_m \leq \int_{\mathbb{D}^l(2)} \chi \cdot dd^c(\varphi \circ f^m \circ \eta) \wedge \omega_0^{l-1} = \int_{\mathbb{D}^l(2)} \varphi \circ f^m \circ \eta \cdot dd^c\chi \wedge \omega_0^{l-1} \leq \| \varphi \|_\infty \| \chi \|_{C^2}.$$

Hence there exists $c_{1,1} \geq 1$ such that $\| f^{m*}\omega \|_{\rho_{1,1}} \leq c_{1,1} d^m$, which proves (H_1).

Assume now that (H_q) holds for $1 \leq q \leq l-1$, and let us prove (H_{q+1}). For that purpose, we show $(H_{q+1,r})$ by induction on r. Given $0 \leq r \leq q$, we shall deduce $(H_{q+1,r+1})$ from $(H_{q,r})$ and $(H_{q+1,r})$. Let us set $1 < \rho_{q+1,r+1} < \tau_{q+1,r+1} < \min\{\rho_{q,r}, \rho_{q+1,r}\}$ and let

χ be a cut-off function with support in $\mathbb{D}^l(\tau_{q+1,r+1})$ such that $\chi \equiv 1$ on $\mathbb{D}^l(\rho_{q+1,r+1})$. We obtain using (13):

$$T^{q+1-(r+1)} \wedge f^{m*}\omega^{r+1} = T^{q-r} \wedge f^{m*}\omega^r \wedge \big(d^m T + dd^c(\varphi \circ f^m)\big) = d^m\, S_1 + S_2, \quad (15)$$

where $S_1 := T^{q+1-r} \wedge f^{m*}\omega^r$ and $S_2 := T^{q-r} \wedge f^{m*}\omega^r \wedge dd^c(\varphi \circ f^m)$. Now $(H_{q+1,r})$ and $(H_{q,r})$ respectively imply (use Stokes' theorem as before for the second line):

$$d^m \parallel S_1 \parallel_{\rho_{q+1,r+1}} \leq d^m \parallel S_1 \parallel_{\rho_{q+1,r}} \leq c_{q+1,r}\, d^{m(r+1)},$$

$$\parallel S_2 \parallel_{\rho_{q+1,r+1}} \leq \parallel \varphi \parallel_\infty \parallel \chi \parallel_{C^2} \left\| T^{q-r} \wedge f^{m*}\omega^r \right\|_{\rho_{q,r}} \leq \parallel \varphi \parallel_\infty \parallel \chi \parallel_{C^2} c_{q,r}\, d^{mr}.$$

Using (15) we get $\left\| T^{q+1-(r+1)} \wedge f^{m*}\omega^{r+1} \right\|_{\rho_{q+1,r+1}} \leq c_{q+1,r+1}\, d^{m(r+1)}$ for some $c_{q+1,r+1} \geq 1$. That completes the proof of the growth lemma. □

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