

Bernoulli coding map and almost sure invariance principle for endomorphisms of \mathbb{P}^k

C. Dupont

November 26, 2008

Abstract

Let f be an holomorphic endomorphism of \mathbb{P}^k and μ be its measure of maximal entropy. We prove an Almost Sure Invariance Principle for the systems (\mathbb{P}^k, f, μ) . Our class \mathcal{U} of observables includes the Hölder functions and unbounded ones which present analytic singularities. The proof is based on a geometric construction of a Bernoulli coding map $\omega : (\Sigma, s, \nu) \rightarrow (\mathbb{P}^k, f, \mu)$. We obtain the invariance principle for an observable ψ on (\mathbb{P}^k, f, μ) by applying Philipp-Stout's theorem for $\chi = \psi \circ \omega$ on (Σ, s, ν) . The invariance principle implies the Central Limit Theorem as well as several statistical properties for the class \mathcal{U} . As an application, we give a *direct* proof of the absolute continuity of the measure μ when it satisfies Pesin's formula. This approach relies on the Central Limit Theorem for the unbounded observable $\log \text{Jac } f \in \mathcal{U}$.

Key Words: holomorphic dynamics, Bernoulli coding map, almost sure invariance principle.
MSC: 37F10, 37C40, 60F17.

1 Introduction

Let $f : \mathbb{P}^k \rightarrow \mathbb{P}^k$ be an holomorphic endomorphism of algebraic degree $d \geq 2$. Its equilibrium measure μ is the limit of the probability measures $d_t^{-n} (f^n)^* \eta^k$, where $d_t := d^k$ is the topological degree of f and η^k is the standard volume form on \mathbb{P}^k . We refer to the survey article of Sibony [Sib] for an introduction to the dynamical systems (\mathbb{P}^k, f, μ) . Fornæss-Sibony proved that μ is mixing [FS] and Briend-Duval that μ is the unique measure of maximal entropy [BrDu2].

Przytycki-Urbański-Zdunik [PUZ] introduced coding techniques for (\mathbb{P}^1, f, μ) . This allowed them to prove the Almost Sure Invariance Principle (ASIP) for Hölder and singular observables, like $\log |f'|$. In the present article, we extend the coding techniques to (\mathbb{P}^k, f, μ) and obtain the ASIP for observables which allow analytic singularities. As an application, we obtain a direct proof of the absolute continuity of μ when it satisfies Pesin's formula. We review our results in subsections 1.1 - 1.4, subsection 1.5 is devoted to related results.

1.1 Bernoulli coding maps

Let us endow $\Sigma := \{1, \dots, d_t\}^{\mathbb{N}}$ with the natural product measure $\nu := \otimes_{n=0}^{\infty} \bar{\nu}$, where $\bar{\nu}$ is equidistributed on $\{1, \dots, d_t\}$. We denote by $\tilde{\alpha}$ the elements of Σ and by s the left shift acting on Σ . Let \mathcal{J} be the support of μ . The following theorem yields coding maps $\omega : \Sigma \rightarrow \mathcal{J}$ up to zero measure sets. The set $\mathcal{S} \subset \mathbb{P}^k$ will be defined in section 4, it has zero Lebesgue measure.

Theorem A: *Let $z \in \mathbb{P}^k \setminus \mathcal{S}$. There exist an s -invariant set $\Sigma' \subset \Sigma$ of full ν -measure and an f -invariant set $\mathcal{J}' \subset \mathcal{J}$ of full μ -measure satisfying the following properties. For any $\tilde{\alpha} \in \Sigma'$, the point $\omega(\tilde{\alpha}) := \lim_{n \rightarrow \infty} z_n(\tilde{\alpha}) \in \mathcal{J}'$ is well defined. We have $\omega_*\nu = \mu$ and the following diagram commutes:*

$$\begin{array}{ccc} \Sigma' & \xrightarrow{s} & \Sigma' \\ \omega \downarrow & & \downarrow \omega \\ \mathcal{J}' & \xrightarrow{f} & \mathcal{J}' \end{array}$$

Moreover there exist $\theta, \epsilon > 0$, $n_z \geq 1$ and $\tilde{n} : \Sigma' \rightarrow \mathbb{N}$ larger than n_z such that:

1. $d(z_n(\tilde{\alpha}), \omega(\tilde{\alpha})) \leq \tilde{c}_\epsilon d^{-\epsilon n}$ for every $\tilde{\alpha} \in \Sigma'$ and $n \geq \tilde{n}(\tilde{\alpha})$,
2. $\nu(\{\tilde{n} \leq q\}) \geq 1 - c_\theta d^{-\theta q}$ for every $q \geq n_z$.

We note that Σ' , \mathcal{J}' and ω depend on $z \in \mathbb{P}^k \setminus \mathcal{S}$, but $\omega_*\nu = \mu$ holds true for any such z . Observe also that ω is not necessarily injective. The proof of theorem A (see section 4) is based on the construction of a geometric coding tree, following the approach of Przytycki-Urbański-Zdunik [PUZ] for (\mathbb{P}^1, f, μ) . The point z is the *root* of the tree, and the set $\{z_n(\tilde{\alpha}), \tilde{\alpha} \in \Sigma\}$ is a suitable enumeration of the d_t^{n+1} points of $f^{-(n+1)}(z)$, these are *vertices* of the tree. The convergence of $(z_n(\tilde{\alpha}))_n$ for a generic $\tilde{\alpha} \in \Sigma$ is obtained by constructing d_t good paths joining z to $w \in f^{-1}(z)$, whose inverse images decrease exponentially. In the context of (\mathbb{P}^1, f, μ) , that property was obtained in [PUZ] by using Koebe distortion theorem. The difficulty in higher dimensions is to substitute this argument. We establish for that purpose a quantified version of a theorem of Briend-Duval (see section 3).

1.2 The class \mathcal{U} and approximation by cylinders

Definition : *An observable $\psi : \mathbb{P}^k \rightarrow \mathbb{R} \cup \{-\infty\}$ belongs to the class \mathcal{U} if:*

- e^ψ is h -Hölder for some $h > 0$,
- $\mathcal{N}_\psi := \{\psi = -\infty\}$ is a (possibly empty) proper algebraic set of \mathbb{P}^k ,
- $\psi \geq \log d(\cdot, \mathcal{N}_\psi)^\rho$ for some $\rho > 0$.

For instance, the Hölder functions are in \mathcal{U} , as well as the unbounded function $\log \text{Jac } f$. We will show that $\mathcal{U} \subset L^p(\mu)$ for any $1 \leq p < +\infty$ (see subsection 2.2).

Theorem B: *Let $\psi \in \mathcal{U}$ be a μ -centered observable and ω be a coding map provided by theorem A. Let $\chi := \psi \circ \omega$ and $1 \leq p < +\infty$. We denote by $\mathbb{E}(\chi|\mathcal{C}_n)$ the conditional expectation of χ with respect to the $(n+1)$ -cylinders.*

1. *there exist $\hat{c}_p, \lambda_p > 0$ such that $\|\chi - \mathbb{E}(\chi|\mathcal{C}_n)\|_p \leq \hat{c}_p e^{-n\lambda_p}$ for every $n \geq 0$.*
2. *$R_j(\chi) := \int_{\Sigma} \chi \cdot \chi \circ s^j d\nu$ satisfies $|R_j(\chi)| \leq 2 \|\chi\|_2 \hat{c}_2 e^{-(j-1)\lambda_2}$ for every $j \geq 1$.*

The proof occupies section 5, it is based on the regularity properties of ω (namely the points 1, 2 of theorem A) and on the fact that μ is a Monge-Ampère mass with Hölder potentials. Theorem B allows us to prove theorem C below.

1.3 Almost Sure Invariance Principle

Let $\psi \in L^2(\mu)$ be a μ -centered observable and $S_n(\psi) := \sum_{j=0}^{n-1} \psi \circ f^j$. We say that ψ satisfies the Almost Sure Invariance Principle (ASIP) if there exist, on an extended probability space, a sequence of random variables $(\mathcal{S}_n)_{n \geq 0}$ together with a Brownian motion \mathcal{W} such that for some $\gamma > 0$:

- $\mathcal{S}_n = \mathcal{W}(n) + o(n^{1/2-\gamma})$ almost everywhere,
- $(\mathcal{S}_0(\psi), \dots, \mathcal{S}_n(\psi))$ and $(\mathcal{S}_0, \dots, \mathcal{S}_n)$ have the same distribution for any $n \geq 0$.

We shall denote σ -ASIP to specify the variance of Brownian motion.

Theorem C: *For every μ -centered observable $\psi \in \mathcal{U}$, we have:*

1. *$\sigma := \lim_{n \rightarrow \infty} \frac{1}{\sqrt{n}} \|S_n(\psi)\|_2$ exists, and $\sigma^2 = \int_{\mathbb{P}^k} \psi^2 d\mu + 2 \sum_{j \geq 1} \int_{\mathbb{P}^k} \psi \cdot \psi \circ f^j d\mu$.*
2. *If $\sigma = 0$, then $\psi = u - u \circ f$ holds μ -a.e. for some $u \in L^2(\mu)$.*
3. *If $\sigma > 0$, then ψ satisfies the σ -ASIP.*

The ASIP implies classical limit theorems related to Brownian motion: the Central Limit Theorem (CLT), the Law of Iterated Logarithm, Kolmogorov integral tests (see [De], [PS]). The ASIP also implies the almost sure version of the CLT, meaning that $\frac{1}{\log n} \sum_{k=1}^n \frac{1}{k} \delta_{\frac{1}{\sqrt{k}} S_k(\psi)(x)}$ converges μ -a.e. to the normal law $\mathcal{N}(0, \sigma^2)$ (see [LP], [CG]).

Let us outline the proof of theorem C (see section 6). Let $\omega : \Sigma \rightarrow \mathbb{P}^k$ be a coding map provided by theorem A and $\psi \in \mathcal{U}$. Since ω satisfies $f \circ \omega = \omega \circ s$ and $\omega_* \nu = \mu$, we are reduced to prove the assertions for $\chi = \psi \circ \omega$ on (Σ, s, ν) . The points 1 and 2 follow from theorem B(2) and classical arguments. The point 3 is a consequence of theorem B(1) and Philipp-Stout's theorem ([PS], section 7). That result relies on an approximation of the partial sums of $(\chi \circ s^j)_{j \geq 0}$ by a sequence of martingale differences defined with respect to the increasing filtration $(\mathcal{C}_n)_{n \geq 0}$.

1.4 An application to smooth ergodic theory

Let $\chi_1 \leq \dots \leq \chi_k$ be the Lyapunov exponents of μ . Briend-Duval [BrDu1] proved that they are larger than or equal to $\log d^{1/2}$. Since μ has entropy $\log d^k$, Pesin's formula $h(\mu) = 2(\chi_1 + \dots + \chi_k)$ holds if and only if these exponents are minimal. We proved in a previous article that μ is then absolutely continuous with respect to Lebesgue measure [Du]. We there followed the classical approach of Sinai-Pesin-Ledrappier, based on the construction of a suitable invariant partition which is dilated and realizes entropy (see [P1], [Le]). We propose in section 7 a new proof, based on the CLT for the unbounded μ -centered observable $J := \log \text{Jac } f - 2(\chi_1 + \dots + \chi_k) \in \mathcal{U}$. We obtain the following result, where $\sigma_J := \lim_{n \rightarrow \infty} \frac{1}{\sqrt{n}} \|S_n(J)\|_2$.

Theorem D: *If the Lyapunov exponents are minimal equal to $\log d^{1/2}$, then $\sigma_J = 0$, and μ is absolutely continuous with respect to Lebesgue measure.*

A crucial fact for the proof is that for any holomorphic endomorphism of \mathbb{P}^k and any μ -generic point $x \in \mathbb{P}^k$, the minimal dilation rate of f^n at x (i.e. $\|(d_x f^n)^{-1}\|^{-1}$) is bounded below by $d^{n/2}$ up to the multiplicative factor $1/n$. In other words, the usual $e^{-n\epsilon}$ -correction, due to the non-uniform hyperbolicity of (\mathbb{P}^k, f, μ) , can be replaced here by $1/n$. This was proved by Berteloot-Dupont [BeDu], using a pluripotential result of Briend-Duval [BrDu1] and the fact that μ is a Monge-Ampère mass. In particular, the product of the dilation rates satisfies $\text{Jac } f^n(x) \geq \|(d_x f^n)^{-1}\|^{-2k} \geq (d^{n/2}/n)^{2k} = d^{kn}/n^{2k}$. Now if we assume $\sigma_J > 0$, then the function $\log \text{Jac } f^n$ would present non trivial oscillations around its mean value $\log d^{kn}$, due to the CLT. More precisely, it would imply $\log \text{Jac } f^n \leq \log d^{kn} - \sigma_J \sqrt{n}$ on a subset of μ -measure $\simeq \int_{-\infty}^{-1} e^{-u^2/2}$. That contradicts the preceding estimate, hence $\sigma_J = 0$. We deduce the absolute continuity of μ from the cocycle property $J = u - u \circ f$ μ -a.e. and a linearization property of the dynamics along typical negative orbits [BeDu].

1.5 Related results

The systems (\mathbb{P}^k, f, μ) and (Σ, s, ν) are actually conjugated by a bimeasurable map up to zero measure subsets, that property was proved by Briend [Br]. However, the regularity of the conjugacy seems difficult to handle. Let us also mention that finite-to-one coding maps $(\mathbb{P}^k, f, \mu) \rightarrow (\Sigma, s, \nu)$ were constructed by Buzzi [Bu] by means of suitable partitions of \mathbb{P}^k .

The ASIP has been proved for many dynamical systems: for piecewise monotonic maps by Hofbauer-Keller [HK], for Anosov maps by Denker-Philipp [DP] and for partially and non-uniformly hyperbolic systems by Dolgopyat [Do] and Melbourne-Nicol [MN]. We refer to the survey articles of Chernov [C] and Denker [De] for limit theorems and statistical properties concerning dynamical systems.

The ASIP implies the CLT. Nevertheless, the latter can be directly proved via coding techniques and Ibragimov's theorem [I]. That method was employed by Sinai

[Sin] and Ratner [R] for the geodesic flow in negative curvature, and by Bowen [Bo] for Anosov maps. In the present article, Ibragimov's condition is fulfilled by theorem B.

The Gordin's theorem provides another method for proving the CLT (see [G], [Li]). It relies on an approximation of $(\psi \circ f^j)_{j \geq 0}$ by a sequence of reverse martingale differences. In our context, this can be done if $\sum_{n \geq 0} \|\Lambda^n \psi\|_2$ (denoted (\star)) converges, where Λ denotes the Ruelle-Perron-Frobenius operator (we have $\Lambda^n \psi(z) = \frac{1}{d^n} \sum_{y \in f^{-n}(z)} \psi(y)$ for every $z \in \mathbb{P}^k$). Let us note that the reverse martingale mentioned is defined with respect to the decreasing filtration $(f^{-n}\mathcal{B})_{n \geq 0}$, where \mathcal{B} is the Borel σ -algebra of \mathbb{P}^k .

The exponential decay of correlations ensures the convergence of (\star) . This was proved in the context of (\mathbb{P}^k, f, μ) by Fornæss-Sibony [FS] for C^2 observables and by Dinh-Sibony for Hölder observables [DS2]. Dinh-Nguyen-Sibony have recently extended that property for differences of quasi-plurisubharmonic functions (the so-called *dsh* functions) [DNS2]. The proof relies on exponential estimates for plurisubharmonic functions with respect to μ . They also obtained in that article a Large Deviations Theorem for bounded dsh and Hölder observables. In [DNS1], Dinh-Nguyen-Sibony proved the local CLT for (\mathbb{P}^1, f, μ) by using the theory of perturbed operators.

Denker-Przytycki-Urbański [DPU] employed a geometric method to prove the convergence of (\star) for (\mathbb{P}^1, f, μ) and Hölder observables. The idea was to compare $\Lambda^n \psi(z)$ to $\Lambda^n \psi(z')$ by using the contraction of most of the inverse branches of f^n . The cornerstone is a precise analysis of the dynamics near the critical points in the support of μ . Cantat-Leborgne [CL] extended this approach to (\mathbb{P}^k, f, μ) . A crucial ingredient was a polynomial estimate for the μ -measure of postcritical neighbourhoods (lemma 5.7 of [CL]). The original proof of that lemma contains a gap, the authors have recently proposed another one. Cantat-Leborgne also established in [CL] a quantified version of the Briend-Duval theorem. Our version is similar, but we shall give a different proof.

The systems (\mathbb{P}^k, f, μ) whose measure μ is absolutely continuous with respect to Lebesgue measure were characterized by Berteloot, Dupont and Loeb [BeDu], [BL]. In that case, f is semi-conjugated to an affine dilation on a complex torus, these maps are the so-called *Lattès examples*. We note that theorem D characterizes these maps by the minimality of the Lyapunov exponents. Another characterization of Lattès examples involves the *Hausdorff dimension* of μ , defined as the infimum of the Hausdorff dimension of Borel sets with full μ -measure (see Pesin's book [P2]): Dinh-Dupont [DD] proved that $\dim_{\mathcal{H}}(\mu) = 2k$ if and only if the exponents are minimal. In the context of (\mathbb{P}^1, f, μ) , Mañé [Mañ] proved that $\log d = \dim_{\mathcal{H}}(\mu) \cdot \chi$, where χ denotes the Lyapunov exponent of μ . In particular, the function $L := \log d - \dim_{\mathcal{H}}(\mu) \cdot \log |f'|$ is a μ -centered observable. Zdunik [Z] proved that $\sigma_L = 0$ if and only if f is a Lattès example, a Tchebychev polynomial or a power $z^{\pm d}$. The proof relies on the classification of critically finite fractions with parabolic Thurston's orbifold.

2 Generalities

2.1 The holomorphic systems (\mathbb{P}^k, f, μ)

We introduce in this section the systems (\mathbb{P}^k, f, μ) . We refer to the articles [BrDu1], [BrDu2], [FS] and [Sib] for definitions and properties. Here \mathbb{P}^k denotes the complex projective space of dimension k . We denote by η the Fubini-Study form on \mathbb{P}^k . This is a $(1, 1)$ -form defined in homogeneous coordinates by $\frac{i}{2\pi} \partial \bar{\partial} \log \|z\|^2$. It induces the standard metric on \mathbb{P}^k , the volume of \mathbb{P}^k with respect to this metric is equal to 1. The form η induces on every complex line $L \subset \mathbb{P}^k$ the spherical metric with area 1. Let f be an holomorphic endomorphism of \mathbb{P}^k with algebraic degree $d \geq 2$. It is defined in homogeneous coordinates by $[P_0 : \dots : P_k]$ where the P_i are homogeneous polynomials of degree d (without common zero except the origin). The topological degree of f is $d_t := d^k$. An inverse branch of f^n on $U \subset \mathbb{P}^k$ is an injective holomorphic map g_n satisfying $f^n \circ g_n = \text{Id}_U$. We let $\text{Perf} := \cup_{n \geq 1} \{x \in \mathbb{P}^k, f^n(x) = x\}$, this set is at most countable. Let \mathcal{C} be the critical set of f , $\mathcal{V} := \cup_{i=0}^{\infty} f^i(\mathcal{C})$ and $\mathcal{V}_n := \cup_{i=1}^n f^i(\mathcal{C})$. The degree of \mathcal{V}_n , denoted τ_n , is equal to $(d + \dots + d^n) \deg \mathcal{C}$ counted with multiplicity.

The equilibrium measure μ is defined as the limit of $\mu_{n,z} := \frac{1}{d_t^n} \sum_{f^n(y)=z} \delta_y$, where δ_y denotes the Dirac mass at y . In that definition, z has to be taken outside a totally invariant algebraic set $\mathcal{E} \subset \mathcal{V}$, the so-called exceptional set of f . We denote by \mathcal{J} the support of μ . The measure μ is mixing and satisfies $\mu(f(B)) = d_t \mu(B)$ whenever f is injective on B . It is the unique measure of maximal entropy (equal to $\log d_t$). The Lyapunov exponents $\chi_1 \leq \dots \leq \chi_k$ of μ are larger than or equal to $\log d^{1/2}$. They satisfy the classical formula $\int_{\mathbb{P}^k} \log \text{Jac } f \, d\mu = 2(\chi_1 + \dots + \chi_k)$, where $\text{Jac } f$ is the non-negative \mathcal{C}^∞ function on \mathbb{P}^k satisfying $f^* \eta^k = \text{Jac } f \cdot \eta^k$. The latter is the real jacobian of f , it vanishes on the critical set \mathcal{C} of f .

The measure μ can also be defined *via* pluripotential theory: we have $\mu = T^k$, where T is the Green current of f . The latter is a closed positive $(1, 1)$ current on \mathbb{P}^k with Hölder potentials. In particular, for any algebraic subset $A \subset \mathbb{P}^k$, there exist $c, \gamma > 0$ such that the r -neighbourhood of A satisfies $\mu(A[r]) \leq c r^\gamma$ for any $r > 0$ (see [DS4], Prop. 2.3.7). For any $\delta > 0$ and $\tilde{c} > 0$, we set $c_\delta := (1 - d^{-\delta})^{-1}$ and $\tilde{c}_\delta := \tilde{c}(1 - d^{-\delta})^{-1}$. In the sequel, $c > 0$ is a constant independent of n , it may differ from a line to another.

2.2 The class \mathcal{U}

Let us recall the definition of the class \mathcal{U} (see subsection 1.2).

Definition 2.1 *Let \mathcal{U} be the set of functions $\psi : \mathbb{P}^k \rightarrow \mathbb{R} \cup \{-\infty\}$ satisfying:*

- e^ψ is h -Hölder on \mathbb{P}^k for some $h > 0$,
- $\mathcal{N}_\psi := \{\psi = -\infty\}$ is a (possibly empty) proper algebraic set of \mathbb{P}^k ,
- $\psi \geq \log d(\cdot, \mathcal{N}_\psi)^\rho$ on \mathbb{P}^k for some $\rho > 0$.

The Hölder functions belong to \mathcal{U} . Examples of unbounded observables are:

- the functions $\psi = \log |Q| - q \log \|\cdot\|$, where Q is a q -homogeneous polynomial on \mathbb{C}^{k+1} . Here the algebraic subset \mathcal{N}_ψ is the zero set of Q .
- the functions $\psi = \log \|\Lambda^j d_x f\|$ ($1 \leq j \leq k$), where $\Lambda^j d_x f$ is the j -exterior power of the differential $d_x f$. In particular, $\log \text{Jac } f \in \mathcal{U}$ (take $j = k$).

The conditions of definition 2.1 are easy to verify for these functions, the last one is a consequence of Lojasiewicz's inequality (see [Lo], §4.7). We prove below that $\psi \in L^p(\mu)$ for any $\psi \in \mathcal{U}$ and $1 \leq p < +\infty$. Actually, we establish an estimate for $\int_{\mathcal{N}_\psi[r]} |\psi|^p$, useful to prove theorem B. We recall that $\mu(\mathcal{N}_\psi[r]) \leq c r^\gamma$ for some $c, \gamma > 0$ (see subsection 2.1).

Proposition 2.2 *Let $\psi \in \mathcal{U}$ and $1 \leq p < +\infty$. There exists $\kappa > 0$ such that:*

$$\forall 0 < r < 1/2 \quad , \quad \int_{\mathcal{N}_\psi[r]} |\psi|^p d\mu \leq \kappa r^{\gamma/2}.$$

In particular $\psi \in L^p(\mu)$.

PROOF: Let $\psi \in \mathcal{U}$ and $\mathcal{N} := \mathcal{N}_\psi$. We may assume that $0 \leq e^\psi \leq 1$ by adding some constant to ψ . Let $r < 1/2$ and $\mathcal{Q}_j := \mathcal{N}[r/2^j] \setminus \mathcal{N}[r/2^{j+1}]$. Since $e^\psi \geq (r/2^{j+1})^\rho$ on \mathcal{Q}_j , we obtain:

$$\int_{\mathcal{N}[r]} |\psi|^p d\mu = \sum_{j \geq 0} \int_{\mathcal{Q}_j} |\log e^\psi|^p d\mu \leq \sum_{j \geq 0} \left| \rho \log \left(\frac{r}{2^{j+1}} \right) \right|^p \cdot \mu(\mathcal{Q}_j).$$

The inequalities $\mu(\mathcal{Q}_j) \leq c(r/2^j)^\gamma$ and $|\log \frac{r}{2^{j+1}}| = (j+1) \log 2 + \log \frac{1}{r} \leq (j+2) \log \frac{1}{r}$ yield:

$$\int_{\mathcal{N}[r]} |\psi|^p d\mu \leq \left[c \rho^p \sum_{j \geq 0} \frac{(j+2)^p}{2^{\gamma j}} \right] \left(\log \frac{1}{r} \right)^p r^\gamma = M_{\rho, \gamma} \cdot \left(\log \frac{1}{r} \right)^p r^{\gamma/2} \cdot r^{\gamma/2}.$$

The lemma follows with $\kappa := M_{\rho, \gamma} \cdot \sup_{0 < r < 1/2} \left(\log \frac{1}{r} \right)^p r^{\gamma/2}$. □

2.3 The Bernoulli space (Σ, s, ν)

We endow $\mathcal{A} := \{1, \dots, d_t\}$ with the equidistributed probability measure $\bar{\nu}$. We set $\Sigma := \mathcal{A}^{\mathbb{N}}$, $s : \Sigma \rightarrow \Sigma$ the left shift and $\nu := \otimes_{n=0}^{\infty} \bar{\nu}$. We denote by $\tilde{\alpha} := (\alpha_n)_{n \geq 0}$ the elements of Σ , by \mathcal{C}_n the set of cylinders of length $n+1$, and by $\pi_n : \Sigma \rightarrow \mathcal{A}^{n+1}$ the projection $\pi_n(\tilde{\alpha}) := (\alpha_0, \dots, \alpha_n)$. For any $\tilde{\alpha} \in \Sigma$, we set $\mathcal{C}_n(\tilde{\alpha}) := \pi_n^{-1}(\alpha_0, \dots, \alpha_n)$. We denote by $\mathbb{E}(\chi | \mathcal{C}_n)$ the conditional expectation of $\chi \in L^2(\nu)$ with respect to \mathcal{C}_n . If $\mathcal{L} = \{A_1, \dots, A_p\} \subset \mathcal{C}_n$, we set $\mathcal{L}^* := \cup_{1 \leq j \leq p} A_j$.

2.4 Almost Sure Invariance Principle

Let (X, g, m) be either (Σ, s, ν) or (\mathbb{P}^k, f, μ) . For any observable $\varphi \in L^2(m)$, we set $S_n(\varphi) := \sum_{j=0}^{n-1} \varphi \circ g^j$ and $R_j(\varphi) := \int_X \varphi \cdot \varphi \circ g^j dm$. We say that φ is m -centered if $\int_X \varphi dm = 0$ and that φ is a cocycle if $\varphi = u - u \circ g$ m -a.e. for some $u \in L^2(m)$.

An observable φ on (X, g, m) satisfies the Almost Sure Invariance Principle (ASIP) if there exist on a probability space (\tilde{X}, \tilde{m}) a sequence of random variables $(\mathcal{S}_n)_{n \geq 0}$ and a Brownian motion \mathcal{W} such that:

- $\mathcal{S}_n = \mathcal{W}(n) + o(n^{1/2-\gamma})$ \tilde{m} -a.e. for some $\gamma > 0$,
- $(\mathcal{S}_0(\psi), \dots, \mathcal{S}_n(\psi))$ and $(\mathcal{S}_0, \dots, \mathcal{S}_n)$ have the same distribution for any $n \geq 0$.

We denote σ -ASIP to specify the variance of Brownian motion. The σ -ASIP implies the σ -Central Limit Theorem (σ -CLT), meaning that:

$$\forall t \in \mathbb{R}, \lim_{n \rightarrow \infty} m \left(\frac{S_n(\varphi)}{\sigma \sqrt{n}} \leq t \right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^t e^{-u^2/2} du.$$

Remark 2.3 Suppose that $\omega : \Sigma \rightarrow \mathbb{P}^k$ is a coding map provided by theorem A. Since $\omega_* \nu = \mu$ and $f \circ \omega = \omega \circ s$, a μ -centered observable $\psi \in L^2(\mu)$ satisfies the σ -ASIP if and only if the ν -centered observable $\chi := \psi \circ \omega \in L^2(\nu)$ satisfies the σ -ASIP.

We shall use Philipp-Stout's theorem ([PS], Section 7) to prove the ASIP for $\chi := \psi \circ \omega$ on the Bernoulli space (Σ, s, ν) . The version below comes from the original one by using the s -invariance of ν and the independence of the random process $(\xi_n)_{n \geq 0}$ defined by $\xi_n(\tilde{\alpha}) = \alpha_n$.

Theorem (Philipp-Stout) Let χ be a ν -centered observable on Σ satisfying:

1. $\chi \in L^{2+\delta}(\nu)$ for some $\delta > 0$,
2. $\|\chi - \mathbb{E}(\chi | \mathcal{C}_n)\|_{2+\delta} \leq c \beta^n$ for some $c > 0$ and $\beta < 1$.

Then the sequence $\frac{1}{\sqrt{n}} \|S_n(\chi)\|_2$ has a limit σ . If $\sigma > 0$, then χ satisfies the σ -ASIP.

Let us compare that result with Ibragimov's theorem (see [I], Theorem 2.1), which only requires moments of order 2 and a summability condition:

Theorem (Ibragimov) Let χ be a ν -centered observable on Σ satisfying:

$$\sum_{n \geq 0} \|\chi - \mathbb{E}(\chi | \mathcal{C}_n)\|_2 < \infty.$$

Then the sequence $\frac{1}{\sqrt{n}} \|S_n(\chi)\|_2$ has a limit σ . If $\sigma > 0$, then χ satisfies the σ -CLT.

3 A quantified version of Briend-Duval theorem

This section is devoted to the proof of theorem 3.2 (see subsection 3.2). That result will be crucial to establish theorem A.

3.1 Briend-Duval theorem

We recall that $\mathcal{V}_l = \cup_{i=1}^l f^i(\mathcal{C})$, $\mathcal{V} = \cup_{i=0}^\infty f^i(\mathcal{C})$ and that $d_t = d^k$ is the topological degree of f (see subsection 2.1). We set $\tau_* := 2 \deg \mathcal{V}_1 / (1 - 1/d)$.

Theorem (Briend-Duval [BrDu2]) *Let $\eta > 0$ and $l \geq 1$ be such that $\tau_*/d^l < \eta$. Let L be a complex line in \mathbb{P}^k not contained in \mathcal{V} , and $\Delta \Subset \tilde{\Delta}$ be topological discs in $L \setminus \mathcal{V}_l$. Then, for any $n \geq l$, there exist $(1 - \eta)d_t^n$ inverse branches g_n on Δ satisfying:*

$$\text{diam } g_n(\Delta) \leq \frac{\tilde{c} d^{-n/2}}{\eta^{1/2} \text{mod}(\tilde{\Delta} \setminus \Delta)^{1/2}},$$

where \tilde{c} is a universal constant, and $\text{mod}(\tilde{\Delta} \setminus \Delta)$ is the modulus of the annulus $\tilde{\Delta} \setminus \Delta$.

Let us recall the definition of the modulus (see Ahlfors book [A], chapters 1 and 2). Let Λ denote the family of curves joining the boundary components of $A := \tilde{\Delta} \setminus \Delta$. For any conformal metric ρ on A , we respectively denote by area_ρ and by l_ρ the area and the length with respect to ρ . We denote by $\text{conf}(A)$ the set of conformal metrics giving finite area to A . The modulus of the annulus A is then defined by:

$$\text{mod}(A) := \sup_{\rho \in \text{conf}(A)} \frac{l_\rho(\Lambda)^2}{\text{area}_\rho(A)},$$

where $l_\rho(\Lambda) := \inf_{\lambda \in \Lambda} l_\rho(\lambda)$.

3.2 Statement of the quantified version

We begin with some notations. Let $0 < \theta < 1$ and $\theta_n := [\theta n + \frac{\log \tau_*}{\log d}] + 1$. We introduce this integer in view of applying Briend-Duval theorem with $\eta = d^{-\theta n}$ and $l = \theta_n$ (indeed, $\tau_*/d^{\theta_n} < d^{-\theta n}$). Since the degree of $\mathcal{V}_{\theta_n} = \cup_{i=1}^{\theta_n} f^i(\mathcal{C})$ is at most $\tau_{\theta_n} = (d + \dots + d^{\theta_n}) \deg \mathcal{C}$, we have $\tau_{\theta_n} < d^{\theta n}$ up to a multiplicative constant.

We let $0 < \theta < \theta' < 1$ and consider $n_0 \geq 1$ satisfying:

$$\forall n \geq n_0 \quad , \quad \theta_n < \theta' n \quad \text{and} \quad \tau_{\theta_n} < d^{\theta' n}. \quad (1)$$

Let us recall that $\mathcal{V}_{\theta_n}[\delta]$ is the δ -neighbourhood of \mathcal{V}_{θ_n} in \mathbb{P}^k . We fix $\theta'/2 < \zeta < 1$ and define $\mathcal{D} := \limsup_{n \geq n_0} \mathcal{V}_{\theta_n}[d^{-\zeta n}]$.

Proposition 3.1 *The set \mathcal{D} satisfies $\text{Vol}(\mathcal{D}) = 0$.*

The proof is postponed to subsection 3.5. We now state the quantified version. The constant \tilde{c} has been introduced in the statement of Briend-Duval's theorem, and we denote by L the complex line containing z and w .

Theorem 3.2 *There exists $\epsilon > 0$ such that for every distinct points $(z, w) \notin \mathcal{D} \cup \mathcal{V}$, there exist an injective smooth path $\gamma : [0, 1] \rightarrow L \setminus \mathcal{V}$ joining z and w , a decreasing family of topological discs $(\Delta_n)_n \subset L$ and an integer $n_{z,w}$ such that for any $n \geq n_{z,w}$:*

1. $\gamma[0, 1] \subset \Delta_n \subset L \setminus \mathcal{V}_{\theta_n}$,
2. there exist $(1 - d^{-\theta_n})d_t^n$ inverse branches of f^n on Δ_n ,
3. these branches satisfy $\text{diam } g_n(\Delta_n) \leq \tilde{c}d^{-\epsilon n}$.

We note that θ, ϵ and \tilde{c} do not depend on $(z, w) \in \mathbb{P}^k \setminus (\mathcal{D} \cup \mathcal{V})$.

3.3 Construction of good paths in the complex line $L \subset \mathbb{P}^k$

Let (z, w) be distinct points in $\mathbb{P}^k \setminus (\mathcal{D} \cup \mathcal{V})$. We identify the complex line L containing z and w with the 2-dimensional sphere. We recall that the Fubini-Study metric induces on L the standard spherical metric s with area 1. We assume with no loss of generality that z and w are the North and South pole of L . Let E be the equator of L . For any $y \in E$, we denote by M_y the meridian containing y , and by $M_y\{\delta\}$ the δ -neighbourhood of M_y in L for the spherical metric. The constants $0 < \theta < \theta' < 2\zeta$ have been defined in subsection 3.2. Now we let $0 < \zeta < \zeta' < \zeta'' < 1$ satisfying:

$$\theta' < \zeta'' - \zeta' \quad \text{and} \quad \theta + \zeta'' < 1. \quad (2)$$

We may take for $(\theta, \theta', \zeta, \zeta', \zeta'')$ suitable multiples of a small $\theta > 0$. The second inequality of (2) will be used in next subsection. The integer n_0 has been defined in subsection 3.2.

Proposition 3.3 *Let (z, w) be distinct points in $\mathbb{P}^k \setminus (\mathcal{D} \cup \mathcal{V})$. With the above notations, there exists a subset $F \subset E$ of full Lebesgue measure satisfying the following properties. For any $y \in F$, there exists $n_{z,w}(y) \geq n_0$ such that:*

1. the meridian M_y does not intersect \mathcal{V} ,
2. the neighbourhood $M_y\{d^{-\zeta''n}\}$ does not intersect \mathcal{V}_{θ_n} for any $n \geq n_{z,w}(y)$.

Let us now prove proposition 3.3. We start with some notations. Let H^+ and H^- be the (open) North and South hemispheres of L , these sets induce a partition $L = H^+ \sqcup E \sqcup H^-$. We denote by Leb the Lebesgue measure on E and by p_1 (resp. p_2) the spherical projection from z (resp. w) to E . For any $y \in E$ and $\delta > 0$, let $\mathcal{I}(y, \delta)$ be the interval in E centered at y with length 2δ . We also denote by $D(c, \delta) \subset L$ the disc with center c and radius δ . We define $p_\kappa(c) := p_1(c)$ if $c \in H^+ \cup E$ and $p_\kappa(c) := p_2(c)$ if $c \in H^-$. The same convention holds for the projection of $D(c, \delta)$ to E : we use p_1 or

p_2 depending on $c \in H^+ \cup E$ or $c \in H^-$.

Let $\{c_i, 1 \leq i \leq l_{\theta_n}\} := \mathcal{V}_{\theta_n} \cap L$, where $l_{\theta_n} \leq \deg(\mathcal{V}_{\theta_n}) \leq \tau_{\theta_n}$. Since the Fubini-Study metric induces s on L , the set $\mathcal{L}_{\theta_n} := \cup_{i=1}^{l_{\theta_n}} D(c_i, d^{-\zeta^n})$ is a subset of $\mathcal{V}_{\theta_n}[d^{-\zeta^n}]$. We recall that $\mathcal{D} = \limsup_{n \geq n_0} \mathcal{V}_{\theta_n}[d^{-\zeta^n}]$ and that $(z, w) \notin \mathcal{D}$. Thus there exists $n_1 \geq n_0$ depending on (z, w) such that:

$$\forall n \geq n_1, (z, w) \notin \mathcal{V}_{\theta_n}[d^{-\zeta^n}]. \quad (3)$$

In particular $(z, w) \notin \mathcal{L}_{\theta_n}$. Since $\zeta < \zeta' < \zeta''$, we may increase n_1 so that $d^{-\zeta'n} + d^{-\zeta''n} < d^{-\zeta^n}$ for any $n \geq n_1$. We have therefore, for $\rho = z$ or w :

$$\forall 1 \leq i \leq l_{\theta_n}, \forall n \geq n_1, D(\rho, d^{-\zeta'n}) \cap D(c_i, d^{-\zeta''n}) = \emptyset.$$

This implies, with $e_i := p_\kappa(c_i) \in E$ and c a positive constant:

$$\forall 1 \leq i \leq l_{\theta_n}, p_\kappa\left(D(c_i, d^{-\zeta''n})\right) \subset \mathcal{I}_i := \mathcal{I}(e_i, c d^{-\zeta''n} \cdot d^{\zeta'n}). \quad (4)$$

Hence $\mathcal{I}(\theta_n) := \cup_{i=1}^{l_{\theta_n}} \mathcal{I}_i$ satisfies $\text{Leb } \mathcal{I}(\theta_n) \leq \tau_{\theta_n} \cdot c d^{-(\zeta''-\zeta')n} \leq c d^{(\theta'-(\zeta''-\zeta'))n}$. Since $\sum_n \text{Leb } \mathcal{I}(\theta_n) < \infty$ (see (2)), the Borel-Cantelli lemma yields, for every y in a full Lebesgue measure subset $F' \subset E$, an integer $n_{z,w}(y) \geq n_1$ satisfying:

$$y \notin \bigcup_{n \geq n_{z,w}(y)} \mathcal{I}(\theta_n). \quad (5)$$

Let us prove the point 2 of proposition 3.3 (the point 1 will be proved below, F is a subset of F'). Let $y \in F'$ and $\mathcal{I} := \mathcal{I}(y, d^{-(\zeta''-\zeta')n})$. Since the intervals \mathcal{I}_i defining $\mathcal{I}(\theta_n)$ are centered at $e_i = p_\kappa(c_i)$, the set $p_1^{-1}(\mathcal{I})$ does not intersect any point $c_i \in H^+ \cup E$. The same property holds for $p_2^{-1}(\mathcal{I})$ with the $c_i \in H^-$. This implies that $M_y\{d^{-\zeta''n}\}$ does not intersect $\mathcal{V}_{\theta_n} \cap L$ for any $n \geq n_{z,w}(y)$, and yields the point 2.

For the point 1, it suffices to verify that $p_\kappa(\mathcal{V} \cap L)$ has zero Lebesgue measure. Let $\mathcal{W} := \mathcal{V} \cap L$. Since $(z, w) \in L$ and $(z, w) \notin \mathcal{V} = \cup_{i=0}^{\infty} f^i(\mathcal{C})$, the complex line L is not an algebraic subset of the hypersurface $f^i(\mathcal{C})$ for any $i \geq 0$. In particular, $\mathcal{W}_i := f^i(\mathcal{C}) \cap L$ is finite for every $i \geq 0$. Hence $\mathcal{W} = \cup_{i \geq 0} \mathcal{W}_i$ satisfies $\text{Leb}(p_\kappa(\mathcal{W})) = 0$. We finally set $F := F' \setminus p_\kappa(\mathcal{W})$, that completes the proof of proposition 3.3.

3.4 Proof of theorem 3.2

We set $\epsilon := \frac{1}{2}(1 - (\theta + \zeta'')) > 0$ (see (2)). Let (z, w) be distinct points in $\mathbb{P}^k \setminus (\mathcal{D} \cup \mathcal{V})$ and consider some $y \in F$ provided by proposition 3.3: the meridian M_y does not intersect \mathcal{V} and its neighbourhood $M_y\{d^{-\zeta''n}\}$ in L does not intersect \mathcal{V}_{θ_n} for every $n \geq n_{z,w}(y)$.

We set $n_{z,w} := n_{z,w}(y)$ and denote $M := M_y$ for sake of simplicity. Let $\gamma : [0, 1] \rightarrow L$ be the natural parametrization of M . We define $\Delta_n := M\{d^{-\zeta''n}/2\}$ and $\tilde{\Delta}_n := M\{d^{-\zeta''n}\}$. Let us apply Briend-Duval's theorem with $\eta = d^{-\theta n}$, $l = \theta_n$ and $\Delta_n \Subset$

$\tilde{\Delta}_n \subset L \setminus \mathcal{V}_{\theta_n}$. Since $n > \theta'n \geq \theta_n = l$ and $\tau_*/d^{\theta_n} < d^{-\theta_n}$ (see (1)), there exist $(1 - d^{-\theta_n})d_t^n$ inverse branches on the disc Δ_n satisfying:

$$\text{diam } g_n(\Delta_n) \leq \tilde{c} d^{-n/2} \left(d^{-\theta_n} \bmod \left[\tilde{\Delta}_n \setminus \Delta_n \right] \right)^{-1/2}. \quad (6)$$

It remains to bound the modulus of $A_n := \tilde{\Delta}_n \setminus \Delta_n$. Let Λ_n be the set of curves joining the boundary components of A_n . We denote by area_s and by l_s the area and the length in L with respect to the spherical metric s . The following estimates hold up to multiplicative constants. We have $l_s(\lambda) \geq d^{-\zeta''n}$ for any $\lambda \in \Lambda_n$, hence $l_s(\Lambda_n) = \inf_{\lambda \in \Lambda_n} l_s(\lambda) \geq d^{-\zeta''n}$. The inequalities $\text{area}_s(A_n) \leq \text{area}_s(\tilde{\Delta}_n) \leq d^{-\zeta''n}$ then imply:

$$\text{mod}(A_n) = \sup_{\rho \in \text{conf } A_n} \frac{l_\rho(\Lambda)^2}{\text{area}_\rho(A_n)} \geq \frac{l_s(\Lambda_n)^2}{\text{area}_s(A_n)} \geq \frac{d^{-2\zeta''n}}{d^{-\zeta''n}} = d^{-\zeta''n}. \quad (7)$$

From (6), (7) and $\epsilon = \frac{1}{2}(1 - (\theta + \zeta''))$, we deduce that $\text{diam } g_n(\Delta_n) \leq \tilde{c} d^{-\epsilon n}$. That completes the proof of theorem 3.2.

3.5 Volume of neighbourhoods

This subsection is devoted to the proof of proposition 3.1: we want to show $\text{Vol}(\mathcal{D}) = 0$, where $\mathcal{D} = \bigcap_{n \geq n_0} \bigcup_{p \geq n} \mathcal{V}_{\theta_p}[d^{-\zeta p}]$. We recall that $\mathcal{V}_{\theta_p}[d^{-\zeta p}]$ is the $d^{-\zeta p}$ -neighbourhood of $\bigcup_{i=1}^{\theta_p} f^i(\mathcal{C})$ and that $\zeta > \theta'/2$. The proof is based on the following lemma (see [DS4], lemma 2.3.8).

Lemma 3.4 *Let $X \subset \mathbb{P}^k$ be an algebraic subvariety of dimension m and degree q . Then $\text{Vol } X[\delta] \leq q \delta^{2(k-m)}$ for any $\delta > 0$, up to a multiplicative constant independent of X .*

We deduce $\text{Vol}(\mathcal{D}) = 0$ as follows. We set $p \geq n \geq n_0$ and apply lemma 3.4 with $X = \mathcal{V}_{\theta_p}$ and $\delta = d^{-\zeta p}$ (here $k - m = 1$ and $q = \deg \mathcal{V}_{\theta_p} \leq \tau_{\theta_p}$). We obtain with $\tau_{\theta_p} \leq d^{\theta'p}$ (see (1)): $\text{Vol } \mathcal{V}_{\theta_p}[d^{-\zeta p}] \leq \tau_{\theta_p} (d^{-\zeta p})^2 \leq d^{-(2\zeta - \theta')p}$. Hence:

$$\forall n \geq n_0, \text{Vol}(\mathcal{D}) \leq \text{Vol} \bigcup_{p \geq n} \mathcal{V}_{\theta_p}[d^{-\zeta p}] \leq c_{2\zeta - \theta'} d^{-(2\zeta - \theta')n}.$$

This yields $\text{Vol}(\mathcal{D}) = 0$ when n tends to infinity.

PROOF OF LEMMA 3.4: The argument is based on Lelong's inequality. Let \mathcal{E} be a maximal δ -separated set in X for the ambient metric: this means that $d(a, b) \geq \delta$ for any pair of distinct elements of \mathcal{E} , and that for any $x \in X$ there exists $a \in \mathcal{E}$ satisfying $d(a, x) < \delta$. Since $X[\delta] \subset \bigcup_{a \in \mathcal{E}} B_a(2\delta)$, we get up to a multiplicative constant:

$$\text{Vol } X[\delta] \leq (2\delta)^{2k} \text{Card } \mathcal{E}. \quad (8)$$

We now give an upper bound for $\text{Card } \mathcal{E}$. Observe that $\text{Vol } X$ is equal to the degree of X , and that the balls $(B_a(\delta/2))_{a \in \mathcal{E}}$ are mutually disjoint. Thus:

$$q = \text{Vol } X \geq \sum_{a \in \mathcal{E}} \text{Vol} (X \cap B_a(\delta/2)).$$

Now Lelong's inequality asserts that $\text{Vol}(X \cap B_a(\delta/2)) \geq \delta^{2m}$ for any $a \in \mathcal{E}$, up to a multiplicative constant. Hence $\text{Card } \mathcal{E} \leq q \delta^{-2m}$, as desired. \square

4 Proof of theorem A

We set $\mathcal{S} := \mathcal{V} \cup \mathcal{D} \cup f(\mathcal{D}) \cup \text{Per}(f)$, where \mathcal{D} is defined in subsection 3.2. We have $\text{Vol}(\mathcal{S}) = 0$ since $\text{Vol}(\mathcal{D}) = 0$. Let us recall the statement of theorem A.

Theorem A: *Let $z \in \mathbb{P}^k \setminus \mathcal{S}$. There exist an s -invariant set $\Sigma' \subset \Sigma$ of full ν -measure and an f -invariant set $\mathcal{J}' \subset \mathcal{J}$ of full μ -measure satisfying the following properties. For any $\tilde{\alpha} \in \Sigma'$, the point $\omega(\tilde{\alpha}) := \lim_{n \rightarrow \infty} z_n(\tilde{\alpha}) \in \mathcal{J}'$ is well defined. We have $\omega_* \nu = \mu$ and the following diagram commutes:*

$$\begin{array}{ccc} \Sigma' & \xrightarrow{s} & \Sigma' \\ \omega \downarrow & & \downarrow \omega \\ \mathcal{J}' & \xrightarrow{f} & \mathcal{J}' \end{array}$$

Moreover there exist $\theta, \epsilon > 0$, $n_z \geq 1$ and $\tilde{n} : \Sigma' \rightarrow \mathbb{N}$ larger than n_z such that:

1. $d(z_n(\tilde{\alpha}), \omega(\tilde{\alpha})) \leq \tilde{c}_\epsilon d^{-\epsilon n}$ for every $\tilde{\alpha} \in \Sigma'$ and $n \geq \tilde{n}(\tilde{\alpha})$,
2. $\nu(\{\tilde{n} \leq q\}) \geq 1 - c_\theta d^{-\theta q}$ for every $q \geq n_z$.

We shall use theorem 3.2 and the method of coding trees introduced in [PUZ] for (\mathbb{P}^1, f, μ) . We recall that $\mathcal{A} = \{1, \dots, d_t\}$. Let $z \notin \mathcal{S}$ and $\{w_\alpha, \alpha \in \mathcal{A}\} := f^{-1}(z)$. By the very definition of \mathcal{S} , the cardinal of $f^{-1}(z)$ is equal to d_t and $w_\alpha \neq z$, $w_\alpha \notin \mathcal{V} \cup \mathcal{D}$ for every $\alpha \in \mathcal{A}$. We denote by L_α the projective line in \mathbb{P}^k containing (z, w_α) and apply theorem 3.2: let γ_α be an injective smooth path joining (z, w_α) and $(\Delta_n(\alpha))_n \subset L_\alpha$ be a decreasing sequence of discs containing γ_α provided by that theorem. We set $n_z := \max\{n_{z, w_\alpha}, \alpha \in \mathcal{A}\}$.

Let us fix $\tilde{\alpha} = (\alpha_n)_{n \geq 0} \in \Sigma$. We define inductively injective smooth paths $\gamma_n(\tilde{\alpha}) : [0, 1] \rightarrow \mathbb{P}^k \setminus \mathcal{V}$ and points $z_n(\tilde{\alpha}) \in \mathbb{P}^k \setminus \mathcal{V}$. We first set $\gamma_0(\tilde{\alpha}) := \gamma_{\alpha_0}$. This path joins $z = \gamma_0(\tilde{\alpha})(0)$ and $w_{\alpha_0} = \gamma_0(\tilde{\alpha})(1) =: z_0(\tilde{\alpha})$. Assume that the paths $\gamma_j(\tilde{\alpha})$ and the points $z_j(\tilde{\alpha})$ have been defined for $0 \leq j \leq n-1$. We let $\gamma_n(\tilde{\alpha})$ to be the lift of γ_{α_n} by f^n with starting point $\gamma_n(\tilde{\alpha})(0) = z_{n-1}(\tilde{\alpha})$. This path is well defined since γ_{α_n} does not intersect \mathcal{V} . We finally let $z_n(\tilde{\alpha}) := \gamma_n(\tilde{\alpha})(1)$.

We note that $z_{n-1}(\tilde{\alpha})$ and $z_n(\tilde{\alpha})$ are the endpoints of $\gamma_n(\tilde{\alpha})$ and that $z_n(\Sigma) = f^{-(n+1)}(z)$ has cardinal d_t^{n+1} . The reader will easily check the relation $f \circ z_n(\tilde{\alpha}) = z_{n-1} \circ s(\tilde{\alpha})$. Observe also that $\gamma_n(\tilde{\alpha})$ and $z_n(\tilde{\alpha})$ depend only on $\pi_n(\tilde{\alpha}) = (\alpha_0, \dots, \alpha_n)$. The following lemma is a consequence of theorem 3.2 and the fact that $\gamma_\alpha[0, 1] \subset \Delta_n(\alpha)$.

Lemma 4.1 *For every $\alpha \in \mathcal{A}$ and $n \geq n_z$, there exist at least $(1 - d^{-\theta n})d_t^n$ elements $(\alpha_0, \dots, \alpha_{n-1}) \in \mathcal{A}^n$ such that $\text{diam } \gamma_n(\alpha_0, \dots, \alpha_{n-1}, \alpha) \leq \tilde{c} d^{-\epsilon n}$.*

Let $\Omega_n := \{\tilde{\alpha} \in \Sigma, \text{diam } \gamma_n(\tilde{\alpha}) > \tilde{c} d^{-\epsilon n}\}$ and \mathcal{B}_n be the collection of $(n+1)$ -cylinders $\{C_n(\tilde{\alpha}), \tilde{\alpha} \in \Omega_n\}$. We have $\Omega_n = \mathcal{B}_n^*$. Let us also define:

$$\Omega(n) := \bigcup_{p \geq n} \Omega_p = \bigcup_{p \geq n} \mathcal{B}_p^*.$$

Lemma 4.2 *For any $n \geq n_z$, we have:*

1. $\text{Card}(\mathcal{B}_n) \leq d_t^{n+1} d^{-\theta n}$.
2. $\nu(\Omega_n) \leq d^{-\theta n}$, hence $\nu(\Omega(n)) \leq c_\theta d^{-\theta n}$.
3. if $\tilde{\alpha} \notin \Omega(n)$, then $d(z_{m-1}(\tilde{\alpha}), z_m(\tilde{\alpha})) \leq \tilde{c} d^{-\epsilon m}$ for any $m \geq n$.

PROOF: We have $\mathcal{B}_n = \{C_n(\tilde{\alpha}), \text{diam } \gamma_n(\tilde{\alpha}) > \tilde{c} d^{-\epsilon n}\}$. For every $\alpha \in \mathcal{A}$, we set $\mathcal{B}_n(\alpha) \subset \mathcal{B}_n$ to be the collection of $(n+1)$ -cylinders whose last coordinate is equal to α . The lemma 4.1 implies that $\text{Card}(\mathcal{B}_n(\alpha)) \leq d_t^n d^{-\theta n}$ and thus $\text{Card}(\mathcal{B}_n) = \sum_{\alpha \in \mathcal{A}} \text{Card}(\mathcal{B}_n(\alpha)) \leq d_t^{n+1} d^{-\theta n}$, which is the point 1. The point 2 follows:

$$\nu(\Omega_n) = \nu(\mathcal{B}_n^*) = \text{Card}(\mathcal{B}_n) / d_t^{n+1} \leq d^{-\theta n}.$$

For the point 3, observe that $d(z_{m-1}(\tilde{\alpha}), z_m(\tilde{\alpha})) \leq \text{diam } \gamma_m(\tilde{\alpha})$. If $\tilde{\alpha} \notin \Omega(n)$, then $\tilde{\alpha} \notin \Omega_m$ for any $m \geq n$, hence $\text{diam } \gamma_m(\tilde{\alpha}) \leq \tilde{c} d^{-\epsilon m}$. \square

Let $\Omega := \bigcap_{n \geq n_z} \Omega(n) = \limsup_{n \geq n_z} \Omega_n$. The set $\Sigma'' := \Sigma \setminus \Omega$ has full ν -measure since $\nu(\Omega) \leq \nu(\Omega(n)) \leq c_\theta d^{-\theta n}$ for any $n \geq n_z$. For every $\tilde{\alpha} \in \Sigma''$, we define $\tilde{n}(\tilde{\alpha})$ to be the least integer $n \geq n_z$ satisfying $\tilde{\alpha} \notin \Omega(n)$. Let $\Theta_q := \{\tilde{n} \leq q\}$.

Lemma 4.3

1. $\omega(\tilde{\alpha}) = \lim_{n \rightarrow \infty} z_n(\tilde{\alpha})$ is well defined for every $\tilde{\alpha} \in \Sigma''$.
2. $d(z_n(\tilde{\alpha}), \omega(\tilde{\alpha})) \leq \tilde{c}_\epsilon d^{-\epsilon n}$ for every $n \geq \tilde{n}(\tilde{\alpha})$.
3. $\omega : \Sigma'' \rightarrow \mathbb{P}^k$ satisfies $\omega_* \nu = \mu$.
4. $\nu(\Theta_q) \geq 1 - c_\theta d^{-\theta q}$ for any $q \geq n_z$.

PROOF: The points 1, 2 and 4 come from lemma 4.2(3,2) and the definition of $\tilde{n}(\tilde{\alpha})$. Now we prove the point 3. Let us consider the surjective map $z_n : \Sigma'' \rightarrow f^{-(n+1)}(z)$. Since $z_n(\tilde{\alpha})$ depends only on $\underline{\alpha} := (\alpha_0, \dots, \alpha_n) \in \mathcal{A}^{n+1}$, the measure $z_{n*}\nu$ is equal to:

$$z_{n*}\nu = \sum_{\underline{\alpha} \in \mathcal{A}^{n+1}} \nu(\Sigma'' \cap C_n(\underline{\alpha})) \delta_{z_n(\underline{\alpha})} = \frac{1}{d_t^{n+1}} \sum_{f^{n+1}(y)=z} \delta_y = \mu_{n+1,z}.$$

Since $z \notin \mathcal{S}$ and $\mathcal{E} \subset \mathcal{V} \subset \mathcal{S}$, the sequence of probability measures $(\mu_{n,z})_n$ converges to μ (see subsection 2.1). Hence it remains to prove $z_{n*}\nu \rightarrow \omega_*\nu$, meaning that $\int_{\Sigma''} \varphi \circ z_n d\nu \rightarrow \int_{\Sigma''} \varphi \circ \omega d\nu$ for every test function $\varphi : \mathbb{P}^k \rightarrow \mathbb{R}$. But this follows from point 1 and Lebesgue convergence theorem. \square

It remains to define Σ' , \mathcal{J}' and to verify the relation $f \circ \omega = \omega \circ s$ on Σ' . The lemma 4.3(3) implies that $\Sigma_* := \omega(\Sigma'')$ satisfies $\mu(\Sigma_*) = \nu(\omega^{-1}\Sigma_*) \geq \nu(\Sigma'') = 1$. We define $\mathcal{J}' := \bigcap_{n \in \mathbb{Z}} f^n(\mathcal{J} \cap \Sigma_*)$ and $\Sigma' := \bigcap_{n \in \mathbb{Z}} s^n(\Sigma'' \cap \omega^{-1}\mathcal{J}')$. These are invariant subsets of full measure. We obtain $f \circ \omega = \omega \circ s$ on Σ' by taking limits in $f \circ z_n(\tilde{\alpha}) = z_{n-1} \circ s(\tilde{\alpha})$. That completes the proof of theorem A.

5 Proof of theorem B

Let us recall the statement.

Theorem B: *Let $\psi \in \mathcal{U}$ be a μ -centered observable and ω be a coding map provided by theorem A. Let $\chi := \psi \circ \omega$ and $1 \leq p < +\infty$.*

1. *there exist $\hat{c}_p, \lambda_p > 0$ such that $\|\chi - \mathbb{E}(\chi | \mathcal{C}_n)\|_p \leq \hat{c}_p e^{-n\lambda_p}$ for every $n \geq 0$.*
2. *$R_j(\chi) := \int_{\Sigma} \chi \cdot \chi \circ s^j d\nu$ satisfies $|R_j(\chi)| \leq 2 \|\chi\|_2 \hat{c}_2 e^{-(j-1)\lambda_2}$ for every $j \geq 1$.*

5.1 Proof of theorem B(1)

We set $\chi_B := \chi \cdot 1_B$ for any $B \subset \Sigma$ and use the following estimates provided by theorem A. We recall that $\Theta_n = \{\tilde{n}(\tilde{\alpha}) \leq n\}$.

$$(\star) \quad d(z_n(\tilde{\alpha}), \omega(\tilde{\alpha})) \leq \tilde{c}_\epsilon d^{-\epsilon n} \text{ for every } \tilde{\alpha} \in \Theta_n,$$

$$(\star\star) \quad \nu(\Theta_n) \geq 1 - c_\theta d^{-n\theta} \text{ for every } n \geq n_z.$$

We will need the following lemma, which is a direct consequence of (\star) .

Lemma 5.1 *Let $\tilde{\alpha} \in \Theta_n$ and $\tilde{\beta} \in C_n(\tilde{\alpha}) \cap \Theta_n$. Then $d(\omega(\tilde{\alpha}), \omega(\tilde{\beta})) \leq 2 \tilde{c}_\epsilon d^{-\epsilon n}$.*

5.1.1 The Hölder case

Let ψ be an h -Hölder and μ -centered observable on \mathbb{P}^k . We set $\chi := \psi \circ \omega$. The theorem B(1) is a consequence of the following estimates, which hold for every $n \geq n_z$.

Lemma 5.2 $\|\chi_{\Theta_n^c} - \mathbb{E}(\chi_{\Theta_n^c} | \mathcal{C}_n)\|_p \leq 2 \|\chi\|_\infty (c_\theta d^{-n\theta})^{1/p}$.

PROOF: The left hand side is less than $2 \|\chi_{\Theta_n^c}\|_p$ by Jensen inequality. Then the conclusion follows from (★★). \square

Lemma 5.3 $\|\chi_{\Theta_n} - \mathbb{E}(\chi_{\Theta_n} | \mathcal{C}_n)\|_p \leq c d^{-n\tau}$ for some $c, \tau > 0$.

PROOF: We denote $\varphi := \chi_{\Theta_n} - \mathbb{E}(\chi_{\Theta_n} | \mathcal{C}_n)$ and estimate $\|\varphi_{\Theta_n^c}\|_p, \|\varphi_{\Theta_n}\|_p$. Since $\varphi_{\Theta_n^c} = -\mathbb{E}(\chi_{\Theta_n} | \mathcal{C}_n) \cdot 1_{\Theta_n^c}$, we have:

$$\|\varphi_{\Theta_n^c}\|_p \leq \|\mathbb{E}(\chi_{\Theta_n} | \mathcal{C}_n)\|_{2p} \cdot \nu(\Theta_n^c)^{1/2p} \leq \|\chi\|_{2p} \cdot (c_\theta d^{-n\theta})^{1/2p}.$$

We now deal with $\|\varphi_{\Theta_n}\|_p$. For every $\tilde{\alpha} \in \Theta_n$, let $\nu_{\tilde{\alpha}}$ be the conditional measure of ν on the cylinder $C_n(\tilde{\alpha})$. We have for every $\tilde{\alpha} \in \Theta_n$:

$$\varphi_{\Theta_n}(\tilde{\alpha}) = \int_{C_n(\tilde{\alpha}) \cap \Theta_n} (\chi(\tilde{\alpha}) - \chi(\tilde{\beta})) d\nu_{\tilde{\alpha}}(\tilde{\beta}) + \chi(\tilde{\alpha}) \cdot \nu_{\tilde{\alpha}}(C_n(\tilde{\alpha}) \cap \Theta_n^c). \quad (9)$$

We deduce from $\chi = \psi \circ \omega$, lemma 5.1 and the fact that ψ is h -Hölder:

$$\forall \tilde{\alpha} \in \Theta_n, |\varphi_{\Theta_n}(\tilde{\alpha})| \leq (2\tilde{c}_\epsilon d^{-n\epsilon})^h + \|\chi_{\Theta_n}\|_\infty \cdot \nu_{\tilde{\alpha}}(C_n(\tilde{\alpha}) \cap \Theta_n^c).$$

Hence we get for every $p \geq 1$ up to a multiplicative constant:

$$\forall \tilde{\alpha} \in \Theta_n, |\varphi_{\Theta_n}(\tilde{\alpha})|^p \leq d^{-nhp\epsilon} + \|\chi_{\Theta_n}\|_\infty^p \cdot \nu_{\tilde{\alpha}}(C_n(\tilde{\alpha}) \cap \Theta_n^c).$$

By integrating over Θ_n and using (★★), we deduce:

$$\|\varphi_{\Theta_n}\|_p^p \leq d^{-nhp\epsilon} + \|\chi\|_\infty^p \cdot c_\theta d^{-n\theta}.$$

That completes the proof of the lemma. \square

5.1.2 The general case $\psi \in \mathcal{U}$

Let $\psi : \mathbb{P}^k \rightarrow \mathbb{R} \cup \{-\infty\}$ be a μ -centered observable in \mathcal{U} : the function e^ψ is h -Hölder and satisfies $\psi \geq \log d(\cdot, \mathcal{N}_\psi)^\rho$ on \mathbb{P}^k (see definition 2.1). Observe in particular that ψ is bounded from above. We recall that $\mathcal{N}_\psi[r]$ is the r -neighbourhood of $\{\psi = -\infty\}$ and that $\chi = \psi \circ \omega$. We consider the following subsets of Σ :

$$\mathcal{S}_n := \Theta_n^c \setminus \mathcal{N}_n, \quad \Gamma_n = \Theta_n \setminus \mathcal{N}_n, \quad \mathcal{N}_n := \omega^{-1}(\mathcal{N}_\psi[d^{-n(h\epsilon/2\rho)}]).$$

We shall need the following observations. First, we have $\nu(\mathcal{N}_n) = \mu(\mathcal{N}_\psi[d^{-n(h\epsilon/2\rho)}]) \leq d^{-n\gamma(h\epsilon/2\rho)}$ up to a multiplicative constant (see subsection 2.1). We deduce from (**):

$$\nu(\Gamma_n^c) = \nu(\Theta_n^c \cup \mathcal{N}_n) \leq c_\theta d^{-n\theta} + d^{-n\gamma(h\epsilon/2\rho)} \leq c d^{-n\eta} \quad (10)$$

for some $c, \eta > 0$. Second, for every $\tilde{\alpha} \in \mathcal{N}_n^c = \mathcal{S}_n \cup \Gamma_n$, we have $\chi(\tilde{\alpha}) \geq \log d(\omega(\tilde{\alpha}), \mathcal{N}_\psi)^\rho \geq \log d^{-\rho n(h\epsilon/2\rho)}$, hence:

$$\|\chi_{\mathcal{S}_n \cup \Gamma_n}\|_\infty \leq n(h\epsilon \log d)/2. \quad (11)$$

The theorem B(1) is now a consequence of the following estimates.

Lemma 5.4 $\|\chi_{\mathcal{N}_n} - \mathbb{E}(\chi_{\mathcal{N}_n} | \mathcal{C}_n)\|_p \leq (\kappa d^{-n(h\epsilon/2\rho) \cdot (\gamma/2)})^{1/p}$.

PROOF: The left hand side is less than $2\|\chi_{\mathcal{N}_n}\|_p$. Proposition 2.2 yields $\|\chi_{\mathcal{N}_n}\|_p = \|\psi \circ \omega \cdot 1_{\mathcal{N}_n}\|_p \leq (\kappa d^{-n(h\epsilon/2\rho) \cdot (\gamma/2)})^{1/p}$ for every n such that $d^{-n(h\epsilon/2\rho)} < 1/2$. \square

Lemma 5.5 $\|\chi_{\mathcal{S}_n} - \mathbb{E}(\chi_{\mathcal{S}_n} | \mathcal{C}_n)\|_p \leq n(h\epsilon \log d) \cdot (c d^{-n\eta})^{1/p}$.

PROOF: The left hand side is less than $2\|\chi_{\mathcal{S}_n}\|_p$. We conclude by using (10) and (11) (observe that $\mathcal{S}_n \subset \Gamma_n^c$). \square

Lemma 5.6 $\|\chi_{\Gamma_n} - \mathbb{E}(\chi_{\Gamma_n} | \mathcal{C}_n)\|_p \leq c d^{-n\tau}$ for some $c, \tau > 0$.

PROOF: We follow the proof of lemma 5.3: we set $\varphi := \chi_{\Gamma_n} - \mathbb{E}(\chi_{\Gamma_n} | \mathcal{C}_n)$ and estimate $\|\varphi_{\Gamma_n^c}\|_p, \|\varphi_{\Gamma_n}\|_p$. The line (10) yields:

$$\|\varphi_{\Gamma_n^c}\|_p \leq \|\mathbb{E}(\chi_{\Gamma_n} | \mathcal{C}_n)\|_{2p} \cdot \nu(\Gamma_n^c)^{1/2p} \leq \|\chi\|_{2p} \cdot (c d^{-n\eta})^{1/2p}.$$

Now we deal with $\|\varphi_{\Gamma_n}\|_p$. We can write as in (9):

$$\forall \tilde{\alpha} \in \Gamma_n, \varphi(\tilde{\alpha}) = \int_{\mathcal{C}_n(\tilde{\alpha}) \cap \Gamma_n} (\chi(\tilde{\alpha}) - \chi(\tilde{\beta})) d\nu_{\tilde{\alpha}}(\tilde{\beta}) + \chi(\tilde{\alpha}) \cdot \nu_{\tilde{\alpha}}(C_n(\tilde{\alpha}) \cap \Gamma_n^c). \quad (12)$$

Let $\tilde{\alpha} \in \Gamma_n$ and $\tilde{\beta} \in C_n(\tilde{\alpha}) \cap \Gamma_n$. We deduce from $(\tilde{\alpha}, \tilde{\beta}) \notin \mathcal{N}_n$ that $e^\psi \circ \omega(\tilde{\alpha})$ and $e^\psi \circ \omega(\tilde{\beta})$ are larger than $d^{-nh\epsilon/2}$. This implies:

$$|\chi(\tilde{\alpha}) - \chi(\tilde{\beta})| = |\log e^\psi \circ \omega(\tilde{\alpha}) - \log e^\psi \circ \omega(\tilde{\beta})| \leq d^{nh\epsilon/2} |e^\psi \circ \omega(\tilde{\alpha}) - e^\psi \circ \omega(\tilde{\beta})|.$$

Using lemma 5.1 and the fact that e^ψ is h -Hölder, the last term is less than $d^{nh\epsilon/2} \cdot (2\tilde{c}_\epsilon d^{-n\epsilon})^h$. Then we deduce from (12), up to a multiplicative constant:

$$\forall \tilde{\alpha} \in \Gamma_n, |\varphi(\tilde{\alpha})| \leq d^{-nh\epsilon/2} + \|\chi_{\Gamma_n}\|_\infty \cdot \nu_{\tilde{\alpha}}(C_n(\tilde{\alpha}) \cap \Gamma_n^c).$$

Taking the p -th power, integrating over Γ_n and using (10), (11), we obtain up to a multiplicative constant:

$$\|\varphi_{\Gamma_n}\|_p^p \leq d^{-nhp\epsilon/2} + (n(h\epsilon \log d)/2)^p \cdot c d^{-n\eta}.$$

That completes the proof of the lemma. \square

5.2 Proof of theorem B(2)

Let $\psi \in \mathcal{U}$ be a μ -centered observable and $\chi = \psi \circ \omega$. Let $j \geq 1$ and $n \geq 0$ to be specified later. We set $\chi_n := \mathbb{E}(\chi | \mathcal{C}_n)$ and write:

$$\chi \cdot \chi \circ s^j = (\chi - \chi_n) \cdot \chi \circ s^j + \chi_n \cdot (\chi \circ s^j - \chi_n \circ s^j) + \chi_n \cdot \chi_n \circ s^j.$$

By using the s -invariance of ν and Jensen inequality $\|\chi_n\|_2 \leq \|\chi\|_2$, we deduce:

$$|R_j(\chi)| = \left| \int_{\Sigma} \chi \cdot \chi \circ s^j d\nu \right| \leq 2 \|\chi\|_2 \|\chi - \chi_n\|_2 + \left| \int_{\Sigma} \chi_n \cdot \chi_n \circ s^j d\nu \right|. \quad (13)$$

The variables χ_n and $\chi_n \circ s^j$ respectively depend on (ξ_0, \dots, ξ_n) and $(\xi_j, \dots, \xi_{n+j})$, where $\xi_n : \Sigma \rightarrow \mathcal{A}$ is the projection $\xi_n(\tilde{\alpha}) = \alpha_n$. These are independent variables when $n = j - 1$, hence $\int_{\Sigma} \chi_n \cdot \chi_n \circ s^j d\nu = \int_{\Sigma} \chi_n d\nu \int_{\Sigma} \chi_n \circ s^j d\nu$ in that case. But this product is zero since χ is ν -centered. The conclusion then follows from (13) with $n = j - 1$ and theorem B(1).

6 Proof of theorem C

Let us recall the statement.

Theorem C: *For every μ -centered observable $\psi \in \mathcal{U}$, we have:*

1. $\sigma := \lim_{n \rightarrow \infty} \frac{1}{\sqrt{n}} \|S_n(\psi)\|_2$ exists, and $\sigma^2 = \int_{\mathbb{P}^k} \psi^2 d\mu + 2 \sum_{j \geq 1} \int_{\mathbb{P}^k} \psi \cdot \psi \circ f^j d\mu$.
2. If $\sigma = 0$, then $\psi = u - u \circ f$ μ -a.e. for some $u \in L^2(\mu)$.
3. If $\sigma > 0$, then ψ satisfies the σ -ASIP.

The points 1 and 2 are consequences of classical lemma 6.1 below, whose condition $\sum_{j \geq 1} j |R_j(\varphi)| < \infty$ is fulfilled by theorem B(2). The point 3 follows from proposition 2.2, theorem B(1) and Philipp-Stout's theorem (see subsection 2.4).

Lemma 6.1 *Let (X, g, m) be a dynamical system and $\varphi \in L^2(m)$ be a m -centered observable. We denote $S_n(\varphi) = \sum_{j=0}^{n-1} \varphi \circ g^j$ and $R_j(\varphi) = \int_X \varphi \cdot \varphi \circ g^j dm$. Let $\sigma^2 := R_0(\varphi) + 2 \sum_{j \geq 1} R_j(\varphi)$. If $\sum_{j \geq 1} j |R_j(\varphi)| < \infty$, then σ^2 is finite and we have:*

1. $\|S_n(\varphi)\|_2^2 = n\sigma^2 + O(1)$. In particular, $\lim_{n \rightarrow \infty} \frac{1}{n} \|S_n(\varphi)\|_2^2 = \sigma^2$.
2. $\sigma^2 = 0$ if and only if $\varphi = u - u \circ g$ m -a.e. for some $u \in L^2(m)$.

PROOF: Let $S_n := S_n(\varphi)$ and $R_j := R_j(\varphi)$. Since m is g -invariant, we have $\|S_n\|_2^2 = nR_0 + 2 \sum_{j=1}^{n-1} (n-j) R_j$. We deduce for every $n \geq 1$:

$$\|S_n\|_2^2 = n \left(R_0 + 2 \sum_{j=1}^{\infty} R_j \right) + (-2) \left(\sum_{j=1}^{n-1} j R_j + \sum_{j=n}^{\infty} n R_j \right) = n\sigma^2 + A_n, \quad (14)$$

where $|A_n| \leq 2 \sum_{j \geq 1} j |R_j|$. That proves the point 1. Let us show the point 2. Suppose $\sigma^2 = 0$. In view of (14), the function $u_p := \frac{1}{p} \sum_{n=1}^p S_n$ satisfies $\|u_p\|_2 \leq (2 \sum_{j \geq 1} j |R_j|)^{1/2}$ for every $p \geq 1$. Let $u := \lim_{j \rightarrow \infty} u_{p_j}$ be a weak cluster point in $L^2(m)$ and observe that:

$$\forall j \geq 1, u_{p_j} - u_{p_j} \circ g = \frac{1}{p_j} \sum_{n=0}^{p_j-1} (\varphi - \varphi \circ g^n) = \varphi - \frac{1}{p_j} S_{p_j}.$$

We deduce $\varphi = u - u \circ g$ m -a.e. by taking limits in $L^2(m)$: $\lim_{j \rightarrow \infty} u_{p_j} \circ g = u \circ g$ since m is g -invariant, and $\lim_{j \rightarrow \infty} \frac{1}{p_j} S_{p_j} = \int_X \varphi dm = 0$ by Von Neumann theorem. The reverse implication of the point 2 comes from $\sigma^2 = \lim_{n \rightarrow \infty} \frac{1}{n} \|S_n(\varphi)\|_2^2 = \lim_{n \rightarrow \infty} \frac{1}{n} \|u - u \circ g^n\|_2^2 = 0$. \square

7 Proof of theorem D

We recall that $J := \log \text{Jac } f - \int_{\mathbb{P}^k} \log \text{Jac } f d\mu$, this is an unbounded μ -centered observable in \mathcal{U} . We set $\sigma_J := \lim_n \frac{1}{\sqrt{n}} \|S_n(J)\|_2$, which is well defined by theorem C. We denote by $\chi_1 \leq \dots \leq \chi_k$ the Lyapunov exponent of μ , they are larger than or equal to $\log d^{1/2}$.

Theorem D: *If the Lyapunov exponents of μ are minimal equal to $\log d^{1/2}$, then $\sigma_J = 0$ and μ is absolutely continuous with respect to Lebesgue measure.*

The first part $\sigma_J = 0$ will be proved in subsection 7.2. The second part is a consequence of theorem 7.1 below (that theorem will be proved in subsection 7.3 by using $\sigma_J = 0$). In the sequel, the maps f^n and $d_x f^n$ are implicitly written in some fixed charts of \mathbb{P}^k .

Theorem 7.1 *Assume that the Lyapunov exponents are minimal. Then for μ almost every $x \in \mathbb{P}^k$, there exists $\rho(x) > 0$ and a subsequence $(n_j(x))_{j \geq 1}$ such that $f^{n_j} \circ (x + d^{-n_j/2} \cdot \text{Id}_{\mathbb{C}^k}) : B(\rho(x)) \rightarrow \mathbb{P}^k$ is injective.*

PROOF OF THE SECOND PART OF THEOREM D (ABSOLUTE CONTINUITY): We use the notations of theorem 7.1. Let $x \in \mathbb{P}^k$ be a μ -generic point and set $n_j := n_j(x)$. Since f^{n_j} is injective on the ball $B_j := B_x(\rho(x)d^{-n_j/2})$ and μ has constant jacobian d^k (see subsection 2.1), we obtain $\mu(B_j) = \mu(f^{n_j}(B_j))d^{-kn_j}$. Observe also that $\text{Leb}(B_j) = \rho(x)^{2k} (d^{-n_j/2})^{2k} = \rho(x)^{2k} d^{-kn_j}$ up to a multiplicative constant. We obtain therefore for μ -a.e. $x \in \mathbb{P}^k$:

$$\liminf_{r \rightarrow 0} \frac{\mu(B_x(r))}{\text{Leb}(B_x(r))} \leq \liminf_{j \rightarrow \infty} \frac{\mu(B_j)}{\text{Leb}(B_j)} = \liminf_{j \rightarrow \infty} \frac{\mu(f^{n_j}(B_j))}{\rho(x)^{2k}} \leq \frac{1}{\rho(x)^{2k}} < \infty.$$

That proves the absolute continuity of μ (see [Mat], theorem 2.12). \square

7.1 Preliminaries

Observe that $J = \log \text{Jac } f - \log d^k$ when the Lyapunov exponents are equal to $\log d^{1/2}$. Since the jacobian is a multiplicative function, we have in that case:

$$S_n(J) = \sum_{i=0}^{n-1} J \circ f^i(x) = \log \text{Jac } f^n - \log d^{kn}. \quad (15)$$

The *singular values* $\delta_1 \leq \dots \leq \delta_k$ of the linear map $A := d_x f^n$ are defined as the eigenvalues of $\sqrt{AA^*}$. In particular, there exist unitary matrices (U, V) such that $d_x f^n = U \text{Diag}(\delta_1, \dots, \delta_k) V$. We have therefore:

$$\delta_1 = \|(d_x f^n)^{-1}\|^{-1} \quad \text{and} \quad \prod_{i=1}^k \delta_i^2 = \text{Jac } f^n(x) \geq \delta_1^{2k}. \quad (16)$$

For any $\rho, \tau > 0$ and $n \geq 1$, we define:

$$\mathcal{B}_n(\rho) := \{x \in \mathbb{P}^k, f^n \circ (x + d_x f^n)^{-1} : B(\rho) \rightarrow \mathbb{P}^k \text{ is an injective map}\},$$

$$\mathcal{R}_n(\tau) := \left\{x \in \mathbb{P}^k, \|(d_x f^n)^{-1}\|^{-1} \geq d^{n/2}/\tau\right\}.$$

The following estimates were proved by Berteloot-Dupont [BeDu]. They hold for every system (\mathbb{P}^k, f, μ) whose Lyapunov exponents satisfy $\chi_k < 2\chi_1$.

Theorem 7.2 *There exists $\alpha :]0, 1] \rightarrow \mathbb{R}_+^*$ satisfying $\lim_{\rho \rightarrow 0} \alpha(\rho) = 1$ and for $n \geq 1$:*

1. $\mu(\mathcal{B}_n(\rho)) \geq \alpha(\rho)$,
2. $\mu(\mathcal{B}_n(\rho) \cap \mathcal{R}_n(\tau)^c) \leq (\rho\tau)^{-2}$.

That result implies the following lemma.

Lemma 7.3 *Let $\rho \in]0, 1]$. There exists $\mathcal{H} \subset \mathbb{P}^k$ satisfying $\mu(\mathcal{H}) = 1$ and:*

$$\forall x \in \mathcal{H}, \exists n(x) \geq 1, \forall n \geq n(x), x \notin \mathcal{B}_n(\rho) \text{ or } \text{Jac } f^n(x) \geq d^{kn}/n^{2k}.$$

PROOF: We apply proposition 7.2(2) with $\tau = n$ to get $\mu(\mathcal{B}_n(\rho) \cap \mathcal{R}_n(n)^c) \leq (\rho n)^{-2}$. Since $\sum_{n \geq 1} \mu(\mathcal{B}_n(\rho) \cap \mathcal{R}_n(n)^c) < \infty$, there exists by Borel-Cantelli lemma a subset \mathcal{H} of full μ -measure satisfying:

$$\forall x \in \mathcal{H}, \exists n(x) \geq 1, \forall n \geq n(x), x \notin \mathcal{B}_n(\rho) \text{ or } x \in \mathcal{R}_n(n).$$

But $x \in \mathcal{R}_n(n)$ implies by (16): $\text{Jac } f^n(x) \geq (d^{n/2}/n)^{2k} = d^{kn}/n^{2k}$. □

7.2 Proof of the first part of theorem D ($\sigma_J = 0$)

Suppose that the exponents are minimal and that $\sigma_J = \lim_n \frac{1}{\sqrt{n}} \|S_n(J)\|_2 > 0$. Then J satisfies the CLT: if $V := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-1} e^{-u^2/2} du$, we get $\mu\left(\mathcal{G}_n := \left\{\frac{S_n(J)}{\sqrt{n}} \leq -\sigma_J\right\}\right) \geq V/2$ for n larger than some N (see subsection 2.4).

Let $\rho > 0$ be such that $\mu(\mathcal{B}_n(\rho)) > 1 - V/4$ for every $n \geq 1$. If we set $\mathcal{F}_n := \mathcal{B}_n(\rho) \cap \mathcal{G}_n$, then $\mathcal{F} := \limsup_{n \geq N} \mathcal{F}_n$ satisfies $\mu(\mathcal{F}) \geq V/4$. Let $x \in \mathcal{F} \cap \mathcal{H}$, where \mathcal{H} is provided by lemma 7.3. Let $(n_j(x))_{j \geq 1}$ be such that $x \in \mathcal{F}_{n_j}$ for every $j \geq 1$. The inclusion $\mathcal{F}_{n_j} \subset \mathcal{G}_{n_j}$ yields $S_{n_j}(J)(x) \leq -\sigma_J \sqrt{n_j}$ for every $j \geq 1$. Since $S_{n_j}(J) = \log \text{Jac } f^{n_j} - \log d^{kn_j}$ (the exponents are indeed minimal, see (15)), we deduce:

$$\forall j \geq 1, \text{Jac } f^{n_j}(x) \leq d^{kn_j} e^{-\sigma_J \sqrt{n_j}}. \quad (17)$$

But $\text{Jac } f^{n_j}(x) \geq d^{kn_j}/n_j^{2k}$ for every $n_j \geq n(x)$, following from $x \in \mathcal{B}_{n_j}(\rho) \cap \mathcal{H}$ and lemma 7.3. That contradicts (17) when j tends to infinity.

7.3 Proof of theorem 7.1

We proved in subsection 7.2 that $\sigma_J = 0$. Hence $J = u - u \circ f$ μ -a.e. for some $u \in L^2(\mu)$ by theorem C. We obtain therefore:

$$u - u \circ f^n(x) = \sum_{i=0}^{n-1} J \circ f^i(x) = \log \text{Jac } f^n(x) - \log d^{kn}. \quad (18)$$

Let $\epsilon > 0$ and $m \geq 1$ such that $\mathcal{M} := \{|u| \leq \log m\}$ satisfies $\mu(\mathcal{M}) \geq (1 - \epsilon)^{1/2}$. Since μ is mixing, $\mathcal{M}_n := \mathcal{M} \cap f^{-n} \mathcal{M}$ satisfies $\mu(\mathcal{M}_n) \geq \mu(\mathcal{M})^2 - \epsilon \geq 1 - 2\epsilon$ for n larger than some N' . Let ρ be small and τ be large enough such that $\mu(\mathcal{B}_n(\rho) \cap \mathcal{R}_n(\tau)) \geq 1 - 2\epsilon$ for every $n \geq 1$. We define $\mathcal{T}_n := \mathcal{B}_n(\rho) \cap \mathcal{R}_n(\tau) \cap \mathcal{M}_n$ and $\mathcal{T} := \limsup_{n \geq N'} \mathcal{T}_n$. Observe that $\mu(\mathcal{T}) \geq 1 - 4\epsilon$. Let $x \in \mathcal{T}$ and $(n_j)_j$ (depending on x) such that $x \in \mathcal{T}_{n_j}$ for every $j \geq 1$. Since $x \in \mathcal{T}_{n_j} \subset \mathcal{B}_{n_j}(\rho)$, the map $f^{n_j} \circ (x + (d_x f^{n_j})^{-1}) : B(\rho) \rightarrow \mathbb{P}^k$ is injective.

Let $\Lambda_n = d^{-n/2} \cdot \text{Id}_{\mathbb{C}^k}$. It is enough to prove that $d_x f^{n_j} = (U_j P_j V_j) \Lambda_{n_j}^{-1}$, where (U_j, V_j) are unitary matrices and P_j is a diagonal matrix with entries in $[a, b] \subset \mathbb{R}_+^*$ ((a, b) being independent of j). Indeed, this implies that $f^{n_j} \circ (x + \Lambda_{n_j})$ is injective on $B(\rho/b)$, completing the proof of theorem 7.1. We shall omit the subscript j for simplification, and denote by $\delta_1 \leq \dots \leq \delta_k$ the singular values of $d_x f^n$. Let (U, V) be unitary matrices such that $d_x f^n = U \text{Diag}(\delta_1, \dots, \delta_k) V$ (see subsection 7.1). The fact that $x \in \mathcal{R}_n(\tau)$ yields:

$$\delta_1 = \left\| (d_x f^n)^{-1} \right\|^{-1} \geq d^{n/2}/\tau. \quad (19)$$

Now we give an upper bound for δ_k . Since $x \in \mathcal{T}_n \subset \mathcal{M}_n$, we have $(x, f^n(x)) \in \mathcal{M} = \{|u| \leq \log m\}$. This implies by (18):

$$d^{kn/2}/m \leq \prod_{i=1}^k \delta_i = \text{Jac } f^n(x)^{1/2} \leq d^{kn/2} m.$$

We deduce from (19):

$$\delta_k \leq \frac{\delta_1 \dots \delta_{k-1}}{\delta_1^{k-1}} \delta_k = \frac{\text{Jac } f^n(x)^{1/2}}{\delta_1^{k-1}} \leq \frac{d^{kn/2}m}{(d^{n/2}/\tau)^{k-1}} = d^{n/2}\tau^{k-1}m.$$

Thus $\text{Diag}(\delta_1, \dots, \delta_k) = \Lambda_n^{-1} P$, where P is diagonal with entries in $[1/\tau, \tau^{k-1}m]$. We obtain finally $d_x f^n = U \Lambda_n^{-1} P V = (U P V) \Lambda_n^{-1}$, as desired.

Acknowledgements: We thank the referees for their careful readings and useful suggestions. Their comments enabled us to improve the exposition of the article.

References

- [A] L. Ahlfors, *Lectures on quasiconformal mappings* (second edition), University Lecture Series, **38**. American Mathematical Society (2006).
- [BeDu] F. Berteloot, C. Dupont, *Une caractérisation des endomorphismes de Lattès par leur mesure de Green*, Comment. Math. Helv., **80** (2005), no. 2, 433-454.
- [BL] F. Berteloot, J.J. Loeb, *Une caractérisation géométrique des exemples de Lattès de $\mathbb{C}\mathbb{P}^k$* , Bull. Soc. Math. Fr., **129** (2001), no. 2, 175-188.
- [Bo] R. Bowen, *Equilibrium states and the ergodic theory of Anosov diffeomorphisms*, Lecture Notes in Mathematics, **470**, (1975).
- [Br] J.Y. Briend, *La propriété de Bernoulli pour les endomorphismes de $\mathbb{P}^k(\mathbb{C})$* , Ergodic Theory Dynamical Systems, **22** (2002), no. 2, 323-327.
- [BrDu1] J.Y. Briend, J. Duval, *Exposants de Liapounoff et distribution des points périodiques d'un endomorphisme de $\mathbb{C}\mathbb{P}^k$* , Acta Math., **182** (1999), no. 2, 143-157.
- [BrDu2] J.Y. Briend, J. Duval, *Deux caractérisations de la mesure d'équilibre d'un endomorphisme de $\mathbb{C}\mathbb{P}^k$* , Publ. Math. Inst. Hautes Études Sci., **93** (2001), 145-159.
- [Bu] J. Buzzi, *The coding of non-uniformly expanding maps with application to endomorphisms of $\mathbb{C}\mathbb{P}^k$* , Ergodic Theory Dynamical Systems, **23** (2003), no. 4, 1015-1024.
- [CL] S. Cantat, S. Leborgne, *Théorème limite central pour les endomorphismes holomorphes et les correspondances modulaires*, Int. Math. Res. Not., **56** (2005), 3479-3510.
- [CG] J.-R. Chazottes, S. Gouëzel, *On almost-sure versions of classical limit theorems for dynamical systems*, Probab. Theory Related Fields, **138** (2007), no. 1-2, 195-234.

- [C] N.I. Chernov, *Limit theorems and Markov approximations for chaotic dynamical systems*, Probab. Theory Related Fields, **101** (1995), no. 3, 321-362.
- [De] M. Denker, *The central limit theorem for dynamical systems*, Dynamical systems and ergodic theory (Warsaw, 1986), 33-62, Banach Center Publ., **23**, PWN, Warsaw, 1989.
- [DP] M. Denker, W. Philipp, *Approximation by Brownian motion for Gibbs measures and flows under a function*, Ergodic Theory Dynamical Systems, **4** (1984), no. 4, 541-552.
- [DPU] M. Denker, F. Przytycki, M. Urbański, *On the transfer operator for rational functions on the Riemann sphere*, Ergodic Theory Dynamical Systems, **16** (1996), no. 2, 255-266.
- [DD] T.C. Dinh, C. Dupont, *Dimension de la mesure d'équilibre d'applications méromorphes*, J. Geom. Anal., **14** (2004), no. 4, 613-627.
- [DNS1] T.C. Dinh, V.A. Nguyen, N. Sibony, *On thermodynamics of rational maps on the Riemann sphere*, Ergodic Theory Dynamical Systems, **27** (2007), no. 4, 1095-1109.
- [DNS2] T.C. Dinh, V.A. Nguyen, N. Sibony, *Exponential estimates for plurisubharmonic functions and stochastic dynamics*, preprint (2008).
- [DS2] T.C. Dinh, N. Sibony, *Decay of correlations and the central limit theorem for meromorphic maps*, Comm. Pure Appl. Math., **59** (2006), no. 5, 754-768.
- [DS4] T.C. Dinh, N. Sibony, *Super-potentials of positive closed currents, intersection theory and dynamics*, to appear in Acta Math.
- [Do] D. Dolgopyat, *Limit theorems for partially hyperbolic systems*, Trans. Amer. Math. Soc., **356** (2004), no. 4, 1637-1689.
- [Du] C. Dupont, *Formule de Pesin et applications méromorphes*, Bull. Braz. Math. Soc. (N.S.), **37** (2006), no. 3, 393-418.
- [FS] J.E. Fornæss, N. Sibony, *Complex dynamics in higher dimensions*, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., 439, Complex potential theory (Montreal, PQ, 1993), 131-186, Kluwer Acad. Publ., Dordrecht, 1994.
- [G] M. I. Gordin, *The central limit theorem for stationary processes*, Soviet Math. Dokl., **10** (1969), 1174-1176.
- [HK] F. Hofbauer, G. Keller, *Ergodic properties of invariant measures for piecewise monotonic transformations*, Math. Z., **180** (1982), no. 1, 119-140
- [I] I.A. Ibragimov, *Some limit theorems for stationary processes*, Theory Prob. Appl., **7** (1962), no. 4, 349-382.

- [LP] M. Lacey, W. Philipp, *A note on the almost sure central limit theorem*, Statist. Probab. Lett. **9** (1990), no. 3, 201-205.
- [Le] F. Ledrappier, *Propriétés ergodiques des mesures de Sinai*, Inst. Hautes Études Sci. Publ. Math., **59** (1984), 163-188.
- [Li] C. Liverani, *Central limit theorem for deterministic systems*, International Conference on Dynamical Systems (Montevideo, 1995), Pitman Res. Notes Math. Ser., **362** (1996).
- [Lo] S. Lojasiewicz, *Introduction to complex analytic geometry*, Birkhäuser, 1991.
- [Mañ] R. Mañé, *The Hausdorff dimension of invariant probabilities of rational maps*, Lecture Notes in Math., **1331**, Springer, 1988.
- [Mat] P. Mattila, *Geometry of sets and measures in euclidian spaces*, Cambridge University Press, 1995.
- [MN] I. Melbourne, M. Nicol, *Almost sure invariance principle for nonuniformly hyperbolic systems*, Comm. Math. Phys., **260** (2005), no.1, 131-146.
- [P1] Ja. B. Pesin, *Characteristic Lyapounoff exponents and smooth ergodic theory*, Russ. Math. Surveys, **32** (1977), no. 4, 55-114.
- [P2] Ja. B. Pesin, *Dimension theory in dynamical systems*, Chicago Lectures in Mathematics, 1997.
- [PS] W. Philipp, W. Stout, *Almost sure invariance principles for partial sums of weakly dependent random variables*, Mem. Amer. Math. Soc. **2** (1975), issue 2, no. 161.
- [PUZ] F. Przytycki, M. Urbański, A. Zdunik, *Harmonic, Gibbs and Hausdorff measures on repellers for holomorphic maps. I*, Ann. of Math. (2) **130** (1989), no. 1, 1-40.
- [R] M. Ratner, *The central limit theorem for geodesic flows on n -dimensional manifolds of negative curvature*, Israel J. Math. **16** (1973), 181-197.
- [Sib] N. Sibony, *Dynamique des applications rationnelles de \mathbb{P}^k* , in Dynamique et Géométrie Complexes, Panoramas et Synthèses No **8**, SMF, 1999.
- [Sin] Y. G. Sinai, *The central limit theorem for geodesic flows on manifolds of constant negative curvature*, Soviet Math. Dokl., **1** (1960), 983-987.
- [Z] A. Zdunik, *Parabolic orbifolds and the dimension of the maximal measure for rational maps*, Invent. Math., **99** (1990), no. 3, 627-649.

Christophe Dupont
Université Paris XI
CNRS UMR 8628
Mathématique, Bâtiment 425
91405 Orsay Cedex, France
christophe.dupont@math.u-psud.fr