UNIVERSITÉ DE RENNES 1

Microlocal Analysis

Terminal Examination - the 12/12/2022 (2h)

Documents are not allowed

Exercice 1 [About the basic rules of pseudo-differential calculus]**.** Let $m \in \mathbb{Z}$ and let $a(x,\xi) \in S^m_{1,0}(\mathbb{R}^n)$ be a symbol of order m. recall that the action of the pseudo-differential operator $\mathrm{Op}(a) \equiv a(x,D)$ is given by

$$\mathrm{Op}(a)u(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix\cdot\xi}\, a(x,\xi)\, \hat{u}(\xi)\, d\xi\,.$$

1.1. We denote by $[\mathrm{Op}(a),\partial_j]$ the commutator of $\mathrm{Op}(a)$ with the partial derivative with respect to the $j^{\text{èm}}$ direction. Prove that $[\mathrm{Op}(a),\partial_j]$ is a pseudo-differential operator and compute its symbol in terms of a.

$$\begin{aligned}\mathrm{Op}(a)(\partial_j u) - \partial_j(\mathrm{Op}(a)(u)) &= \frac{1}{(2\pi)^n}\int e^{ix\xi}a(x,\xi)(i\xi_j)\hat{u}(\xi)d\xi - \frac{1}{(2\pi)^n}\int \frac{\partial}{\partial x_j}(e^{ix\xi}a(x,\xi))\hat{u}(\xi)d\xi \\ &= \frac{-1}{(2\pi)^n}\int e^{ix\xi}\frac{\partial}{\partial x_j}a(x,\xi)\hat{u}(\xi)d\xi\end{aligned}$$

In other words

$$[\mathrm{Op}(a),\partial_j] = \mathrm{Op}(-\partial_j a),$$

which can be tested on differential operators.

1.2. Same question for $[\mathrm{Op}(a),x_j]$ where x_j is the multiplication operator by x_j.

$$\begin{aligned}\mathrm{Op}(a)(x_j u) - x_j\mathrm{Op}(a)(u) &= \frac{i}{(2\pi)^n}\int e^{ix\xi}a(x,\xi)\partial_j\hat{u}(\xi)d\xi - \frac{1}{(2\pi)^n}\int x_j e^{ix\xi}a(x,\xi)\hat{u}(\xi)d\xi \\ &= \frac{-i}{(2\pi)^n}\int \frac{\partial}{\partial\xi_j}\left(e^{ix\xi}a(x,\xi)\right)\hat{u}(\xi)d\xi - \frac{1}{(2\pi)^n}\int x_j e^{ix\xi}a(x,\xi)\hat{u}(\xi)d\xi \\ &= \frac{-i}{(2\pi)^n}\int e^{ix\xi}\frac{\partial}{\partial\xi_j}a(x,\xi)\hat{u}(\xi)d\xi\end{aligned}$$

In other words

$$[\mathrm{Op}(a),x_j] = \mathrm{Op}(-i\partial_{\xi_j}a),$$

which again can be tested on differential operators.

Exercice 2 [About the localization of the wave front set]**.** Let $u \in \mathcal{E}'(\mathbb{R}^n)$ be a compactly supported distribution. We say that a direction $\xi \neq 0$ is in $\Upsilon(u)$ when there exists a conic neighborhood $\mathcal{C}_1$ of ξ such that $\hat{u}$ is rapidly decreasing inside $\mathcal{C}_1$. The complement of $\Upsilon(u)$ is denoted by $\Sigma(u) := \Upsilon(u)^c$. In what follows, we fix some $\xi \neq 0$ inside $\Upsilon(u)$.

2.0. Explain the sense of the sentence "*$\hat{u}$ is of at most polynomial growth* ", and then recall why $\hat{u}$ is a smooth function of at most polynomial growth.

This means that

$$\exists\, p \in \mathbb{N}\,, \qquad |\hat{u}(\zeta)| \leq C\,\langle \zeta \rangle^p\,.$$

In the present case, since u is compactly supported, we have

$$\hat{u}(\zeta) = \langle u, e^{i\zeta\cdot} \rangle_{\mathcal{E}',\mathcal{E}} = \langle u, \chi\, e^{i\zeta\cdot} \rangle_{\mathcal{E}',\mathcal{E}}\,, \qquad \mathcal{C}_c^\infty(\mathbb{R}^n) \ni \chi \equiv 1 \text{ on } \operatorname{supp} u\,.$$

On the other hand, since u is a tempered distribution, we can find some $p \in \mathbb{N}$ *such that*

$$|\langle u, \chi\, e^{i\zeta\cdot} \rangle_{\mathcal{E}',\mathcal{E}}| \leq \mathcal{N}_p(\chi\, e^{i\zeta\cdot})\,, \qquad \mathcal{N}_p(\varphi) := \sup_{|\alpha|,|\beta|\leq p} \parallel x^\alpha\, \partial_x^\beta \varphi \parallel_\infty\,.$$

Now, with R such that $\operatorname{supp} \chi \subset B(0,R]$, *it suffices to remark that*

$$\parallel x^\alpha\, \partial_x^\beta (\chi\, e^{i\zeta x}) \parallel_\infty \lesssim R^p\, \langle \zeta \rangle^p.$$

2.1. Explain the sense of the sentence "$\hat{u}$ *is rapidly decreasing inside* $\mathcal{C}_1$".

With $\langle \zeta \rangle := (1+ \parallel \zeta \parallel^2)^{1/2}$, *this means that*

$$\forall\, N \in \mathbb{N}\,, \quad \exists\, C_N\,; \quad \forall\, \zeta \in \mathcal{C}_1\,, \quad \parallel \hat{u}(\zeta) \parallel \leq C_N\, \langle \zeta \rangle^{-N}\,. \tag{1}$$

2.2. Prove that there is a conic neighborhood $\mathcal{C}_2$ of ξ and a constant $c \in]0,1[$ such that

$$\forall\, \eta \in \mathcal{C}_2\,, \quad \parallel \eta - \zeta \parallel \leq c \parallel \eta \parallel \implies \zeta \in \mathcal{C}_1\,.$$

Indication : interpret the condition in terms of $\check{\xi} := \dfrac{\xi}{\parallel \xi \parallel}$, $\tilde{\eta} := \dfrac{\eta}{\parallel \eta \parallel}$ *and* $\tilde{\zeta} := \dfrac{\zeta}{\parallel \eta \parallel}$.

We have $\zeta \in \mathcal{C}_1$ *whenever* $\tilde{\zeta} \in \mathcal{C}_1$. *Thus, the statement can be reformulated as the existence of a conic neighborhood* $\mathcal{C}_2$ *of* $\check{\xi}$ *(which is a conic neighborhood of* ξ*) such that*

$$\forall\, \tilde{\eta} \in \mathcal{C}_2 \cap B(0,1]\,, \quad \parallel \tilde{\eta} - \tilde{\zeta} \parallel \leq c \implies \tilde{\zeta} \in \mathcal{C}_1\,.$$

Since $\mathcal{C}_1$ *is open, we can find some* $c \in]0,1[$ *such that* $B(\check{\xi}, 2c] \subset \mathcal{C}_1$. *Define* $\mathcal{C}_2$ *as the conic neighborhood generated by* $B(\check{\xi}, c[$ *so that* $\tilde{\eta} \in \mathcal{C}_2 \cap B(0,1]$ *means that* $\parallel \tilde{\eta} - \check{\xi} \parallel < c$. *Then*

$$\left.\begin{array}{l} \parallel \tilde{\zeta} - \check{\xi} \parallel \leq \parallel \tilde{\zeta} - \eta \parallel + \parallel \tilde{\eta} - \check{\xi} \parallel \leq \parallel \tilde{\zeta} - \eta \parallel + c \\ \parallel \tilde{\eta} - \tilde{\zeta} \parallel \leq c \end{array}\right\} \implies \parallel \tilde{\zeta} - \check{\xi} \parallel \leq 2c\,,$$

which implies that $\tilde{\zeta} \in B(\check{\xi}, 2c] \subset \mathcal{C}_1$.

2.3. Let ϕ be in the Schwartz space $\mathcal{S}(\mathbb{R}^n)$.

2.3.1. Prove and give a sense to the formula $\widehat{\phi u}(\eta) = F(\eta) + G(\eta)$ where

$$F(\eta) := \int_{\parallel \eta - \zeta \parallel \leq c \parallel \eta \parallel} \hat{\phi}(\eta - \zeta)\, \hat{u}(\zeta)\, d\zeta\,, \qquad G(\eta) := \int_{\parallel \eta - \zeta \parallel \geq c \parallel \eta \parallel} \hat{\phi}(\eta - \zeta)\, \hat{u}(\zeta)\, d\zeta\,.$$

We have $\widehat{\phi u} = \hat{\phi} \star \hat{u}$ in the sense of distributions. Since $\hat{\phi} \in \mathcal{S}(\mathbb{R}^n)$, the value of $\widehat{\phi u}(\eta)$ is well defined according to

$$\widehat{\phi u}(\eta) = \langle \hat{u}(\cdot), \hat{\phi}(\eta - \cdot) \rangle_{\mathcal{S}', \mathcal{S}}. \tag{2}$$

The Fourier transform of a distribution with compact support (like in the case of u) is (see the question **2.0***) a smooth function of at most polynomial growth (the same applies to the derivatives), that is*

$$\exists\, p \in \mathbb{N}, \quad |\hat{u}(\zeta)| = |\langle u, e^{i\zeta \cdot} \rangle_{\mathcal{E}', \mathcal{E}}| \leq C \, \langle \zeta \rangle^p \, . \tag{3}$$

On the other hand, $\hat{\phi}$ is in the Schwartz space $\mathcal{S}(\mathbb{R}^n)$. This property is conserved under the action of a (fixed) translation : $\hat{\phi}(\eta - \cdot)$ is still in the Schwartz space $\mathcal{S}(\mathbb{R}^n)$. The growth (3) is compensated by the decreasing of $\hat{\phi}(\eta - \cdot)$ so that the product $\hat{\phi}(\eta - \cdot)\, \hat{u}(\cdot)$ is in $L^1(\mathbb{R}^n)$, and the dual product (2) can be interpreted as a usual integration which can then be separated into the above two integrals.

2.3.2. Prove that F is rapidly decreasing on $\mathcal{C}_2$.

The condition $\| \eta - \zeta \| \leq c \| \eta \|$ implies that

$$\| \zeta \| \geq \| \eta \| - \| \eta - \zeta \| \geq (1 - c) \ \| \eta \| \ .$$

In view of the question 2.2, knowing that $\eta \in \mathcal{C}_2$, it also means that $\zeta \in \mathcal{C}_1$ so that we can use (1) to obtain

$$|F(\eta)| \leq C_N \int_{\|\eta - \zeta\| \leq c \|\eta\|} |\hat{\phi}(\eta - \zeta)| \, (1 - c)^{-N} \, \langle \eta \rangle^{-N} \, d\zeta \leq \tilde{C}_N \ \langle \eta \rangle^{-N}.$$

with

$$\tilde{C}_N := \frac{C_N}{(1 - c)^N} \ \| \hat{\phi} \|_{L^1(\mathbb{R}^n)} \ .$$

This holds true for all $N \in \mathbb{N}$ and for all $\eta \in \mathcal{C}_2$, which gives the expected result.

2.3.3. By using Peetre's inequality

$$\forall\, t \in \mathbb{R}, \qquad \langle \eta \rangle^t \leq 2^{|t|} \, \langle \zeta \rangle^t \, \langle \eta - \zeta \rangle^{|t|} \, ,$$

prove that G is rapidly decreasing.

Since $\hat{\phi}$ is rapidly decreasing and due to (3), we can assert that

$$\forall\, N \in \mathbb{N}, \qquad |G(\eta)| \lesssim \int_{\|\eta - \zeta\| \geq c \|\eta\|} \langle \eta - \zeta \rangle^{-N} \, \langle \zeta \rangle^p \, d\zeta \, .$$

We take N in the form $N = n + 1 + p + q$ with q large. This becomes

$$\forall\, q \in \mathbb{N}, \qquad |G(\eta)| \lesssim \int_{\|\eta - \zeta\| \geq c \|\eta\|} \langle \eta - \zeta \rangle^{-q} \, \langle \eta - \zeta \rangle^{-n-1} \, \Big(\frac{\langle \zeta \rangle}{\langle \eta - \zeta \rangle} \Big)^p \, d\zeta \, .$$

On the domain of integration, we have $\langle \eta - \zeta \rangle^{-1} \leq c^{-1} \langle \eta \rangle^{-1}$ as well as (using Peetre's inequality with $t = -1$)

$$\frac{\langle \zeta \rangle}{\langle \eta - \zeta \rangle} \leq 2 \, \langle \eta \rangle \, .$$

There remains

$$\forall \, q \in \mathbb{N} \, , \qquad |G(\eta)| \lesssim \int_{\|\eta - \zeta\| \geq c \|\eta\|} c^{-q} \, \langle \eta \rangle^{-q} \, \langle \eta - \zeta \rangle^{-n-1} \, 2^p \, \langle \eta \rangle^p \, d\zeta \lesssim \langle \eta \rangle^{p-q} \, .$$

Just take $q = N + p$ with any $N \in \mathbb{N}$.

2.3.4. Show that $\Sigma(\phi u) \subset \Sigma(u)$.

From questions 2.2.1, 2.2.2 and 2.2.3, we can infer that $\widehat{\phi u}$ is rapidly decreasing on $\mathcal{C}_2$ which implies that $\xi \in \Upsilon(\phi u)$. This is verified for all $\xi \in \Upsilon(u)$ so that $\Upsilon(u) \subset \Upsilon(\phi u)$ which, passing to the complement, is equivalent to $\Sigma(\phi u) \subset \Sigma(u)$.

2.3.5. Let $\chi \in \mathcal{D}(\mathbb{R}^n)$, $\psi \in C^\infty(\mathbb{R}^n)$ and $v \in \mathcal{D}'(\mathbb{R}^n)$. Prove that $\Upsilon(\chi v) \subset \Upsilon(\psi \chi v)$.

We cannot use directly the question 2.3.4 because $\psi \notin \mathcal{S}(\mathbb{R}^n)$. However, we can find some $\tilde{\chi} \in \mathcal{D}(\mathbb{R}^n)$ such that $\tilde{\chi}$ is equal to one on the support of χ so that $\chi v \equiv \tilde{\chi} \chi v \in \mathcal{E}'(\mathbb{R}^n)$. Since $\tilde{\chi} \psi \in \mathcal{S}(\mathbb{R}^n)$, from the question 2.2.4, we can assert that

$$\Upsilon(\chi v) \subset \Upsilon\big((\tilde{\chi} \psi) \chi v\big) \equiv \Upsilon\big(\psi (\tilde{\chi} \chi) v\big) \equiv \Upsilon(\psi \chi v) \, .$$

2.4. Below, the symbol "*WF*" is for "*Wave Front set*". From the foregoing, deduce that

$$\forall \, \psi \in C^\infty(\mathbb{R}^n) \, , \qquad \forall \, v \in \mathcal{D}'(\mathbb{R}^n) \, , \qquad WF(\psi v) \subset WF(v) \, .$$

It suffices to show that

$$(x, \xi) \notin WF(v) \implies (x, \xi) \notin WF(\psi \, v) \, .$$

Fix some $(x, \xi) \notin WF(v)$. By definition, we can find some cutoff function $\chi \in \mathcal{D}(\mathbb{R}^n)$ such that $\chi(x) \neq 0$ and such that $\widehat{\chi v}$ is rapidly decreasing in a conic neighborhood of ξ. In other words, $\xi \in \Upsilon(\chi v) \subset \Upsilon(\psi \chi v) \equiv \Upsilon\big(\chi (\psi v)\big)$. It follows that $(x, \xi) \notin WF(\psi \, v)$.

Exercice 3 [About the square root of an elliptic operator]**.** Let a be a symbol which is in $S^m_{1,0}(\mathbb{R}^n; \mathbb{R})$ with $m \in \mathbb{R}$ and $n \in \mathbb{N}$. We assume that

$$\exists \, (c, R) \in (\mathbb{R}^*_+)^2 \, ; \qquad a(x, \xi) \geq c \, (1 + \| \, \xi \, \|^2)^{m/2} \quad \text{if} \quad \| \, \xi \, \| \geq R \, .$$

This is a classical proof (almost done during the course) using the symbolic calculus.

3.1. Prove that we can find an elliptic operator $b_0 \in S^{(m/2)}_{1,0}(\mathbb{R}^n)$ such that

$$Op(a) - Op(b_0) \circ Op(b_0) \in S^{m-1}_{1,0}(\mathbb{R}^n) \, .$$

Let $\chi \in C^\infty(\mathbb{R}^n)$ be such that

$$\chi(\xi) = \begin{cases} 0 & \text{if} \quad |\xi| \leq R, \\ 1 & \text{if} \quad 2\,R \leq |\xi|. \end{cases}$$

Take $b_0(x,\xi) := \sqrt{a}(x,\xi)\,\chi(\xi)$. Remark that a can take negative values for $|\xi| \leq R$. Thus, it is important here to localize out of the ball of radius R to be sure that $\sqrt{a}$ is well defined. It is clear that $b_0 \in S^{m/2}_{1,0}(\mathbb{R}^n;\mathbb{R}^*_+)$ and that b_0 is elliptic since

$$b_0(x,\xi) = \sqrt{a}(x,\xi) \geq \sqrt{c}\,\left(1+ \parallel \xi \parallel^2\right)^{m/4} \quad \text{if} \quad \parallel \xi \parallel \geq 2\,R\,.$$

On the other hand

$$Op(a) - Op(b_0) \circ Op(b_0) = Op(a - b_0^2) + Op\big(S^{m-1}_{1,0}(\mathbb{R}^n)\big).$$

By construction, we have $a - b_0^2 = a\ (1-\chi^2)$ so that

$$\partial_x^\alpha \partial_\xi^\beta a(x,\xi) = \sum_{\gamma \leq \beta} C^\gamma_\beta\ \partial_x^\alpha \partial_\xi^\gamma a(x,\xi)\ \partial_\xi^{\beta-\gamma}(1-\chi^2)(\xi),$$

which is a sum of products of functions in $\in S^{m-|\gamma|}_{1,0}(\mathbb{R}^n)$ and functions in $\in S^{-\infty}_{1,0}(\mathbb{R}^n)$ (because $\chi \equiv 1$ for large $|\xi|$). This implies that

$$Op(a) - Op(b_0) \circ Op(b_0) \in Op\big(S^{-\infty}_{1,0}(\mathbb{R}^n)\big) + Op\big(S^{m-1}_{1,0}(\mathbb{R}^n)\big) \subset Op\big(S^{m-1}_{1,0}(\mathbb{R}^n)\big).$$

3.2. We fix some $N \in \mathbb{N}$ with $N \geq 2$. Show by induction that we can find symbols $b_k \in S^{(m/2)-k}_{1,0}(\mathbb{R}^n)$ with $0 \leq k \leq N$ which are adjusted such that

$$Op(a) - Op(b_0 + \cdots + b_N) \circ Op(b_0 + \cdots + b_N) \in S^{m-N-1}_{1,0}(\mathbb{R}^n)\,.$$

By induction, with $b' = b_0 + \cdots + b_k$, we can start with

$$Op(a) - Op(b') \circ Op(b') = Op(c)\,, \qquad c \in Op\big(S^{m-k-1}_{1,0}(\mathbb{R}^n)\big)\,.$$

It follows in particular that

$$R = Op(r) := Op(a) - Op(b' \sharp b') \in Op\big(S^{m-k-1}_{1,0}(\mathbb{R}^n)\big)\,.$$

Due to the elliptic property of $b_0 \in S^{(m/2)}_{1,0}(\mathbb{R}^n)$, we know that

$$|b_0(x,\xi)| \geq \sqrt{c}\,\left(1+ \parallel \xi \parallel^2\right)^{m/4} \quad \text{if} \quad \parallel \xi \parallel \geq 2\,R.$$

To obtain b', we add to b_0 more decreasing symbols b_j (when $j \geq 1$). As a consequence

$$\exists\,(c',R') \in (\mathbb{R}^*_+)^2\,; \qquad |b'(x,\xi)| \geq c'\,\left(1+ \parallel \xi \parallel^2\right)^{m/4} \quad \text{if} \quad \parallel \xi \parallel \geq R'\,.$$

We seek $b_{k+1} \in S_{1,0}^{(m/2)-k-1}(\mathbb{R}^n)$ *such that*

$$Op(a) - Op(b' + b_{k+1}) \circ Op(b' + b_{k+1}) \in Op\big(S_{1,0}^{m-k-2}(\mathbb{R}^n)\big)\,,$$

or equivalently such that

$$r - 2\, b'\, b_{k+1} + b_{k+1}^2 \in S_{1,0}^{m-k-2}(\mathbb{R}^n)\,.$$

Since $b_{k+1}^2 \in S_{1,0}^{m-2k-2}(\mathbb{R}^n) \subset S_{1,0}^{m-k-2}(\mathbb{R}^n)$, *this is the same as*

$$r - 2\, b'\, b_{k+1} = g \in S_{1,0}^{m-k-2}(\mathbb{R}^n)\,.$$

For $\parallel \xi \parallel \geq R'$, *it suffices to take* $b_{k+1} = (r-g)/2b'$ *and extend this function smoothly for smaller values of* $\parallel \xi \parallel$ *to recover some symbol* $b_{k+1} \in S_{1,0}^{(m/2)-k-1}(\mathbb{R}^n)$ *allowing to recover the expected property.*

Problème [About the canonical commutation relations]**.** We consider two unbounded self-adjoint operators A and B on the Hilbert space $\mathcal{H}$ satisfying the exponentiated commutation relation

$$(ECR) \qquad \forall\,(s,t) \in \mathbb{R}^2\,, \qquad e^{isA}\, e^{itB} = e^{-ist\hbar}\, e^{itB}\, e^{isA}\,,$$

where $\hbar$ is the reduced Planck constant. In what follows, we consider a function f which is in the Schwarz space $\mathcal{S}(\mathbb{R}^2)$ and which is real valued. We denote by $\hat{f}$ its Fourier transform. We define the bounded operator $Q(f)$ by the formula

$$Q(f) := \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \hat{f}(s,t)\; U(s,t)\; ds\, dt\,.$$

P.1. Define $U(s,t) := e^{ist\hbar/2}\, e^{isA}\, e^{itB}$. Prove that

$$(CCR) \qquad \forall\,(s,t,s',t') \in \mathbb{R}^4\,, \quad U(s,t)\, U(s',t') = e^{-i\hbar(st'-ts')/2}\, U(s+s',t+t')\,.$$

It suffices to apply the ECR to find

$$\begin{aligned} U(s,t)\, U(s',t') &= e^{i(st+s't')\hbar/2}\, e^{isA}\, \big(e^{itB}\, e^{is'A}\big)\, e^{it'B} \\ &= e^{i(st+s't')\hbar/2}\, e^{isA}\, \big(e^{is't\hbar}\, e^{is'A}\, e^{itB}\big)\, e^{it'B} \\ &= e^{i(st+2s't+s't')\hbar/2}\, e^{i(s+s')A}\, e^{i(t+t')B} \\ &= e^{i(st+2s't+s't')\hbar/2}\, e^{-i(s+s')(t+t')\hbar/2}\, U(s+s',t+t')\,. \end{aligned}$$

P.2. Show that $U(s,t)^* = U(-s,-t)$ (where the star $*$ is for the adjoint operation).

There are two possible proofs. Either, we can use the ECR to see that

$$U(s,t)^* = e^{-ist\hbar/2}\, e^{-itB}\, e^{-isA} = e^{-ist\hbar/2}\, e^{ist\hbar}\, e^{-isA}\, e^{-itB} = U(-s,-t)\,.$$

Or we can remark that $U(s,t)$ is by construction a unitary operator whose inverse is the adjoint. Now, from the CCR, we have directly access to $U(s,t)\,U(-s,-t) = Id$.

P.3. Recall that f is real valued. Explain why $Q(f)$ is well defined and self-adjoint.

Since $U(s,t)$ is a unitary operator and f is in $\mathcal{S}(\mathbb{R}^2)$, we can find some constant C such that $\| \hat{f}(s,t)\ U(s,t) \| \leq (1+s^2+t^2)^{-2}$ where $\| \cdot \|$ is for the norm operator. The integral is then absolutely convergent (in the sense of a Bochner integral), and we can compute

$$Q(f) - Q(f)^* = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} [\hat{f}(s,t)\ U(s,t) - \bar{\hat{f}}(s,t)\ U(s,t)^*]\ ds\, dt\,.$$

Since f is real valued, we have $\bar{\hat{f}}(s,t) = f(-s,-t)$. From question P.2, we get

$$Q(f) - Q(f)^* = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} [\hat{f}(s,t)\ U(s,t) - \hat{f}(-s,-t)\ U(-s,-t)]\ ds\, dt = 0\,,$$

just by changing (s,t) into $(-s,-t)$ in the second integral.

P.4. Prove that $U(s,t)\,Q(f) := Q(f')$ where the function f' is defined by its Fourier transform which is given by

$$\hat{f}'(s',t') := e^{i\hbar(s't-st')/2}\, \hat{f}(s'-s,t'-t).$$

From the CCR, we have

$$\begin{aligned} U(s,t)\,Q(f) &= \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \hat{f}(s',t')\ U(s,t)\,U(s',t')\ ds'\,dt' \\ &= \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \hat{f}(s',t')\ e^{-i\hbar(st'-ts')/2}\, U(s+s',t+t')\ ds'\,dt' \\ &= \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \hat{f}(s''-s,t''-t)\ e^{-i\hbar(s(t''-t)-t(s''-s))/2}\, U(s'',t'')\ ds''\,dt'' \\ &= \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{-i\hbar(s''t-st'')/2}\ \hat{f}'(s'',t'')\ e^{-i\hbar(s(t''-t)-t(s''-s))/2}\, U(s'',t'')\ ds''\,dt''. \end{aligned}$$

After simplification of the exponential factors, we can recognize $Q(f')$.

P.5. Prove that we have

$$U(s,t)^*\,Q(f)\,U(s,t) = U(-s,-t)\,Q(f)\,U(s,t) = Q(g)$$

where the function g is such that $\hat{g}(s',t') = e^{i\hbar(s't-st')}\,\hat{f}(s',t')$.

Exchanging the role of f and g and using question P.4., the above relation is equivalent to

$$U(s,t)\,Q(f)\,U(-s,-t) = Q(f')\,U(-s,-t) = Q(g)\,.$$

It suffices to show that

$$(\sharp) \qquad Q(f')\,U(-s,-t) = Q(f'')\,, \qquad \widehat{f''}(s',t') := e^{i\hbar(s't-st')/2}\,\widehat{f'}(s'+s,t'+t)$$

to recover that

$$\widehat{f''}(s',t') := e^{i\hbar(s't-st')/2}\, e^{i\hbar((s'+s)t-s(t'+t))/2}\, \widehat{f'}(s',t') = e^{i\hbar(s't-st')}\, \hat{f}(s',t')$$

as expected. Now, the proof of (♯) follows the same lines as in question P.4.

P.6. Explain why we have $Q(f)\,Q(g) = Q(f\star g)$ for all $(f,g)\in\mathcal{S}(\mathbb{R}^2)$ where $f\star g$ is the Moyal product described by

$$\widehat{f\star g}(s,t) := \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} e^{-i\hbar(st'-ts')/2}\, \hat{f}(s-s',t-t')\ \hat{g}(s',t')\ ds'\,dt'.$$

We have

$$\begin{aligned} Q(f\star g) &= \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2}\Big(\frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} e^{-i\hbar(st'-ts')/2}\, \hat{f}(s-s',t-t')\ \hat{g}(s',t')\ ds'\,dt'\Big)\, U(s,t)\ ds\,dt \\ &= \frac{1}{(2\pi)^4}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{-i\hbar((s'+s'')t'-(t'+t'')s')/2}\, \hat{f}(s'',t'')\ \hat{g}(s',t')\, U(s'+s'',t'+t'')\ ds'\,dt'\,ds''\,dt''. \end{aligned}$$

We can exploit the CCR in the form

$$U(s'+s'',t'+t'') = e^{i\hbar(s''t'-t''s')/2}\, U(s'',t'')\, U(s',t')\,,$$

to obtain

$$\begin{aligned} Q(f\star g) &= \frac{1}{(2\pi)^4}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \hat{f}(s'',t'')\ \hat{g}(s',t')\, U(s'',t'')\, U(s',t')\ ds'\,dt'\,ds''\,dt'' \\ &= \Big(\frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} \hat{f}(s'',t'')\, U(s'',t'')\, ds''\,dt''\Big)\Big(\frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} \hat{g}(s',t')\, U(s',t')\, ds'\,dt'\Big) \\ &= Q(f)\,Q(g)\,. \end{aligned}$$

P.7. Let ϕ and ψ in $\mathcal{H}$ as well as s and t in $\mathbb{R}$. We assume that f is such that $Q(f)=0$. By exploiting the relation

$$0 = \langle U(s,t)\,\phi, Q(f)\,U(s,t)\,\psi\rangle\,,$$

show that the operator Q is injective on $\mathcal{S}(\mathbb{R}^2)$.

With g as in question P.5, we must have

$$0 = \langle U(s,t)\,\phi, Q(f)\,U(s,t)\,\psi\rangle = \langle \phi, U(-s,-t)\,Q(f)\,U(s,t)\,\psi\rangle = \langle\phi, Q(g)\,\psi\rangle\,.$$

In view of the definition of g, this is the same as

$$0 = \int_{\mathbb{R}^2} e^{i\hbar(s't-st')}\ \hat{f}(s',t')\ \langle\phi, U(s',t')\psi\rangle\ ds'\,dt'\,.$$

We can recognize above the Fourier transform of the continuous function

$$F(s',t') := \hat{f}(s',t')\ \langle\phi, U(s',t')\psi\rangle$$

evaluated at the point $\hbar\,(-t,s)$. This must be zero for all values of (s,t). By Fourier inversion formula, this is possible if and only if F is zero at all positions (s',t'). Now, for $\phi = U(s',t')\psi$ with $\parallel\psi\parallel=1$, we find that

$$0 = F(s',t') = \hat{f}(s',t')\ \langle U(s',t')\psi, U(s',t')\psi\rangle = \hat{f}(s',t')\ \langle\psi,\psi\rangle = \hat{f}(s',t')\,,$$

and therefore $f=0$.