UNIVERSITÉ DE RENNES 1

Microlocal Analysis

Terminal Examination - the 12/12/2022 (2h)

Documents are not allowed

Exercice 1 [About the basic rules of pseudo-differential calculus]**.** Let $m \in \mathbb{Z}$ and let $a(x,\xi) \in S^m_{1,0}(\mathbb{R}^n)$ be a symbol of order m. We recall that the action of the pseudo-differential operator $\mathrm{Op}(a) \equiv a(x,D)$ is given by

$$\mathrm{Op}(a)u(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix\cdot\xi}\, a(x,\xi)\, \hat{u}(\xi)\, d\xi\,.$$

1.1. We denote by $[\mathrm{Op}(a), \partial_j]$ the commutator of $\mathrm{Op}(a)$ with the partial derivative with respect to the $j^{\text{èm}}$ direction. Prove that $[\mathrm{Op}(a), \partial_j]$ is a pseudo-differential operator and compute its symbol in terms of a.

1.2. Same question for $[\mathrm{Op}(a), x_j]$ where x_j is the multiplication operator by x_j.

Exercice 2 [About the localization of the wave front set]**.** Let $u \in \mathcal{E}'(\mathbb{R}^n)$ be a compactly supported distribution. We say that a direction $\xi \neq 0$ is in $\Upsilon(u)$ when there exists a conic neighborhood $\mathcal{C}_1$ of ξ such that $\hat{u}$ is rapidly decreasing inside $\mathcal{C}_1$. The complement of $\Upsilon(u)$ is denoted by $\Sigma(u) := \Upsilon(u)^c$. In what follows, we fix some $\xi \neq 0$ inside $\Upsilon(u)$.

2.0. Explain the sense of the sentence "*$\hat{u}$ is of at most polynomial growth* ", and then recall why $\hat{u}$ is a smooth function of at most polynomial growth.

2.1. Explain the sense of the sentence "*$\hat{u}$ is rapidly decreasing inside $\mathcal{C}_1$*".

2.2. Prove that there is a conic neighborhood $\mathcal{C}_2$ of ξ and a constant $c \in]0,1[$ such that

$$\forall\, \eta \in \mathcal{C}_2\,, \quad \| \eta - \zeta \| \leq c \| \eta \| \implies \zeta \in \mathcal{C}_1\,.$$

Indication : interpret the condition in terms of

$$\check{\xi} := \frac{\xi}{\| \xi \|}\,, \qquad \tilde{\eta} := \frac{\eta}{\| \eta \|}\,, \qquad \tilde{\zeta} := \frac{\zeta}{\| \eta \|}.$$

2.3. Let ϕ be in the Schwartz space $\mathcal{S}(\mathbb{R}^n)$.

2.3.1. Prove and give a sense to the formula $\widehat{\phi u}(\eta) = F(\eta) + G(\eta)$ where

$$F(\eta) := \int_{\|\eta-\zeta\| \leq c\|\eta\|} \hat{\phi}(\eta-\zeta)\, \hat{u}(\zeta)\, d\zeta\,, \qquad G(\eta) := \int_{\|\eta-\zeta\| \geq c\|\eta\|} \hat{\phi}(\eta-\zeta)\, \hat{u}(\zeta)\, d\zeta\,.$$

2.3.2. Prove that F is rapidly decreasing on $\mathcal{C}_2$.

2.3.3. By using Peetre's inequality

$$\forall\, t \in \mathbb{R}\,, \qquad \langle \eta \rangle^t \leq 2^{|t|}\, \langle \zeta \rangle^t\, \langle \eta - \zeta \rangle^{|t|}\,,$$

prove that G is rapidly decreasing.

2.3.4. Show that $\Sigma(\phi u) \subset \Sigma(u)$.

2.3.5. Let $\chi \in \mathcal{D}(\mathbb{R}^n)$, $\psi \in C^\infty(\mathbb{R}^n)$ and $v \in \mathcal{D}'(\mathbb{R}^n)$. Prove that $\Upsilon(\chi v) \subset \Upsilon(\psi \chi v)$.

2.4. Below, the symbol "*WF*" is for "*Wave Front set*". From the foregoing, deduce that

$$\forall\, \psi \in C^\infty(\mathbb{R}^n)\,, \qquad \forall\, v \in \mathcal{D}'(\mathbb{R}^n)\,, \qquad WF(\psi v) \subset WF(v)\,.$$

Exercice 3 [About the square root of an elliptic operator]**.** Let a be a symbol which is in $S^m_{1,0}(\mathbb{R}^n; \mathbb{R}^*_+)$ with $m \in \mathbb{R}$ and $n \in \mathbb{N}$. We assume that

$$\exists\, (c, R) \in (\mathbb{R}^*_+)^2\,; \qquad a(x,\xi) \geq c\,(1+ \parallel \xi \parallel^2)^{m/2} \quad \text{if} \quad \parallel \xi \parallel \geq R\,.$$

3.1. Prove that we can find an elliptic operator $b_0 \in S^{(m/2)}_{1,0}(\mathbb{R}^n)$ such that

$$Op(a) - Op(b_0) \circ Op(b_0) \in S^{m-1}_{1,0}(\mathbb{R}^n)\,.$$

3.2. We fix some $N \in \mathbb{N}$ with $N \geq 2$. Show by induction that we can find symbols $b_k \in S^{(m/2)-k}_{1,0}(\mathbb{R}^n)$ with $0 \leq k \leq N$ which are adjusted such that

$$Op(a) - Op(b_0 + \cdots + b_N) \circ Op(b_0 + \cdots + b_N) \in S^{m-N-1}_{1,0}(\mathbb{R}^n)\,.$$

Problème [About the canonical commutation relations]**.** We consider two unbounded self-adjoint operators A and B on the Hilbert space $\mathcal{H}$ satisfying the exponentiated commutation relation

$$(ECR) \qquad\qquad \forall\, (s,t) \in \mathbb{R}^2\,, \qquad e^{isA}\, e^{itB} = e^{-ist\hbar}\, e^{itB}\, e^{isA}\,,$$

where $\hbar$ is the reduced Planck constant. In what follows, we consider a function f which is in the Schwarz space $\mathcal{S}(\mathbb{R}^2)$ and which is real valued. We denote by $\hat{f}$ its Fourier transform. We define $U(s,t) := e^{ist\hbar/2}\, e^{isA}\, e^{itB}$ together with the bounded operator $Q(f)$ by the formula

$$Q(f) := \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \hat{f}(s,t)\; U(s,t)\; ds\, dt\,.$$

P.1. Prove that

$$(CCR) \qquad \forall\, (s,t,s',t') \in \mathbb{R}^4\,, \quad U(s,t)\, U(s',t') = e^{-i\hbar(st'-ts')/2}\, U(s+s', t+t')\,.$$

$\Longrightarrow$ T.S.V.P.

P.2. Show that $U(s,t)^* = U(-s,-t)$ (where the star $*$ is for the adjoint operation).

P.3. Recall that f is real valued. Explain why $Q(f)$ is well defined and self-adjoint.

P.4. Prove that $U(s,t)\,Q(f) := Q(f')$ where the function f' is defined by its Fourier transform which is given by

$$\hat{f'}(s',t') := e^{i\hbar(s't-st')/2}\,\hat{f}(s'-s,t'-t).$$

P.5. Prove that we have

$$U(s,t)^*\,Q(f)\,U(s,t) = U(-s,-t)\,Q(f)\,U(s,t) = Q(g)$$

where the function g is such that $\hat{g}(s',t') = e^{i\hbar(s't-st')}\,\hat{f}(s',t')$.

P.6. Explain why we have $Q(f)\,Q(g) = Q(f\star g)$ for all $(f,g)\in\mathcal{S}(\mathbb{R}^2)$ where $f\star g$ is the Moyal product described by

$$\widehat{f\star g}(s,t) := \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} e^{-i\hbar(st'-ts')/2}\,\hat{f}(s-s',t-t')\;\hat{g}(s',t')\;ds'\,dt'.$$

P.7. Let ϕ and ψ in $\mathcal{H}$ as well as s and t in $\mathbb{R}$. We assume that f is such that $Q(f)=0$. By exploiting the relation

$$0 = \langle U(s,t)\,\phi, Q(f)\,U(s,t)\,\psi\rangle\,,$$

show that the operator Q is injective on $\mathcal{S}(\mathbb{R}^2)$.