UNIVERSITÉ DE RENNES 1

Spectral Theory

Terminal Examination (2h)

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Exercise 1. Let H be a Hilbert space. We consider an operator $T \in \mathcal{L}(H)$. A complex number λ is said to be :

- $\star$ in the *spectrum* if $T - \lambda Id$ is not invertible ; the spectrum of T is denoted by $\sigma(T)$.
- $\star$ in the *point spectrum* if $T - \lambda Id$ is not injective ; the point spectrum of T is denoted by $\sigma_p(T)$.
- $\star$ An *approached eigenvalue* if there exists a sequence $(x_n) \in H^{\mathbb{N}}$ with $\| x_n \| = 1$ and

$$\lim_{n \to +\infty} \| (T - \lambda Id) x_n \| = 0 \,.$$

The set of approached eigenvalues is denoted by $\sigma_{ap}(T)$.

We recall that the resolvent set $\rho(T)$ is defined by $\rho(T) = \mathbb{C} \setminus \sigma(T)$. Thus, given $\lambda \in \rho(T)$, the operator $T - \lambda Id$ is invertible with inverse denoted by $R_\lambda(T) \equiv (T - \lambda Id)^{-1}$.

1.1. Show that $\sigma_p(T) \subset \sigma_{ap}(T)$.

1.2. Show that $\sigma_{ap}(T) \subset \sigma(T)$.

1.3. Fix $\lambda \in \mathbb{C}$, and consider the following assertions :

(i) $\lambda \notin \sigma_{ap}(T)$.

(ii) There exists a constant $c \in \mathbb{R}_+^*$ such that, for all $x \in H$, we have

$$\| (T - \lambda Id) x \| \geq c \, \| x \| \;.$$

(iii) The operator $T - \lambda Id$ is injective with a closed range.

The purpose of this question 1.3 is to show step by step that the three above assertions are equivalent, and then to deduce some decomposition of $\sigma(T)$.

1.3.1 Prove by contradiction that (i) implies (ii).

1.3.2 Prove that (ii) implies (iii).

1.3.3 Prove that (iii) implies (i).

1.3.4 Deduce that $\sigma(T)$ can be viewed as the following disjoint union

$$\sigma(T) = \sigma_{ap}(T) \cup \{\bar{\lambda} ; \lambda \in \sigma_p(T^*)\} \,.$$

1.4. Look at two complex numbers λ and μ adjusted in the following way

$$\lambda \in \mathbb{C} \setminus \sigma(T) \,, \qquad \mu \in \mathbb{C} \,, \qquad |\mu - \lambda| < \| R_\lambda(T) \|^{-1} \,.$$

1.4.1 Prove that $\mu \in \mathbb{C} \setminus \sigma(T)$.

1.4.2 Prove that

$$\forall \, \lambda \in \mathbb{C} \setminus \sigma(T) \,, \qquad \operatorname{dist}\big(\lambda, \sigma(T)\big)^{-1} \leq \| R_\lambda(T) \| \;.$$

1.5. The boundary of $\sigma(T)$ is defined by $\partial\sigma(T) = \overline{\sigma(T)} \setminus \overbrace{\sigma(T)}^{\circ}$. Select $\lambda \in \partial\sigma(T)$.

1.5.1 Prove that we can find a sequence $(\lambda_n)_{n\in\mathbb{N}^*}$ such that

$$\lambda_n \in \mathbb{C} \setminus \sigma(T)\,, \qquad \lim_{n\to+\infty} \lambda_n = \lambda\,, \qquad \lim_{n\to+\infty} \parallel R_{\lambda_n}(T) \parallel = +\infty\,.$$

1.5.2 Prove that we can find a sequence $(y_n)_{n\in\mathbb{N}^*}$ such that

$$y_n \in H\,, \qquad \parallel y_n \parallel = 1\,, \qquad \lim_{n\to+\infty} \parallel R_{\lambda_n}(T) y_n \parallel = +\infty\,.$$

1.5.3 Let us introduce

$$\forall\, n \in \mathbb{N}\,, \qquad x_n := \frac{R_{\lambda_n}(T) y_n}{\parallel R_{\lambda_n}(T) y_n \parallel}\,.$$

Remark that $\parallel x_n \parallel = 1$, and explain why we have

$$\lim_{n\to+\infty} \parallel (T - \lambda Id) x_n \parallel = 0\,.$$

1.5.4 Conclusion ?

1.6. In this question, we assume that T is compact.

1.6.1 Prove that $\sigma_p(T) = \sigma_{ap}(T)$.

1.6.2 By using the Fredholm alternative, prove that $\sigma(T) = \sigma_{ap}(T)$.

1.7. In this question, we assume that T is a normal operator (satisfying $TT^* = T^*T$).

1.7.1 Assume that $T^*x = \lambda x$. By computing $\parallel (T^* - \lambda Id)x \parallel^2$, prove that $Tx = \bar{\lambda} x$.

1.7.2 Explain why $\sigma(T) = \sigma_{ap}(T)$.

Exercise 2. Let H be a separable Hilbert space with a complete orthonormal sequence (a basis) denoted by $(e_n)_{n\in\mathbb{N}^*}$. Given $(\lambda_n)_{n\in\mathbb{N}^*} \in \mathbb{C}^{\mathbb{N}^*}$, define $T \in \mathcal{L}(H)$ by

$$\forall\, n \in \mathbb{N}^*, \qquad Te_n = \lambda_n e_n\,.$$

2.1. Determine necessary and sufficient conditions on the sequence $(\lambda_n)_n$ to obtain the following properties on T. Justify the answer.

2.1.1 The operator T is bounded.

2.1.2 The operator T is invertible.

2.1.3 The operator T is compact.

2.2. Assume that T is bounded. What is the spectrum of T ?

Exercise 3. The space $L^2([0,1])$ is equipped with the L^2-norm. Consider the unbounded operator $T : L^2([0,1]) \to L^2([0,1])$ with domain the space $\mathcal{C}^0([0,1])$ of continuous functions, and defined by $Tf = f(1/2)$. Is the operator T closable ? Justify the answer.

Exercise 4. Let H be a Hilbert space.

4.1. In this question, we consider a bounded invertible operator $T \in \mathcal{L}(H)$ satisfying $\| T \| \leq 1$ and $\| T^{-1} \| \leq 1$.

4.1.1 Prove that $\sigma(T) \subset \{\lambda \in \mathbb{C}; |\lambda| = 1\}$.

4.1.2 Prove that T is a unitary operator, meaning that $T^* = T^{-1}$.

4.2. We recall that an operator T is positive when

$$\forall\, x \in H\,, \qquad \langle Tx, x\rangle \in \mathbb{R}_+\,.$$

4.2.1 Prove that a bounded operator T is positive if and only if

$$\exists \lambda_0 \in \mathbb{R}_+\,; \qquad \forall \lambda \geq \lambda_0\,, \quad \| \lambda Id - T \| \leq \lambda\,.$$

4.2.2 Consider two positive self-adjoint commuting operators S and T. Prove that the operator ST is positive.