UNIVERSITÉ DE RENNES 1

Microlocal Analysis

Correction of the CC5 on quantization

Documents are not allowed

Surname : **First name :**

We work on $L^2 \equiv L^2(\mathbb{R};\mathbb{C})$ with the two (unbounded essentially) self-adjoint operators

$$X : L^2 \longrightarrow L^2, \quad f \longmapsto x\,f, \qquad P : L^2 \longrightarrow L^2, \quad f \longmapsto -i\partial_x f.$$

1. Compute the commutator $[X,P]$.

$$[X,P] = XP - PX = x(-i\partial_x) - (-i\partial_x)x = -ix\partial_x + ix\partial_x + iId = iId.$$

2. Recall (in terms of X and P) the definition of the Weyl quantization of x^2p.

$$Q_{Weyl}(x^2p) = \frac{1}{(2+1)!}\sum_{\sigma\in\mathcal{S}_3}\sigma(X,X,P) = \frac{1}{3}\,(X^2P + XPX + PX^2).$$

3. Express the above expression in terms of XPX.

$$\begin{aligned} Q_{Weyl}(x^2p) &= \frac{1}{3}\,(X[X,P] + XPX + XPX + [P,X]X + XPX) \\ &= \frac{1}{3}\,(iX + 3\,XPX - iX) = XPX. \end{aligned}$$

4. Let $f(x,p)$ be a function in the Schwartz space, that is in $\mathcal{S}(\mathbb{R}^n\times\mathbb{R}^n;\mathbb{C})$. We denote by $\hat f(a,b)$ its Fourier transform (in both variables x and p).

4.1. Complete the two formulas below for the Weyl quantization of the symbol f :

$$\begin{aligned} Q_{Weyl}(f) &= (2\pi)^{-n}\int\!\!\int \hat f(a,b)\,e^{i(a\cdot X + b\cdot P)}\,dadb \\ &= (2\pi\hbar)^{-n}\int\!\!\int e^{-i(y-x)\xi/\hbar}\,f\Big(\frac{x+y}{2},\xi\Big)\,dyd\xi. \end{aligned}$$

4.2. What can be said about the action on $L^2(\mathbb{R}^n\times\mathbb{R}^n)$ of Q_{Weyl} ?

We have seen (during the course) that Q_{Weyl} is a constant multiple of a unitary map on $L^2(\mathbb{R}^n\times\mathbb{R}^n)$ onto the space $HS(L^2(\mathbb{R}^n))$ of Hilbert-Schmidt operators.

4.3. Complete the following formula : $Q_{Weyl}(f)^* = Q_{Weyl}(\bar f)$.

5. We recall that the Wick-ordered quantization Q_{Wick} of a polynomial in $z = x - ip$ and $\bar{z} = x + ip$ is obtained by putting all lowering operators to the right (acting first) and all raising operators to the left (acting second).

5.1. What is the name of the operator $Q_{Wick}(\bar{z})$?

By definition, we have $Q_{Wick}(\bar{z}) = X + iP = x + \partial_x$ *which is the* <u>*lowering*</u> *(or annihilation) operator. This can be checked by testing* $Q_{Wick}(\bar{z})$ *on the ground state* $e^{-x^2/2}$ *to find that*

$$Q_{Wick}(\bar{z})(e^{-x^2/2}) = xe^{-x^2/2} + \partial_x(e^{-x^2/2}) = 0.$$

5.2. Compute

$$\begin{aligned} Q_{Wick}(\bar{z}z^3 + z)(e^{-x^2/2}) &= [(X - iP)^3(X + iP) + (X - iP)](e^{-x^2/2}) \\ &= (X - iP)^3 Q_{Wick}(\bar{z})(e^{-x^2/2}) + [xe^{-x^2/2} - \partial_x(e^{-x^2/2})] \\ &= 2xe^{-x^2/2}. \end{aligned}$$

6. Compute $Q_{Wick}(x^2)$ in terms of X^2 and Id.

$$\begin{aligned} Q_{Wick}(x^2) &= \frac{1}{4}\, Q_{Wick}((z + \bar{z})^2) = \frac{1}{4}\, Q_{Wick}(z^2 + 2z\bar{z} + \bar{z}^2) \\ &= \frac{1}{4}\left((X - iP)^2 + 2(X - iP)(X + iP) + (X + iP)^2\right) \\ &= \frac{1}{4}\,(X^2 - iPX - iXP - P^2 + 2X^2 - 2iPX \\ &\qquad +2iXP + 2P^2 + X^2 + iXP + iPX - P^2) \\ &= \frac{1}{4}\left(4X^2 + 2i[X, P]\right) = X^2 + \frac{1}{2}i(iId) = X^2 - \frac{1}{2}Id. \end{aligned}$$