UNIVERSITÉ DE RENNES 1

Microlocal Analysis

Correction of the CC4

Let $m \in \mathbb{R}$ and $a(x,\xi) \in S^m_{1,0}(\mathbb{R}^n)$.

1.1. The kernel $K(x,y)$ of the pseudo-differential operator $a(x,D)$ is such that

$$a(x,D)u = \int_{\mathbb{R}^n} K(x,y)\, u(y)\, dy\,, \qquad \forall u \in \mathcal{S}(\mathbb{R}^n)\,.$$

Recall how the kernel K can be computed (at least formally) from the symbol a.

$$K(x,y) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi}\, a(x,\xi)\, d\xi\,. \tag{1}$$

It suffices to know the definition of the action $a(x,D)$ as well as Fubini's theorem since

$$a(x,D)u(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix\cdot\xi}\, a(x,\xi)\, \hat{u}(\xi)\, d\xi = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi}\, a(x,\xi)\, u(y)\, dy\, d\xi\,.$$

1.2. Let $(m_1,m_2) \in \mathbb{R}^2$. Given $a \in S^{m_1}_{1,0}(\mathbb{R}^n)$ and $b \in S^{m_2}_{1,0}(\mathbb{R}^n)$, define

$$a\#b(x,\xi) := \sum_{\alpha\in\mathbb{N}^n} \frac{1}{\alpha!}\, \partial^\alpha_\eta\big(a(x,\eta)\big)_{|\eta=\xi}\; D^\alpha_y\big(b(y,\xi)\big)_{|y=x}\,.$$

What is the sense of the above sum ? Recall the composition formula for pseudo-differential operators $a(x,D)$ and $b(x,D)$ in terms of the symbol $a\#b$.

The sum means that for all $N \in \mathbb{N}$, we have

$$a\#b(x,\xi) - \sum_{|\alpha|\le N} \frac{1}{\alpha!}\, \partial^\alpha_\eta\big(a(x,\eta)\big)_{|\eta=\xi}\; D^\alpha_y\big(b(y,\xi)\big)_{|y=x} \in S^{m_1+m_2-N-1}_{1,0}(\mathbb{R}^n)\,.$$

The composition formula can be written

$$a(x,D)\circ b(x,D) = Op(a\#b)(x,D) + Op(S^{-\infty}_{1,0}(\mathbb{R}^n))\,.$$

1.3. We fix $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^n$ such that $x \neq y$. Select φ and ψ in $C^\infty_c(\mathbb{R}^n)$ such that ϕ is equal to 1 near x, ψ is equal to 1 near y, and the supports of ϕ and ψ are disjoint. We denote by M_ϕ and M_ψ the multiplication operators by ϕ and ψ. Use the question 1.2 to show that $T := M_\phi\, a(x,D)\, M_\psi$ is in $Op(S^{-\infty}_{1,0}(\mathbb{R}^n))$.

From question 1.2, we can assert that

$$T = Op(\phi\, a\#\psi)(x,D) + Op(S^{-\infty}_{1,0}(\mathbb{R}^n))\,,$$

where

$$\phi\, a\#\psi(x,\xi) = \sum_{\alpha\in\mathbb{N}^n} \frac{1}{\alpha!}\ \phi(x)\ (\partial_\xi^\alpha a)(x,\xi)\ D_x^\alpha \psi(x)\,.$$

Now, since $\operatorname{supp}\phi$ *and* $\operatorname{supp} D_x^\alpha\psi \subset \operatorname{supp}\psi$ *are disjoint, all products* $\phi(x)\ D_x^\alpha\psi(x)$ *are equal to* 0.

1.4. Compute the kernel $\tilde K(x,y)$ of T in terms of K.

$$\tilde K(x,y) = \phi(x)\, K(x,y)\, \psi(y)\,.$$

1.5. Prove that $\tilde K$ is a bounded continuous function such that

$$\forall\, N\in\mathbb{N}\,; \quad \exists\, C_N\,; \quad |\tilde K(x,y)| \le C_N\ (1+|x-y|)^{-N}\,.$$

From the question 1.3, we know that $T = Op(\tilde a)$ *with* $\tilde a \in S^{-\infty}_{1,0}(\mathbb{R}^n)$*. In particular, the function* $\tilde a(x,\cdot)$ *is uniformly in* x *in* L^1 *with respect to* ξ*. From (1) and results about the continuity of parameter dependent integrals, we know that* $\tilde K$ *is a continuous bounded function. For all* α*, we have* $\partial_\xi^\alpha \tilde a \in S^{-\infty}_{1,0}(\mathbb{R}^n)$*. Thus, we can iterate this argument at the level of*

$$i^\alpha\,(x-y)^\alpha\,\tilde K(x,y) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \partial_\xi^\alpha\,(e^{i(x-y)\cdot\xi})\,\tilde a(x,\xi)\,d\xi = \frac{(-1)^\alpha}{(2\pi)^n}\int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi}\,\partial_\xi^\alpha\tilde a(x,\xi)\,d\xi$$

to deduce the expected result.

1.6. Show that K is smooth (of class C^∞) near (x,y).

In general, the formula (1) must be interpreted as an oscillatory integral. But, when $\tilde a \in S^{-\infty}_{1,0}(\mathbb{R}^n)$*, we can give a classical sense to*

$$\partial_y^\alpha \tilde K(x,y) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} (-i\xi)^\alpha\, e^{i(x-y)\cdot\xi}\,\tilde a(x,\xi)\,d\xi\,,$$

and similarly (with the general Leibniz rule) for the derivatives with respect to x*. Then, applying the same argument as in question 1.5, we can see that* $\tilde K$ *is of class* C^∞*. Since* K *coincides with* $\tilde K$ *near* (x,y)*, the same holds true concerning* K*.*

1.7. We assume that $a(x,D)$ is a differential operator with smooth coefficients. What can be said about the support of its kernel (viewed as a distribution) ?

We can find smooth functions a_α *such that*

$$a(x,D) = \sum_{|\alpha|\le N} a_\alpha(x)\ D_x^\alpha\,.$$

From (1), we deduce that the kernel associated with $a(x,D)$ *is given by*

$$K(x,y) = \frac{1}{(2\pi)^n}\sum_{|\alpha|\le N} a_\alpha(x)\int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi}\,\xi^\alpha\,d\xi = \frac{1}{(2\pi)^n}\sum_{|\alpha|\le N} a_\alpha(x)\,D_x^\alpha\int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi}\,d\xi\,.$$

The last integral is a Dirac mass at $y=x$*, which implies that the kernel* K *is supported in the diagonal* $x=y$ *of* $\mathbb{R}^n\times\mathbb{R}^n$*.*