UNIVERSITÉ DE RENNES 1

Microlocal Analysis

CC4 about the regularity of kernels, the 02/12/2022 (30mn)

Documents are not allowed

Surname : **First name :**

Let $m \in \mathbb{R}$ and $a(x,\xi) \in S^m_{1,0}(\mathbb{R}^n)$.

1.1. The kernel $K(x,y)$ of the pseudo-differential operator $a(x,D)$ is such that

$$a(x,D)u = \int_{\mathbb{R}^n} K(x,y)\, u(y)\, dy\,, \qquad \forall u \in \mathcal{S}(\mathbb{R}^n)\,.$$

Recall how the kernel K can be computed (at least formally) from the symbol a.

$K(x,y) =$

1.2. Let $(m_1,m_2) \in \mathbb{R}^2$. Given $a \in S^{m_1}_{1,0}(\mathbb{R}^n)$ and $b \in S^{m_2}_{1,0}(\mathbb{R}^n)$, define

$$a\#b(x,\xi) := \sum_{\alpha\in\mathbb{N}^n} \frac{1}{\alpha!}\, \partial^\alpha_\eta \big(a(x,\eta)\big)_{|\eta=\xi}\, D^\alpha_y \big(b(y,\xi)\big)_{|y=x}\,.$$

What is the sense of the above sum ? Recall the composition formula for pseudo-differential operators $a(x,D)$ and $b(x,D)$ in terms of the symbol $a\#b$.

1.3. We fix $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^n$ such that $x \neq y$. Select φ and ψ in $C^\infty_c(\mathbb{R}^n)$ such that ϕ is equal to 1 near x, ψ is equal to 1 near y, and the supports of ϕ and ψ are disjoint. We denote by M_ϕ and M_ψ the multiplication operators by ϕ and ψ. Use the question 1.2 to show that $T := M_\phi\, a(x,D)\, M_\psi$ is in $Op(S^{-\infty}_{1,0}(\mathbb{R}^n))$.

$\Longrightarrow$ T.S.V.P.

1.4. Compute the kernel $\tilde{K}(x,y)$ of T in terms of K, φ and ψ.

$\tilde{K}(x,y) =$

1.5. Using the questions 1.1 and 1.3, prove that $\tilde{K}$ is a bounded continuous function which is such that

$$\forall N \in \mathbb{N}; \quad \exists C_N ; \quad |\tilde{K}(x,y)| \leq C_N \, (1 + |x-y|)^{-N} .$$

1.6. Show that K is smooth (of class C^∞) near (x,y).

1.7. We assume that $a(x,D)$ is a differential operator with smooth coefficients. What can be said about the support of its kernel (viewed as a distribution) ?