UNIVERSITÉ DE RENNES 1

Spectral Theory

CC3, correction

Question. Given two Banach spaces E and F, what does it means to say that $\lambda \in \mathbb{C}$ is in the essential spectrum $\mathrm{sp}_{\mathrm{ess}}(T)$ of $T \in \mathcal{L}(E, F)$?

See the course.

Prove that $\mathrm{sp}_{\mathrm{ess}}(T) \subset \mathrm{sp}(T)$.

See the course.

Exercise 1. Let X_1 and X_2 be Banach spaces. Select $T \in \mathcal{L}(X_1, X_2)$, and define

$$\mathcal{M} = \begin{pmatrix} T & R_- \\ R_+ & 0 \end{pmatrix}, \tag{1}$$

with $R_- : \mathbb{C}^{n_-} \to X_2$ and $R_+ : X_1 \to \mathbb{C}^{n_+}$ bounded. Assume that $\mathcal{M}$ is bijective. We denote by $\mathcal{E}$ its (bounded) inverse :

$$\mathcal{E} = \begin{pmatrix} E & E_+ \\ E_- & E_0 \end{pmatrix}, \qquad \begin{array}{ll} E \in \mathcal{L}(X_2, X_1), & E_+ \in \mathcal{L}(\mathbb{C}^{n_+}, X_1), \\ E_- \in \mathcal{L}(X_2, \mathbb{C}^{n_-}), & E_0 \in \mathcal{L}(\mathbb{C}^{n_+}, \mathbb{C}^{n_-}). \end{array} \tag{2}$$

Assume that T is bijective. Show that E_0 is bijective.

See the course.

Exercise 2. We work on the Banach space $E = C^0([0,1]; \mathbb{R})$ with the sup norm. Let $T : E \to E$ be the operator given by $(Tf)(x) = xf(x)$.

2.1) What is the point spectrum of T ?

It is empty. Indeed, let $\lambda \in \mathbb{C}$ be an eigenvalue. The equation $(T - \lambda)f = 0$ implies $(x - \lambda)f(x) = 0$ for all $x \in [0,1]$, that is $f \equiv 0$.

2.2) What is the spectrum of T ?

It is the interval $[0,1]$. Let $\lambda \notin [0,1]$. Then $T - \lambda$ is invertible with continuous inverse given by $(T - \lambda)^{-1}g(x) = (x - \lambda)^{-1}g(x)$. On the contrary, when $\lambda \in [0,1]$, the operator $T - \lambda$ cannot be surjective since $(T - \lambda)f$ is a function vanishing at λ.

2.3) Is this operator T compact ?

No. The Fredholm alternative says that the non-zero spectrum of T must be discrete, which is not the case.

A direct proof is also available. The sequence $f_n(x) = x^n$ is bounded but its image f_{n+1} has no subsequence converging uniformly on $[0,1]$ since it converges simply to the discontinuous function which is 0 on $[0,1[$ and equal to 1 at 1.

Exercise 3. Let E be a Banach space. Show that a Fredholm operator $T \in \mathcal{L}(E,E)$ is a compact perturbation of an invertible operator if and only if its index vanishes.

$\Longrightarrow$ *Assume that T is a compact perturbation of an invertible operator, that is $T = U + K$ with U invertible and K compact. Then, for all $s \in [0,1]$, the operator $T(s) := U + sK$ is Fredholm, and the function $ind T(\cdot)$ is continuous. It follows that the index of $T(\cdot)$ is constant on $[0,1]$. Therefore*

$$ind T(1) = ind T = ind U = 0.$$

$\Longleftarrow$ *Assume that $dim\, ker T = codim\, ran T = n$. Then*

$$E = ker\, T \oplus F, \qquad E = ran\, T \oplus G, \qquad dim\, ker\, T = dim\, G.$$

Let K be an invertible linear map between $ker\, T$ and G. We still denote by K its extension by zero to the whole space E. Then K is compact (because $ran\, K = G$ is of finite dimension), whereas $T - K$ is (by construction) invertible. It suffices to remark that $T = (T - K) + K$.