UNIVERSITÉ DE RENNES 1

Spectral Theory

CC3, the 12/10/2018 (20mn)

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Surname : **First name :**

Question. Define what is a Fredholm operator $T \in \mathcal{L}(E,F)$.

Exercise 1. Let $E = \mathbb{R}^2$. Given $\lambda \in \mathbb{C}$, define $T \in \mathcal{L}(E)$ and $P_\lambda \in \mathcal{L}(E)$ by

$$T := \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}, \qquad P_\lambda := \frac{1}{2i\pi} \int_{\Gamma_\lambda} (z-T)^{-1}\, dz\,,$$

where Γ_λ is the circle of center λ and radius 1. Determine P_λ by a direct computation.

Let $n \in \mathbb{N}$. Recall that

$$\frac{1}{2i\pi} \int_{\Gamma_\lambda} (z-\lambda)^{-n}\, dz = \frac{1}{2i\pi} \int_{\theta=0}^{2\pi} (z-\lambda)^{-n}\, dz = \frac{1}{2\pi} \int_{\theta=0}^{2\pi} e^{i(1-n)\theta}\, d\theta = \begin{cases} 1 & \text{if} \quad n=1\,, \\ 0 & \text{if} \quad n \neq 1\,. \end{cases}$$

It follows that

$$P_\lambda := \frac{1}{2i\pi} \int_{\Gamma_\lambda} \begin{pmatrix} (z-\lambda)^{-1} & -(z-\lambda)^{-2} \\ 0 & (z-\lambda)^{-1} \end{pmatrix} dz = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Exercise 2. Let E be the Banach space $\mathcal{C}^0([0,1];\mathbb{R})$ equipped with the sup norm. Select a strictly increasing function $\varphi \in E$, and consider the operator $T : X \to X$ which is given by the multiplication $T(f) = \varphi f$. Determine the spectrum $\mathrm{sp}\,(T)$ of T. What is the essential spectrum $\mathrm{sp}_{ess}\,(T)$ of T ? And what is the discrete spectrum $\mathrm{sp}_{dis}\,(T)$ of T ?

Fix $\lambda \in \mathbb{C}$. Since f is continuous, we have

$$\ker(T-\lambda) = \{f \in X; (\varphi-\lambda)f = 0\} = \{f \in X; f_{|\mathbb{R}\setminus\{\lambda\}} = 0\} = \{0\}\,.$$

Introduce $[a,b] = \varphi([0,1])$ with $\varphi(0) = a < \varphi(1) = b$. If $\lambda \notin [a,b]$, the function $(\varphi-\lambda)^{-1} f$ is well-defined on $[0,1]$, in E, and such that

$$\forall f \in E, \quad (T-\lambda)\big((\varphi-\lambda)^{-1} f\big) = f\,,$$

which means that $T-\lambda$ is surjective, and therefore bijective with bounded inverse.
On the contrary, when $\lambda \in [a,b]$, we can find $u \in [0,1]$ such that $\lambda = \varphi(u)$, we find that

$$\mathrm{ran}\,(T-\lambda) \subset F := \{f \in E; f(u) = 0\}\,.$$

The functions $\cos\big(n(x-u)\big)$ with $n \in \mathbb{N}$ are linearly independent and not in F. Therefore, we have codim$\,\mathrm{ran}\,(T-\lambda) = +\infty$, and $T-\lambda$ is not Fredholm. Finally, we have

$$sp_{dis}\,(T) = \emptyset\,, \qquad sp_{ess}\,(T) = [a,b]\,, \qquad sp\,(T) = [a,b]\,.$$