UNIVERSITÉ DE RENNES 1

Spectral Theory

CC3, the 11/10/2019 (20mn)

Documents are not allowed

Surname : **First name :**

Question. Given two Banach spaces E and F, what does it means to say that $\lambda \in \mathbb{C}$ is in the essential spectrum $\mathrm{sp}_{\mathrm{ess}}(T)$ of $T \in \mathcal{L}(E,F)$?

Prove that $\mathrm{sp}_{\mathrm{ess}}(T) \subset \mathrm{sp}(T)$.

Exercise 1. Let X_1 and X_2 be Banach spaces. Select $T \in \mathcal{L}(X_1, X_2)$, and define

$$\mathcal{M} = \begin{pmatrix} T & R_- \\ R_+ & 0 \end{pmatrix}, \tag{1}$$

with $R_- : \mathbb{C}^{n_-} \to X_2$ and $R_+ : X_1 \to \mathbb{C}^{n_+}$ bounded. Assume that $\mathcal{M}$ is bijective. We denote by $\mathcal{E}$ its (bounded) inverse :

$$\mathcal{E} = \begin{pmatrix} E & E_+ \\ E_- & E_0 \end{pmatrix}, \qquad \begin{matrix} E \in \mathcal{L}(X_2, X_1), & E_+ \in \mathcal{L}(\mathbb{C}^{n_+}, X_1), \\ E_- \in \mathcal{L}(X_2, \mathbb{C}^{n_-}), & E_0 \in \mathcal{L}(\mathbb{C}^{n_+}, \mathbb{C}^{n_-}). \end{matrix} \tag{2}$$

Assume that T is bijective. Show that E_0 is bijective.

T.S.V.P $\Longrightarrow$

Exercise 2. We work on the Banach space $E = C^0([0,1];\mathbb{R})$ with the sup norm. Let $T : E \to E$ be the operator given by $(Tf)(x) = xf(x)$.

2.1) What is the point spectrum of T ?

2.2) What is the spectrum of T ?

2.3) Is this operator T compact ?

Exercise 3. Show that a Fredholm operator $T \in \mathcal{L}(E,E)$ is a compact perturbation of an invertible operator if and only if its index vanishes.