UNIVERSITÉ DE RENNES 1

Microlocal Analysis

Correction of the CC3

We consider on the Hilbert space $\mathcal{H} := L^2([-1,1])$ the *position operator* A and the *momentum operator* B defined by

$$A\psi(x) = x\,\psi(x)\,, \qquad B\psi(x) = -i\hbar\,\psi'(x) = -i\hbar\,\frac{d\psi}{dx}(x)\,.$$

1.1. Prove that A is a bounded operator and compute its operator norm $\|A\|$.

Since the multiplicative factor $|x|$ *is bounded by* 1 *on* $[-1,1]$*, we have*

$$\| A\psi \|^2 = \int_{-1}^{1} x^2\,|\psi(x)|^2\,dx \leq \int_{-1}^{1} |\psi(x)|^2\,dx = \| \psi \|^2\ .$$

This means that $\|A\| \leq 1$*. Let* $\varepsilon \in [0,1]$ *and let* $\psi \in \mathcal{H}$ *of norm* 1 *whose support is contained in* $[1-\varepsilon,1]$*. Then*

$$\| A\psi \|^2 = \int_{1-\varepsilon}^{1} x^2\,|\psi(x)|^2\,dx \geq (1-\varepsilon)^2 \int_{1-\varepsilon}^{1} |\psi(x)|^2\,dx = (1-\varepsilon)^2\ \| \psi \|^2\ .$$

Since ε *can be taken arbitrarily small, this implies that* $\|A\| = 1$*.*

1.2. We look at B as an unbounded operator with domain

$$\mathrm{Dom}(B) := \{\, \psi \in C^1([-1,1])\,;\, \psi(-1) = \psi(1)\,\}\,.$$

Check that B is symmetric.

We have to show that

$$\forall(\phi,\psi) \in \mathrm{Dom}(B)^2\,, \qquad \langle \phi, B\psi\rangle = \langle B\phi, \psi\rangle\,.$$

We compute by integration by parts

$$\begin{aligned}\langle \phi, B\psi\rangle &= \int_{-1}^{1} \bar\phi(x) \times (-i\hbar\,\psi'(x))\,dx \\ &= -i\hbar\big[(\bar\phi\,\psi)(1) - (\bar\phi\,\psi)(-1)\big] + i\hbar \int_{-1}^{1} \bar\phi'(x) \times \psi(x)\,dx \\ &= 0 + i\hbar \int_{-1}^{1} \overline{-i\hbar\phi'}(x) \times \psi(x)\,dx = \langle B\phi, \psi\rangle\,,\end{aligned}$$

where we have exploited the boundary conditions.

1.3. For $n \in \mathbb{Z}$, define $\psi_n(x) := e^{\pi i n x}/\sqrt{2}$. Show that ψ_n is in $\mathrm{Dom}(B)$, and that $(\psi_n)_{n\in\mathbb{Z}}$ constitutes an orthonormal basis of eigenvectors for B with real eigenvalues.

The function ψ_n *is smooth (of class* C^∞*) and it satisfies*

$$\psi_n(1) = e^{\pi i n}/\sqrt{2} = (-1)^n/\sqrt{2} = \psi_n(-1)\,.$$

It is therefore in $\mathrm{Dom}(B)$*. We have clearly*

$$B\psi_n = -i\hbar\,(\pi i n)\,e^{\pi i n x}/\sqrt{2} = \hbar\pi n\,\psi_n\,,$$

which indicates that ψ_n *is an eigenvector for* B *with eigenvalue* $\hbar\pi n \in \mathbb{R}$*. The sequence of functions* ψ_n *forms clearly an orthonormal basis of* $\mathcal{H}$*. To show that it is a basis, it suffices to remark that the vector space spanned by the* ψ_n *is dense in* $\mathcal{H}$ *(because a function in* L^2 *whose all Fourier coefficients are zero is simply zero).*

1.4. Let $\psi \in \mathrm{Dom}(AB) \cap \mathrm{Dom}(BA)$. Complete the following formula

$$AB\psi - BA\psi = x \times \big(-i\hbar\,\psi'(x)\big) + i\hbar\,\big(x\psi(x)\big)' = i\hbar\,\psi(x)\,.$$

1.5. Given a self-adjoint operator A on $\mathcal{H}$ and a unit vector $\psi \in \mathcal{H}$, recall that the *uncertainty* of A in the state ψ is defined by

$$\Delta_\psi A := \sqrt{\langle A^2\rangle_\psi - \langle A\rangle_\psi^2}\,, \qquad \langle A\rangle_\psi := \langle \psi, A\psi\rangle\,.$$

Explain why $\Delta_{\psi_n} A$ and $\Delta_{\psi_n} B$ are both unambiguously defined, and compute $\Delta_{\psi_n} B$.

Since A *is bounded, we can define* $A\psi_n \in \mathcal{H}$ *as well as* $A^2\psi_n \in \mathcal{H}$*, and therefore* $\Delta_{\psi_n} A$*. On the other hand* $B\psi_n = \hbar\pi n\,\psi_n \in \mathcal{H}$ *so that*

$$\langle B\rangle_{\psi_n} = \hbar\pi n\,, \qquad \langle B^2\rangle_{\psi_n} = (\hbar\pi n)^2\,,$$

implying that $\Delta_{\psi_n} B = 0$*.*

1.6. Recall precisely the content of the *uncertainty principle*. Is it verified in the case of the above data A, B and ψ ? If not, could you explain why.

Theorem *(uncertainty principle). Under the preceding assumptions, for all unit vector* ψ *such that* $\psi \in \mathrm{Dom}(AB) \cap \mathrm{Dom}(BA)$*, we have* $\Delta_\psi A \times \Delta_\psi B \geq \hbar/2 > 0$*.* □

Now, from the above computations, we get $\Delta_{\psi_n} A \times \Delta_{\psi_n} B = 0$*. This is not a contradiction because* $\psi_n \notin \mathrm{Dom}(BA)$*. Indeed*

$$(A\psi_n)(1) = e^{\pi i n}\,, \qquad (A\psi_n)(-1) = -e^{-\pi i n}\,, \qquad (A\psi_n)(1) \neq (A\psi_n)(-1)\,.$$