UNIVERSITÉ DE RENNES 1

Spectral Theory

CC2, Correction

Question. Let $(\mathrm{Dom}(T), T)$ be a densely defined operator on the Hilbert space H.

a) What is the definition of $\mathrm{Dom}(T^*)$?

$$\mathrm{Dom}(T^*) \;=\; \{x \in \mathsf{H}\,;\, \Phi_x : \mathrm{Dom}(T) \ni y \longmapsto \langle Ty, x\rangle \text{ is continuous for the norm of } \mathsf{H}\}.$$

b) Given $x \in \mathrm{Dom}(T^*)$, explain how is constructed T^*x.

By the Riesz representation theorem, the continuous linear map Φ_x can be represented in the form $\Phi_x(y) = \langle y, z\rangle$ for some $z = T^*x \in \mathsf{H}$.

Exercise 1. Given an operator $(\mathrm{Dom}(T), T)$, consider the two following assertions :

(i) The set $\overline{\Gamma(T)}$ is the graph of an operator.

(ii) For all $(u_n) \in \mathrm{Dom}(T)^{\mathbb{N}}$ such that $u_n \to 0$ and $Tu_n \to v$, we have $v = 0$.

1.1. Prove that (i) implies (ii).

Assume (i)*, that is* $\overline{\Gamma(T)} = \Gamma(R)$ *for some operator* $(\mathrm{Dom}(R), R)$*. Since* $(0, v) \in \overline{\Gamma(T)}$*, we must have* $v = R0 = 0$.

1.2. Prove that (ii) implies (i).

Assume (ii)*, and work by contradiction. The set* $\overline{\Gamma(T)}$ *is not the graph of an operator if we can find* x*,* y *and* z *with* $y \neq z$ *such that* $(x, y) \in \overline{\Gamma(T)}$ *and* $(x, z) \in \overline{\Gamma(T)}$*. It follows that we can find two sequences* $\big((x_n, y_n = Tx_n)\big)_n$ *and* $\big((\tilde{x}_n, \tilde{y}_n = T\tilde{x}_n)\big)_n$ *such that*

$$\lim_{n\to+\infty} x_n = x, \qquad \lim_{n\to+\infty} \tilde{x}_n = x, \qquad \lim_{n\to+\infty} y_n = y, \qquad \lim_{n\to+\infty} \tilde{y}_n = z.$$

Define $u_n = \tilde{x}_n - x_n$ *and* $v_n = \tilde{y}_n - y_n = T(\tilde{x}_n - x_n)$*, so that*

$$\lim_{n\to+\infty} u_n = 0, \qquad \lim_{n\to+\infty} v_n = \lim_{n\to+\infty} Tu_n = z - y.$$

Thus, we must have $z - y = 0$ *which is the expected contradiction.*

1.3. Assume moreover that $(\mathrm{Dom}(T), T)$ is densely defined. Show that T^* is closed.

See Exercice 2.21 in the course.

Exercise 2. Take $\mathsf{H} = \ell^2(\mathbb{N}; \mathbb{C})$. Consider the linear subspace

$$\mathrm{Dom}(T) := \{u = (u_n)_n \in \ell^2(\mathbb{N}; \mathbb{C})\,;\, \sup_{n\in\mathbb{N}} n^3 |u_n| < +\infty\}.$$

Given $u = (u_n)_n \in \ell^2(\mathbb{N};\mathbb{C})$, define $Tu = v = (v_n)_n$ according to

$$v_0 = \sum_{n=0}^{+\infty} n u_n \,, \qquad v_j = 0, \quad \forall j \in \mathbb{N}^*.$$

2.1 Is this operator $(\mathrm{Dom}(T), T)$ closable ? Justify the answer.

No. It suffices to apply the criterion (iii) *of Proposition 1.20. For all* $n \in \mathbb{N}^*$, *define* $u_n := (n^{-1}\delta_{nj})_j$. *The sequence* $(u_n)_n$ *is going to* 0 *whereas* $Tu_n = (\delta_{0j})_j$ *is constant (and not zero).*

2.2 Compute the domain of T^*. Conclusion ?

An element $v \in \mathsf{H}$ *is in the domain of* T^* *if and only if the application*

$$\mathrm{Dom}(T) \ni u \mapsto \langle Tu, v\rangle = \sum_{n=0}^{+\infty} n u_n v_0$$

is continuous on ℓ^2*-norm. But that is never the case except if* $v_0 = 0$. *Thus, we have*

$$\mathrm{Dom}(T^*) = \{v = (v_n)_n \in \ell^2(\mathbb{N};\mathbb{C})\,;\, v_0 = 0\}.$$

The distance from $(\delta_{0j})_j$ *to* $\mathrm{Dom}(T^*)$ *is* 1, *or the codimension of* $\mathrm{Dom}(T^*)$ *is* 1. *It follows that* $\mathrm{Dom}(T^*)$ *is not dense, and the operator* $(\mathrm{Dom}(T), T)$ *cannot be closable. This confirms the result of 2.1.*

Exercise 3. We work here with an operator $(\mathrm{Dom}(T), T)$ whose spectrum $\mathsf{Sp}(T)$ is strictly contained in $\mathbb{C}$. We fix some z in the resolvent set of T.

3.1 Let (x, y) be such that $(T - z)^{-1}(y - zx) = x$. What can we say about (x, y) ?

We must have $(x, y) \in \Gamma(T)$.

3.2 Show that T is closed.

Let $\big((x_n, y_n)\big)_n \in \Gamma(T)^{\mathbb{N}}$ *be such that* $x_n \to x$ *and* $y_n = Tx_n \to y$. *We have*

$$(T - z)^{-1}(y_n - zx_n) = x_n.$$

Since $(T - z)^{-1} : \mathsf{H} \to \mathrm{Dom}(T)$ *is continuous, we can pass to the limit in order to obtain*

$$(T - z)^{-1}(y - zx) = x.$$

From **3.1**, *we can conclude that* $(x, y) \in \Gamma(T)$.