UNIVERSITÉ DE RENNES 1

Spectral Theory

CC2, correction

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Question. Let T be a densely defined operator which is symmetric and closable. Compare the operators T, T^* and $\bar{T}$, and then give a necessary and sufficient condition to be sure that T is self-adjoint.

We have $T \subset \bar{T} \subset T^*$ and T is self-adjoint if and only if $T = \bar{T} = T^*$.

Exercise 1. We work on $L^2(\mathbb{R})$ with the standard inner product

$$\langle f, g \rangle := \int \bar{f}(x) g(x) dx$$

Select some non zero function $\psi_0 \in L^2(\mathbb{R})$, as well as some function $f \in L^\infty(\mathbb{R})$ which is not in $L^2(\mathbb{R})$. Then, define

$$\text{Dom}(T) := \{\psi \in L^2(\mathbb{R}); \int |f(x)\psi(x)| dx < +\infty\}, \qquad T\psi = \langle f, \psi \rangle \psi_0$$

1.1. Explain why $\text{Dom}(T)$ is dense in $L^2(\mathbb{R})$.

Since the function $f \in L^\infty(\mathbb{R})$ is locally integrable, the linear subspace $C_0^\infty(\mathbb{R})$ is contained in $Dom(T)$. On the other hand, $C_0^\infty(\mathbb{R})$ is dense in $L^2(\mathbb{R})$ and therefore the same applies concerning $Dom(T)$.

1.2. Determine the domain $\text{Dom}(T^*)$ of the adjoint T^* of T.

By definition, $\varphi \in Dom(T^*)$ if and only if the application

$$\begin{array}{rcl} Dom(T) & \longrightarrow & L^2(\mathbb{R}) \\ \psi & \longmapsto & \langle T\psi, \varphi \rangle = \langle \psi, f \rangle \langle \psi_0, \varphi \rangle \end{array}$$

is continuous on $L^2(\mathbb{R})$. Above, the number $\langle \psi_0, \varphi \rangle$ is a fixed constant. As a consequence of the Riesz representation theorem, the application $\psi \longmapsto \langle \psi, f \rangle$ is continuous on $L^2(\mathbb{R})$ on condition that $f \in L^2(\mathbb{R})$. But this is not the case. Thus, the only way to recover the continuity is to impose $\langle \psi_0, \varphi \rangle = 0$, which means that $\varphi \in Vec\,(\psi_0)^\perp$. In short, we have

$$Dom(T^*) = Vec\,(\psi_0)^\perp$$

1.3. Is the operator T closable or not ? Justify the answer.

From 1.1, we know that $Dom(T)$ is dense in $L^2(\mathbb{R})$. We have seen in the course that T is closable if and only if $Dom(T^*)$ is dense. This is not the case in view of 1.2.

Exercise 2. Let Ω be an open bounded domain of $\mathbb{R}^N$ with smooth boundary. Define

$$\mathrm{H} := L^2(\Omega;\mathbb{R}), \qquad \langle f,g\rangle := \int_\Omega fg\,dx$$

$$\mathcal{V} := H_0^1(\Omega;\mathbb{R}), \qquad \langle f,g\rangle_{\mathcal{V}} := \int_\Omega \nabla_x f\cdot\nabla_x g\,dx + \int_\Omega fg\,dx$$

with associated norms $\parallel f \parallel := \langle f,f\rangle$ and $\parallel f \parallel_{\mathcal{V}} := \langle f,f\rangle_{\mathcal{V}}$. Select f and V satisfying

$$f\in\mathrm{H} \quad ; \quad V\in(\mathcal{C}^\infty\cap L^\infty)(\Omega;\mathbb{R}^N) \quad ; \quad \operatorname{div} V := \sum_{j=1}^N \partial_{x_j} V \equiv 0.$$

Assume that $u\in\mathcal{V}$ is a solution in the sense of distributions to the equation

$$(V\cdot\nabla_x)u - \Delta u = f \quad \text{in } \mathcal{D}'(\Omega) \tag{1}$$

2.1. Show that there exists a continuous coercive form Q on $\mathcal{V}\times\mathcal{V}$ such that

$$\forall v\in\mathcal{C}_0^\infty(\Omega;\mathbb{R}), \quad Q(u,v) = \langle f,v\rangle.$$

A simple integration by parts indicates that (1) is equivalent to

$$\forall v\in\mathcal{C}_0^\infty(\Omega;\mathbb{R}), \quad Q(u,v) := \int_\Omega (V\cdot\nabla_x)uv\,dx + \int_\Omega \nabla_x u\cdot\nabla_x v\,dx = \langle f,v\rangle.$$

The sesquilinear form Q is continuous on $\mathcal{V}\times\mathcal{V}$ since

$$|Q(u,v)| \leq (1+\parallel V \parallel_{L^\infty}) \parallel u \parallel_{\mathcal{V}} \parallel v \parallel_{\mathcal{V}}$$

The difficulty is to explain why it is coercive. Since V is divergence free, an integration by parts (which can be justified by a density argument) furnishes

$$\int_\Omega (V\cdot\nabla_x)uu\,dx = -\int_\Omega u(V\cdot\nabla_x)u\,dx \quad\Longrightarrow\quad \int_\Omega (V\cdot\nabla_x)uu\,dx = 0$$

and therefore

$$\forall u\in\mathcal{V}, \quad Q(u,u) = \parallel \nabla_x u \parallel^2 .$$

On the other hand, the Poincaré inequality guarantees that

$$\exists C\in[1,+\infty[; \quad \parallel u \parallel \leq C \parallel \nabla_x u \parallel$$

It follows that

$$\forall u\in\mathcal{V}, \quad |Q(u,u)| \geq \frac{1}{1+C^2} \parallel u \parallel_{\mathcal{V}}^2 .$$

2.2. Let $f\in L^2(\Omega;\mathbb{R})$. Show that the equation (1) has a unique solution $u\in\mathcal{V}$.

The Lax-Milgram theorem furnishes the existence of an operator $(\mathrm{Dom}(\mathscr{L}),\mathscr{L})$ *such that*

$$\forall u\in\mathrm{Dom}(\mathscr{L}), \quad \forall v\in\mathcal{V}, \quad Q(u,v) = \langle \mathscr{L}u,v\rangle$$

The operator $\mathscr{L}$ is bijective from $\mathrm{Dom}(\mathscr{L})$ *onto* H. *It suffices to take $u=\mathscr{L}^{-1}f$ to find a solution to (1), which is in* $\mathrm{Dom}(\mathscr{L})\subset\mathcal{V}$ *. Let $\tilde u\in\mathcal{V}$ be another solution. Then*

$$\forall v\in\mathcal{C}_0^\infty(\Omega;\mathbb{R}), \quad Q(\tilde u-u,v)=0 \quad\Longrightarrow\quad 0 = Q(\tilde u-u,\tilde u-u) \geq \frac{1}{1+C^2} \parallel \tilde u - u \parallel_{\mathcal{V}}^2$$

which gives $\tilde u = u$.