UNIVERSITÉ DE RENNES 1

Spectral Theory

CC2, the 27/09/2019 (20mn)

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Surname : **First name :**

Question. Let $(\mathrm{Dom}(T), T)$ be a densely defined operator on the Hilbert space H.

a) What is the definition of $\mathrm{Dom}(T^*)$?

$\mathrm{Dom}(T^*) =$

b) Given $x \in \mathrm{Dom}(T^*)$, explain how is constructed T^*x.

Exercise 1. Given an operator $(\mathrm{Dom}(T), T)$, consider the two following assertions :

(i) The set $\overline{\Gamma(T)}$ is the graph of an operator.

(ii) For all $(u_n) \in \mathrm{Dom}(T)^{\mathbb{N}}$ such that $u_n \to 0$ and $Tu_n \to v$, we have $v = 0$.

1.1. Prove that (i) implies (ii).

1.2. Prove that (ii) implies (i).

1.3. Assume that $(\mathrm{Dom}(T), T)$ is densely defined. Show that T^* is closed.

T.S.V.P $\Longrightarrow$

Exercise 2. Take $\mathsf{H} = \ell^2(\mathbb{N};\mathbb{C})$. Consider the linear subspace

$$\mathrm{Dom}(T) := \{u = (u_n)_n \in \ell^2(\mathbb{N};\mathbb{C})\,;\, \sup_{n\in\mathbb{N}} n^3|u_n| < +\infty\}.$$

Given $u = (u_n)_n \in \ell^2(\mathbb{N};\mathbb{C})$, define $Tu = v = (v_n)_n$ according to

$$v_0 = \sum_{n=0}^{+\infty} n u_n\,, \qquad v_j = 0, \quad \forall j \in \mathbb{N}^*.$$

2.1 Is this operator $(\mathrm{Dom}(T), T)$ closable ? Justify the answer.

2.2 Compute the domain of T^*. Conclusion ?

Exercise 3. We work here with an operator $(\mathrm{Dom}(T), T)$ whose spectrum $\mathsf{Sp}(T)$ is strictly contained in $\mathbb{C}$. We fix some z in the resolvent set of T.

3.1 Let (x, y) be such that $(T - z)^{-1}(y - zx) = x$. What can we say about (x, y) ?

3.2 Show that T is closed.