UNIVERSITÉ DE RENNES 1

Spectral Theory

CC2, the 05/10/2018 (20mn)

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Surname : **First name :**

Question. Let T be a densely defined operator which is symmetric and closable. Compare the operators T, T^* and $\bar{T}$, and then give a necessary and sufficient condition to be sure that T is self-adjoint.

Exercise 1. We work on $L^2(\mathbb{R})$ with the standard inner product

$$\langle f, g\rangle := \int \bar{f}(x) g(x) dx$$

We select some non zero function $\psi_0 \in L^2(\mathbb{R})$, as well as some function $f \in L^\infty(\mathbb{R})$ which is not in $L^2(\mathbb{R})$. Then, we define

$$\mathrm{Dom}(T) := \{\psi \in L^2(\mathbb{R}); \int |f(x)\psi(x)| dx < +\infty\}, \qquad T\psi = \langle f, \psi\rangle \psi_0$$

1.1. Explain why $\mathrm{Dom}(T)$ is dense in $L^2(\mathbb{R})$.

1.2. Determine the domain $\mathrm{Dom}(T^*)$ of the adjoint T^* of T.

1.3. Is the operator T closable or not ? Justify the answer.

T.S.V.P $\Longrightarrow$

Exercise 2. Let Ω be an open bounded domain of $\mathbb{R}^N$ with smooth boundary. Define

$$\begin{aligned} \mathrm{H} &:= L^2(\Omega;\mathbb{R}), \qquad \langle f,g\rangle := \int_\Omega fg dx \\ \mathcal{V} &:= H_0^1(\Omega;\mathbb{R}), \qquad \langle f,g\rangle_{\mathcal{V}} := \int_\Omega \nabla_x f \cdot \nabla_x g dx + \int_\Omega fg dx \end{aligned}$$

with associated norms $\| f \| := \langle f, f\rangle$ and $\| f \|_{\mathcal{V}} := \langle f, f\rangle_{\mathcal{V}}$. Select f and V satisfying

$$f \in \mathrm{H} \quad ; \quad V \in (\mathcal{C}^\infty \cap L^\infty)(\Omega;\mathbb{R}^N) \quad ; \quad \operatorname{div} V := \sum_{j=1}^N \partial_{x_j} V \equiv 0.$$

Assume that $u \in \mathcal{V}$ is a solution in the sense of distributions to the equation

$$(V \cdot \nabla_x) u - \Delta u = f \quad \text{in } \ \mathcal{D}'(\Omega) \tag{1}$$

2.1. Show that there exists a continuous coercive form Q on $\mathcal{V} \times \mathcal{V}$ such that

$$\forall v \in \mathcal{C}_0^\infty(\Omega;\mathbb{R}), \quad Q(u,v) = \langle f, v\rangle.$$

2.2. Let $f \in L^2(\Omega;\mathbb{R})$. Show that the equation (1) has a unique solution $u \in \mathcal{V}$.