UNIVERSITÉ DE RENNES 1

Microlocal Analysis

CC2 on pseudo-differential operators, the 18/11/2022 (45mn)

Documents are not allowed

Surname : **First name :**

We work on $\mathbb{R}^n$. We consider the Laplace operator $\Delta := \sum_{j=1}^{n} \frac{\partial^2}{\partial^2_{x_i}}$.

1.1. What is the symbol $a(x,\xi)$ that is associated to the action of the operator $1-\Delta$ through the relation $1-\Delta = op(a) = a(x,D)$.

$a(x,\xi) =$

1.2. Determine the symbol $b(x,\xi)$ of the pseudo-differential operator allowing to satisfy the relation $b(x,D)\,a(x,D) = Id$. The operator $b(x,D)$ is denoted by $(1-\Delta)^{-1}$. Indicate on the right its symbol class.

$b(x,\xi) =$ $b(x,D) \in S$

1.3. Determine the symbol $c(x,\xi)$ of the pseudo-differential operator $1-(1-\Delta)^{-1}$. Indicate on the right its symbol class.

$c(x,\xi) =$ $c(x,D) \in S$

1.4. Let Ω be a relatively compact open subset of $\mathbb{R}^n$. Fix $s \in \mathbb{R}$ and $f \in H^s(\Omega)$. Given some $s_0 < s$, we consider a distribution $u \in H^{s_0}(\Omega)$ which is such that $\Delta u = f$.

1.4.1. We recall Peetre's inequality :

$$\langle \xi' \rangle^{\tilde{s}} \, \langle \xi \rangle^{-\tilde{s}} \leq 2^{|\tilde{s}|} \, \langle \xi - \xi' \rangle^{|s|} \,, \qquad \forall\, (\tilde{s},\xi,\xi') \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n .$$

We select a cutoff function $\chi \in C_c^\infty(\Omega)$. Show that $\chi\,\Delta u \in H^s(\mathbb{R}^n)$.

1.4.2. Show that $g := (1-\Delta)^{-1}\,\chi\,\Delta u \in H^{s+2}(\mathbb{R}^n)$.

$\Longrightarrow$ T.S.V.P.

1.5. Prove that $g = (1-\Delta)^{-1}\,[\chi, \Delta]u - \left(1 - (1-\Delta)^{-1}\right)\chi u$.

1.6. Explain why $R := [\chi, \Delta]$ is a first-order differential operator.

1.7. Let $\psi \in C_c^\infty(\Omega)$ be a non negative function which is equal to 1 on the ball $\{\xi; |\xi| \leq 1\}$ and equal to 0 out of the ball $\{\xi; |\xi| \leq 2\}$. Prove that the pseudo-differential operator $e(x, D) := \psi(D) + c(x, D)$ (where c is as in question 1.2) is an isomorphism of $H^s(\mathbb{R}^n)$ for all $s \in \mathbb{R}$.

1.8. Show that $\chi u = e(x, D)^{-1}\left[\left(1-\Delta)^{-1}\right) Ru - g + \psi(D)\,(\chi u)\right]$, and deduce from this relation that χu is in $H^{s_0+1}(\mathbb{R}^n)$.