UNIVERSITÉ DE RENNES 1

Spectral Theory

CC1, Correction

Question. Give an example of a Hilbert separable complex space which is of infinite dimension. Write down the corresponding inner product.

Just take $\ell^2(\mathbb{N};\mathbb{C})$ *with*

$$\forall u = (u_n) \in \ell^2, \quad \forall v = (v_n) \in \ell^2, \quad \langle u, v\rangle = \sum_{n=0}^{\infty} \bar{u}_n v_n.$$

Or just take $L^2(\mathbb{R};\mathbb{C})$ *with*

$$\forall u =\in L^2, \quad \forall v \in L^2, \quad \langle u, v\rangle = \int_{\mathbb{R}} \bar{u}(x) v(x)\, dx.$$

Exercise 1. Let $\langle\cdot,\cdot\rangle$ be some inner product on $\mathbb{C}^n$, where $n \in \mathbf{N}^*$. Fix some complex unitary matrix A of size $n \times n$, which means that $A^* = {}^t\bar{A} = A^{-1}$.

1.1. Give an example of a *non diagonal* unitary matrix A which is of size 2×2.

It suffices to give the matrix of a rotation of angle $\theta \neq 0(\pi)$, *like*

$$A = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

1.2. Let λ be an eigenvalue of A. Prove that $|\lambda| = 1$.

Let $x \neq 0$ *be an eigenvector associated with* λ. *Then*

$$\langle Ax, Ax\rangle = \langle \lambda x, \lambda x\rangle = |\lambda|^2 \parallel x \parallel^2 = \langle x, A^* A x\rangle = \langle x, A^{-1} A x\rangle = \parallel x \parallel^2 .$$

It follows that $|\lambda|^2 = 1$ *which is possible only if* $|\lambda| = 1$.

1.3. Show that the eigenvectors corresponding to distinct eigenvalues are orthogonal.

Let $x \neq 0$ *and* $y \neq 0$ *be two eigenvectors which are respectively associated with the distinct eigenvalues* λ *and* μ *(we have* $\lambda \neq \mu$*). Then*

$$\langle Ax, Ay\rangle = \langle \lambda x, \mu y\rangle = \lambda\bar{\mu}\langle x, y\rangle = \langle x, A^* A y\rangle = \langle x, A^{-1} A y\rangle = \langle x, y\rangle.$$

Since $|\mu| = 1$, *we have* $\bar{\mu} = \mu^{-1}$, *and therefore*

$$\mu^{-1}(\lambda - \mu)\langle x, y\rangle = 0,$$

which implies $\langle x, y\rangle = 0$.

1.4. Prove that there is an orthogonal basis of the whole space, consisting of eigenvectors.

Let λ be an eigenvalue of A associated with the eigenvector $x \neq 0$. The vector space

$$F^{\perp}, \quad F := \{\mu x; \mu \in \mathbb{C}\}$$

is of dimension $n-1$ and it is stable under the action of A since

$$\forall y \in F^{\perp}, \quad \langle Ay, x\rangle = \langle y, A^* x\rangle = \langle y, A^{-1}x\rangle = \lambda^{-1}\langle y, x\rangle = 0.$$

Look at the restriction $A_{|F}$ and apply an iterative argument.

Exercise 2. We denote by $\langle \cdot, \cdot \rangle$ the inner product on $L^2(\mathbb{R}; \mathbb{C})$, with associated norm $\| u \|$. By a spectral argument coming from the course, prove that

$$\forall u \in \mathcal{S}(\mathbb{R}), \quad \| u \|^2 \leq \langle (-\partial^2_{xx} + x^2)u, u\rangle,$$

where $\mathcal{S}(\mathbb{R})$ is the Schwartz space.

We have seen in the course that $L^2(\mathbb{R}; \mathbb{C})$ admits some orthogonal basis $(u_n)_{n\geq 1}$ made of eigenvectors of the harmonic operator $-\partial^2_{xx} + x^2$, with eigenvalues $2n-1$. Thus, any L^2 function u can be decomposed according to

$$u = \sum_{n=1}^{\infty} \alpha_n u_n, \quad \alpha_n \in \mathbb{C}.$$

It suffices to remark that

$$\| u \|^2 = \sum_{n=1}^{\infty} |\alpha_n|^2 \leq \sum_{n=1}^{\infty} (2n-1)|\alpha_n|^2 = \langle \sum_{n=1}^{\infty} (2n-1)\alpha_n u_n, \sum_{n=1}^{\infty} \alpha_n u_n\rangle = \langle (-\partial^2_{xx} + x^2)u, u\rangle.$$

Exercise 3. On $\mathcal{S}(\mathbb{R})$, consider the operators $L^+ := -\partial_x + x$ and $L^- := \partial_x + x$. Compute the commutator $[L^+; L^-]$.

Given $\varphi \in \mathcal{S}(\mathbb{R})$, we have to compute

$$\begin{aligned}[L^+; L^-]\varphi &= L^+L^-\varphi - L^-L^+\varphi = L^+(\partial_x\varphi + x\varphi) - L^-(-\partial_x\varphi + x\varphi) \\ &= -\partial_x(\partial_x\varphi + x\varphi) + x(\partial_x\varphi + x\varphi) - \partial_x(-\partial_x\varphi + x\varphi) - x(-\partial_x\varphi + x\varphi) \\ &= -\varphi - x\partial_x\varphi + x\partial_x\varphi + x^2\varphi - \varphi - x\partial_x\varphi + x\partial_x\varphi - x^2\varphi = -2\varphi.\end{aligned}$$

In other words $[L^+; L^-] = -2$.