UNIVERSITÉ DE RENNES 1

Spectral Theory

CC1, correction

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Question. Let E and F be two Banach spaces. Consider a linear subspace $\mathrm{Dom}(T) \subset E$. Define what is a closed operator $(\mathrm{Dom}(T), T)$.

The operator $(Dom(T), T)$ is closed if the graph

$$\Gamma(T) := \{(x, Tx); x \in Dom(T)\}$$

is closed as a subset of $E \times F$.

Exercise 1. Recall that the Sobolev spaces $H \equiv H^0(\mathbf{R})$ and $H^1(\mathbf{R})$ can be defined by

$$H \equiv H^0(\mathbf{R}) \equiv L^2(\mathbf{R}), \qquad \| f \|_H := \Big(\int |f(x)|^2 dx \Big)^{1/2}$$

$$H^1(\mathbf{R}) := \{ f \in L^2(\mathbf{R}); f' \in L^2(\mathbf{R}) \}, \qquad \| f \|_{H^1} := (\| f \|_H^2 + \| f' \|_H^2)^{1/2}$$

On $H \equiv L^2(\mathbf{R})$ consider the two operators $T : H \to H$ and $S : H \to H$ given by

$$\mathrm{Dom}\ (T) = \mathcal{C}_0^\infty(\mathbb{R}), \qquad Tf = f' \tag{1}$$

$$\mathrm{Dom}\ (S) = H^1(\mathbb{R}), \qquad Sf = f' \tag{2}$$

1.1. Show that $(\mathrm{Dom}(S), S)$ is a closed operator.

Let $(f, g) \in \overline{\Gamma(S)}$. Then, we can find a sequence $(f_n)_n \in H^{\mathbf{N}}$ such that, in the sense of the L^2-convergence, we have

$$f_n \longrightarrow f \in H \quad ; \quad f'_n \longrightarrow g \in H$$

Since $(f'_n)_n$ converges to f' in the sense of distributions, with a unique limit, we can assert that $f' = g \in L^2(\mathbf{R})$, and therefore $(f, g) \in \Gamma(S)$. This means that $\Gamma(S)$ is closed.

1.2. What is the smallest closed extension $(\mathrm{Dom}(\bar{T}), \bar{T})$ of $(\mathrm{Dom}(T), T)$? Justify your answer.

The same argument as above with $f_n \in \mathcal{C}_0^\infty(\mathbb{R})$ shows that $\overline{\Gamma(T)} \subset \Gamma(S)$. Since $\mathcal{C}_0^\infty(\mathbb{R})$ is dense in $H^1(\mathbb{R})$, we have the equality $\Gamma(S) \subset \overline{\Gamma(T)}$, and therefore

$$(Dom(\bar{T}), \bar{T}) \equiv (Dom(S), S) \quad ; \quad \bar{T} \equiv S$$

Exercise 2. Let E and F be two Banach spaces. Consider a linear subspace $\mathrm{Dom}(T) \subset E$, and a closable operator $\big(\mathrm{Dom}(T), T\big)$ which is of finite-rank, meaning that

$$d := \dim \,\mathrm{Ran}\, T < +\infty \quad ; \quad \mathrm{Ran}\, T := \big\{T(x); x \in \mathrm{Dom}(T)\big\} \subset F \tag{3}$$

Show that T is bounded.

Assume that T is not bounded. Then, we can find a sequence $(f_n)_n \in Dom(T)^{\mathbf{N}}$ such that

$$\| f_n \| = 1 \quad ; \quad \lim_{n \to +\infty} \| T f_n \| = +\infty$$

Define $u_n := \| T f_n \|^{-1} f_n$ and $v_n := T u_n$, so that

$$\lim_{n \to +\infty} \| u_n \| = 0 \quad ; \quad \| v_n \| = 1$$

The sequence $(v_n)_n$ belongs to the unit sphere $\mathbb{S}^{d-1}$ of $Ran\, T \subset F$, which is compact because $Ran\, T$ is of finite dimension, see (3). Thus, we can find a converging subsequence, still denoted by $(v_n)_n$, such that $(v_n)_n$ converges to some $v \in Ran\, T$ with $\| v \| = 1$. It follows that $v \neq 0$. Then, a criterion viewed during the course guarantees that the operator $\big(Dom(T), T\big)$ is not closable, which is the expected contradiction.