UNIVERSITÉ DE RENNES 1

Spectral Theory

CC1, the 13/09/2019 (20mn)

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Surname : **First name :**

Question. Give an example of a Hilbert separable complex space which is of infinite dimension. Write down the corresponding inner product.

Exercise 1. Let $\langle \cdot, \cdot \rangle$ be some inner product on $\mathbb{C}^n$, where $n \in \mathbf{N}^*$. Fix some complex unitary matrix A of size $n \times n$, which means that $A^* = {}^t\bar{A} = A^{-1}$.

1.1. Give an example of a *non diagonal* unitary matrix A which is of size 2×2.

$$A = \begin{pmatrix} & & & \end{pmatrix}$$

1.2. Let λ be an eigenvalue of A. Prove that $|\lambda| = 1$.

1.3. Show that the eigenvectors corresponding to distinct eigenvalues are orthogonal.

1.4. Prove that there is an orthogonal basis of the whole space, consisting of eigenvectors.

T.S.V.P $\Longrightarrow$

Exercise 2. We denote by $\langle \cdot, \cdot \rangle$ the inner product on $L^2(\mathbb{R})$, with associated norm $\| u \|$. By a spectral argument coming from the course, prove that

$$\forall u \in \mathcal{S}(\mathbb{R}), \quad \| u \|^2 \leq \langle (-\partial_{xx}^2 + x^2) u, u \rangle,$$

where $\mathcal{S}(\mathbb{R})$ is the Schwartz space.

Exercise 3. On $\mathcal{S}(\mathbb{R})$, consider the operators $L^+ := -\partial_x + x$ and $L^- := \partial_x + x$. Compute the commutator $[L^+; L^-]$.