UNIVERSITÉ DE RENNES 1

Spectral Theory

CC1, the 21/09/2018 (20mn)

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Surname : **First name :**

Question. Let E and F be two Banach spaces. Consider a linear subspace $\mathrm{Dom}(T) \subset E$. Define what is a closed operator $(\mathrm{Dom}(T), T)$.

Exercise 1. Recall that the Sobolev spaces $H \equiv H^0(\mathbf{R})$ and $H^1(\mathbf{R})$ can be defined by

$$H \equiv H^0(\mathbf{R}) \equiv L^2(\mathbf{R})\,, \qquad \| f \|_H := \Big(\int |f(x)|^2 dx \Big)^{1/2}$$

$$H^1(\mathbf{R}) := \{ f \in L^2(\mathbf{R}); f' \in L^2(\mathbf{R}) \}\,, \qquad \| f \|_{H^1} := (\| f \|_H^2 + \| f' \|_H^2)^{1/2}$$

On $H \equiv L^2(\mathbf{R})$ consider the two operators $T : H \to H$ and $S : H \to H$ given by

$$\mathrm{Dom}\ (T) = \mathcal{C}_0^\infty(\mathbb{R})\,, \qquad Tf = f' \tag{1}$$

$$\mathrm{Dom}\ (S) = H^1(\mathbb{R})\,, \qquad Sf = f' \tag{2}$$

1.1. Show that $(\mathrm{Dom}(S), S)$ is a closed operator.

1.2. What is the smallest closed extension $(\mathrm{Dom}(\bar{T}), \bar{T})$ of $(\mathrm{Dom}(T), T)$? Justify your answer.

T.S.V.P $\Longrightarrow$

Exercise 2. Let E and F be two Banach spaces. Consider a linear subspace $\mathrm{Dom}(T) \subset E$, and a closable operator $\big(\mathrm{Dom}(T), T\big)$ which is of finite-rank, meaning that

$$d := \dim\,\mathrm{Ran}\,T < +\infty \quad ; \quad \mathrm{Ran}\,T := \big\{T(x); x \in \mathrm{Dom}(T)\big\} \subset F$$

Show that T is bounded.