UNIVERSITÉ DE RENNES 1

Microlocal Analysis

Correction of the CC1 (the 28/10/2022)

Exrecice 1. We denote by $k_B \simeq 10^{-23}$ the Boltzmann's constant and by $\hbar \simeq 10^{-16}$ the reduced Planck constant. Introduce $\beta := 1/(k_B T)$ where T is the temperature. In Planck's model of blackbody radiation, the energy in a given frequency ω of electromagnetic radiation is distributed randomly over all numbers of the form $n\hbar\omega$ with $n = 0, 1, 2, \cdots$. The *likelihood* $p(E = n\hbar\omega)$ of finding energy $n\hbar\omega$ and the *expected value* $\langle E \rangle$ of the total energy are assumed to be

$$p(E = n\hbar\omega) = \frac{1}{Z}\, e^{-\beta n\hbar\omega}\,, \qquad \langle E \rangle := \frac{1}{Z} \sum_{n=0}^{+\infty} n\hbar\omega\, e^{-\beta n\hbar\omega}.$$

1.1. The number Z is a normalization constant which is chosen so that the sum over $n \in \mathbb{N}$ of the probabilities $p(E = n\hbar\omega)$ is 1. What is the value of Z ?

$$Z = \sum_{n=0}^{\infty} (e^{-\beta\hbar\omega})^n = \frac{1}{1 - e^{-\beta\hbar\omega}}\,.$$

1.2. Show that $\langle E \rangle = \dfrac{\hbar\omega}{e^{\beta\hbar\omega} - 1}$.

Beginning with the formula for the sum of a geometric series (for $0 \leq t < 1$*)*

$$\sum_{n=0}^{\infty} t^n = 1/(1-t)\,,$$

by differentiation, we obtain

$$\sum_{n=0}^{\infty} n t^{n-1} = 1/(1-t)^2\,.$$

Multiply this by t *and replace* t *by* $t = e^{-s}$ *to obtain*

$$\sum_{n=0}^{\infty} n e^{-sn} = e^{-s}/(1-e^{-s})^2\,.$$

We can apply this with $s = \beta\hbar\omega$ *to find with question 1.1 that*

$$\langle E \rangle = (1 - e^{-\beta\hbar\omega})\ \hbar\omega \sum_{n=0}^{+\infty} n\, e^{-\beta n\hbar\omega} = (1 - e^{-\beta\hbar\omega})\ \hbar\omega\ \frac{e^{-\beta\hbar\omega}}{(1 - e^{-\beta\hbar\omega})^2} = \frac{\hbar\omega}{e^{\beta\hbar\omega} - 1}\,.$$

1.3. Could you explain why physicists observe for small frequencies ω a linear behavior of $\langle E \rangle$ with respect to the temperature whereas this behavior changes completely for large values of ω (this change of behavior is in connection with the ultraviolet catastrophe).

For "small" frequencies ω, we have $\langle E \rangle \sim \beta^{-1} = k_B T$. Since k_B is a constant whereas T can be modified, physicists have first established (and experimentally verified) linear laws expressing $\langle E \rangle$ in terms of T. On the contrary, for "large" values of ω, we find that $\langle E \rangle \sim 0$ and the preceding linear laws are no longer applicable. This change of behavior is called the ultraviolet catastrophe. At the end of the 19th century, German physicist Max Planck heuristically derived the preceding formula (in question 1.2) for $\langle E \rangle$ precidely by assuming that a hypothetical electrically charged oscillator in a cavity that contained black-body radiation could only change its energy in a minimal increment ($\hbar\omega$). This discovery is of fundamental importance to quantum theory.

Exrecice 2. On the Hilbert space $\mathcal{H} := L^2(\mathbb{R})$, consider the two (unbounded self-adjoint) operators

$$X\, f(x) := x\, f(x)\,, \qquad P\, f(x) := -i\hbar\, \frac{df}{dx}\,.$$

2.1. Let $(r, s) \in \mathbb{R}^2$. What is the name of the theorem allowing to define $e^{irX/\hbar}$ and $e^{isP/\hbar}$ as bounded operators on $\mathcal{H}$ (circle your response) ?

Stone's theorem

2.2. Compute $e^{irX/\hbar}$ on $\mathcal{H}$ as a multiplication operator

$$e^{irX/\hbar}\, f(x) = \sum_{n=0}^{+\infty} \frac{1}{n!}\, (irX/\hbar)^n\, f(x) = \sum_{n=0}^{+\infty} \frac{1}{n!}\, (ir/\hbar)^n\, (X^n f(x)) = \Big(\sum_{n=0}^{+\infty} \frac{1}{n!}\, (irx/\hbar)^n\Big)\, f(x).$$

This means that $e^{irX/\hbar}$ acts on $\mathcal{H}$ by multiplication by the function $e^{irx/\hbar}$, that is

$$e^{irX/\hbar}\, f(x) := e^{irx/\hbar}\, f(x)\,.$$

2.3. Could you explain (at least formally) why $e^{isP/\hbar}$ should be the translation operator given by $e^{isP/\hbar}\, f(x) = f(x+s)$.

$$\begin{aligned} e^{isP/\hbar}\, f(x) &= \sum_{n=0}^{+\infty} \frac{1}{n!}\, (isP/\hbar)^n\, f(x) = \sum_{n=0}^{+\infty} \frac{1}{n!}\, (is/\hbar)^n\, (P^n f(x)) \\ &= \sum_{n=0}^{+\infty} \frac{1}{n!}\, (is/\hbar)^n\, (-i\hbar)^n f^{(n)}(x) = \sum_{n=0}^{+\infty} \frac{1}{n!}\, s^n\, f^{(n)}(x). \end{aligned}$$

We can recognize on the right hand side the Taylor series of $f(x+s)$.

2.4. Show that $e^{irX/\hbar}\, e^{isP/\hbar} = e^{-irs/\hbar}\, e^{isP/\hbar}\, e^{irX/\hbar}$.

On the one side, we have

$$e^{irX/\hbar}\, e^{isP/\hbar}\, f(x) = e^{irX/\hbar}\, f(x+s) = e^{irx/\hbar}\, f(x+s).$$

On the other side, we have

$$e^{isP/\hbar}\, e^{irX/\hbar}\, f(x) = e^{isP/\hbar}\left(e^{irx/\hbar}\, f(x)\right) = e^{ir(x+s)/\hbar}\, f(x+s) == e^{irs/\hbar}\left(e^{irx/\hbar}\, f(x+s)\right).$$

The standard form of the canonical commutation relations is given by $[X, P] = i\hbar I$. *The question 2.4 furnishes the exponentiated form of this canonical commutation relation.*