UNIVERSITÉ DE RENNES 1

Microlocal Analysis

Correction of the CC1 (the 19/11/2021)

Documents are not allowed

Surname : **First name :**

Let $m \in \mathbb{R}$. We consider a symbol $a \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n; \mathbb{C})$ which is in the class $S^m \equiv S^m_{1,0}$ of symbols of order m.

1. Recall the definition of the symbol class S^m.

$$S^m = \{ a \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n; \mathbb{C}) \, ; \quad \forall \, (\alpha, \beta) \in \mathbb{N}^n \times \mathbb{N}^n, \quad \exists \, C_{\alpha,\beta} \in \mathbb{R}_+; \quad \forall (x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n, \\ |\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \leq C_{\alpha,\beta} \, (1+ \parallel \xi \parallel)^{m-|\beta|} \}.$$

2. We assume that we can find $\tilde m < m$ and $R \in \mathbb{R}_+^*$ which are such that

$$\forall \, (\alpha, \beta) \in \mathbb{N}^n \times \mathbb{N}^n, \quad \exists \, C_{\alpha,\beta} \in \mathbb{R}_+; \quad R \leq \parallel \xi \parallel \Longrightarrow |\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \leq C_{\alpha,\beta} \, (1+ \parallel \xi \parallel)^{\tilde m-|\beta|}.$$

Prove that a is in $S^{\tilde m}$.

It suffices to obtain the bound for $|\xi| \leq R$. We already know that

$$\forall \, (\alpha, \beta) \in \mathbb{N}^n \times \mathbb{N}^n, \quad \exists \, \tilde C_{\alpha,\beta} \in \mathbb{R}_+; \quad |\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \leq \tilde C_{\alpha,\beta} \, (1+ \parallel \xi \parallel)^{\tilde m-|\beta|+m-\tilde m}$$

which implies that

$$|\xi| \leq R \implies |\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \leq \tilde C_{\alpha,\beta} \, (1+R)^{m-\tilde m} \, (1+ \parallel \xi \parallel)^{\tilde m-|\beta|}.$$

This yields the expected bound with $C_{\alpha,\beta} = \tilde C_{\alpha,\beta} \, (1+R)^{m-\tilde m}$.

3. We assume in this question that $m < -n$.

3.1 Show that we can find some $K \in L^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ such that

$$\forall u \in \mathcal{S}(\mathbb{R}^n), \qquad op(a)u(x) := \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \, a(x,\xi) \, \hat u(\xi) \, d\xi = \int_{\mathbb{R}^n} K(x,y) \, u(y) \, dy.$$

By construction, we have

$$K(x,y) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi} \, a(x,\xi) \, d\xi.$$

Since $n+m-1<-1$, this yields

$$|K(x,y)| \leq \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} |a(x,\xi)|\, d\xi \leq \frac{C_{0,0}}{(2\pi)^n} \int_{\mathbb{R}^n} (1+ \parallel \xi \parallel)^m \, d\xi$$
$$\leq \frac{C_{0,0}}{(2\pi)^n} \int_0^{+\infty} (1+r)^m \, r^{n-1} \, dr < +\infty.$$

3.2. Let $\alpha \in \mathbb{N}^n$. Prove that $(x-y)^\alpha \, K(x,y) \in L^\infty(\mathbb{R}^n)$.

Using integration by parts in ξ, we get

$$(x-y)^\alpha \, K(x,y) = \frac{(-i)^\alpha}{(2\pi)^n} \int_{\mathbb{R}^n} \big(i(x-y)\big)^\alpha \, e^{i(x-y)\cdot\xi} \, a(x,\xi)\, d\xi = \frac{i^\alpha}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi} \, \partial_\xi^\alpha a(x,\xi)\, d\xi.$$

Since $\partial_\xi^\alpha a \in S^{m-|\alpha|} \subset S^m$ with $m<-n$, the same argument as above yields the expected bound.

3.3. Show that, for all $p \in \mathbb{N}^*$, the map $op(a) : L^p(\mathbb{R}^n) \longrightarrow L^p(\mathbb{R}^n)$ is a bounded operator.

From question 3.2, we can infer that

$$\forall N \in \mathbb{N}, \qquad |K(x,y)| \leq C_N \, (1+|x-y|)^{-N}.$$

This guarantees that

$$|op(a)u(x)| \leq C_N \, (1+|x|)^{-N} * |u|.$$

For $N>n$, the function $(1+|x|)^{-N}$ belongs to L^1. By Young's inequality ($L^1 * L^p \subset L^p$), we get the result.

4. Let A, B and $C \not\equiv 0$ three self-adjoint operators on a Hilbert space $\mathcal{H}$. We assume that $[A,B] = i\, Id$ and $[A,C]=0$. Can we assert that $[B,C] \not\equiv 0$? Justify the answer.

The reply is NO. Just take

$$\mathcal{H} = L^2(\mathbb{R}^2), \qquad A = i\partial_{x_1}, \qquad B = x_1 \times, \qquad C = i\partial_{x_2}.$$