UNIVERSITÉ DE RENNES 1

Microlocal Analysis

CC1 on the origins of Quantum Mechanics, the 28/10/2022 (30mn)

Documents are not allowed

Surname : **First name :**

Exrecice 1. We denote by $k_B \simeq 10^{-23}$ the Boltzmann's constant and by $\hbar \simeq 10^{-16}$ the reduced Planck constant. Introduce $\beta := 1/(k_B T)$ where T is the temperature. In Planck's model of blackbody radiation, the energy in a given frequency ω of electromagnetic radiation is distributed randomly over all numbers of the form $n\hbar\omega$ with $n = 0, 1, 2, \cdots$. The *likelihood* $p(E = n\hbar\omega)$ of finding the energy E at the frequency $n\hbar\omega$ and the *expected value* $\langle E \rangle$ of the total energy are assumed to be

$$p(E = n\hbar\omega) = \frac{1}{Z}\, e^{-\beta n\hbar\omega}, \qquad \langle E \rangle := \frac{1}{Z} \sum_{n=0}^{+\infty} n\hbar\omega\, e^{-\beta n\hbar\omega}.$$

1.1. The number Z is a normalization constant which is chosen so that the sum over $n \in \mathbb{N}$ of the probabilities $p(E = n\hbar\omega)$ is 1. What is the value of Z ?

$Z =$

1.2. Show that $\langle E \rangle = \dfrac{\hbar\omega}{e^{\beta\hbar\omega} - 1}$.

1.3. Could you explain why physicists observe for small frequencies ω a linear behavior of $\langle E \rangle$ with respect to the temperature whereas this behavior changes completely for large values of ω (this change of behavior is in connection with the ultraviolet catastrophe).

$\Longrightarrow$ T.S.V.P.

Exrecice 2. On the Hilbert space $\mathcal{H} := L^2(\mathbb{R})$, consider the two (unbounded self-adjoint) operators

$$X\, f(x) := x\, f(x)\,, \qquad P\, f(x) := -i\hbar\, \frac{df}{dx}\,.$$

2.1. Let $(r, s) \in \mathbb{R}^2$. What is the name of the theorem allowing to define $e^{irX/\hbar}$ and $e^{isP/\hbar}$ as bounded operators on $\mathcal{H}$ (circle your response) ?

Hille-Yosida theorem Von Neumann's theorem Stone's theorem

2.2. Compute $e^{irX/\hbar}$ on $\mathcal{H}$ as a multiplication operator

$e^{irX/\hbar}\, f(x) =$

2.3. Could you explain (at least formally) why $e^{isP/\hbar}$ should be the translation operator given by $e^{isP/\hbar}\, f(x) = f(x + s)$.

2.4. Show that $e^{irX/\hbar}\, e^{isP/\hbar} = e^{-irs/\hbar}\, e^{isP/\hbar}\, e^{irX/\hbar}$.