Linear time solution to the conjugacy problem in RAAGs and their subgroups

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Joint work with John Crisp and Eddy Godelle

1. Conjugacy problem in $\mathbb{F}_n$ and RAAGs
2. Quasiconvex subgroups of RAAGs
3. Example : surface groups in RAAGs
4. Example : graph braid groups in RAAGs
5. Conjugacy problem in subgroups of RAAGs

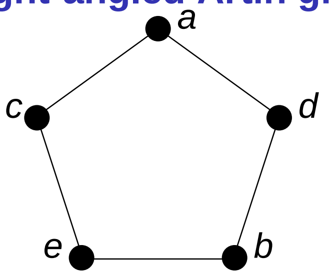
Conjugacy problem in $\mathbb{F}_n$

$$\Updownarrow$$

comparing cyclic words

Linear time
[Knuth-Morris-Pratt, Boyer-Moore, Suffix tree methods, ...]

Right-angled Artin groups



The graph defines a RAAG

$$A := \langle a, b, c, d, e \mid [a, b] = 1, [b, c] = 1, \ldots \rangle$$

In general : any graph without loops, double edges $\rightsquigarrow$ RAAG.

$\forall$ such group A, $\exists$ cubical complex Y, loc. $CAT(0)$ with one n-cube (with opposite faces identified) for each n-tuple of commuting generators, s.t.

$$\pi_1(Y) = A$$

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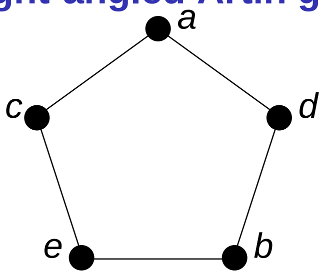

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Solution to conj. problem in RAAG A

1st linear time solution : [Liu, Wrathall, Zeger]

Our solution : given w_1, w_2,

1. build pilings p_1, p_2 ($\Longrightarrow$ lin. time solution to word problem)
2. cyclically reduce p_1, p_2
3. $\xrightarrow{\text{cycling}}$ create *pyramidal* pilings $\widetilde{p}_1, \widetilde{p}_2$
4. $\xrightarrow{\text{extract}}$ canonical cyclic words $\widetilde{w}_1, \widetilde{w}_2$
5. compare them : w_1, w_2 conjugate iff $\widetilde{w}_1 \overset{\text{cyclic}}{=} \widetilde{w}_2$

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Example : $A =$ 

Order generators $a < b < c < d$.

$$w_1 = dc^{-1}b^{-1}ccabb$$

(1) Piling : sticks $\leftrightarrow$ generators, beads of 3 colours : $+, -, 0$
tile : 1 bead $\pm$, connected by threads
to 0-beads on adjct. sticks
0-beads commute, but block $\pm$-beads
cancellation of $+ \leftrightarrow -$ tiles

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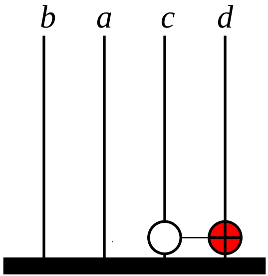

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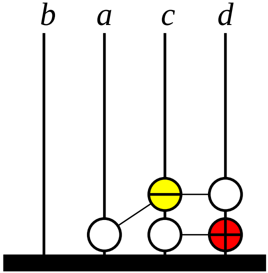

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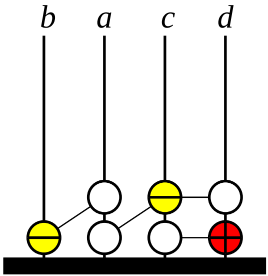

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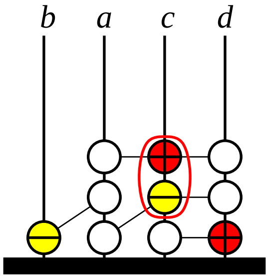

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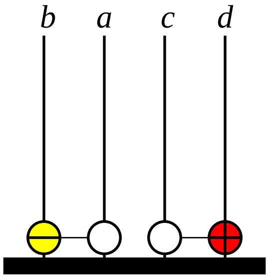

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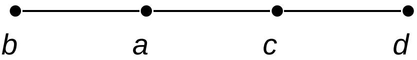

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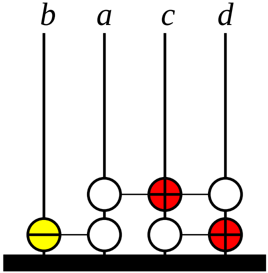

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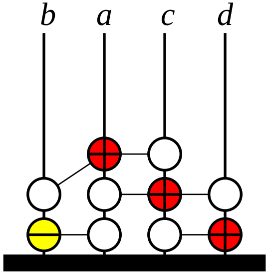

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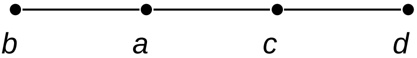

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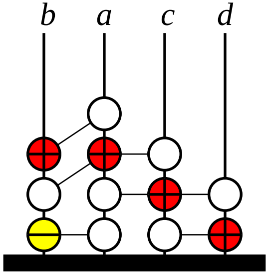

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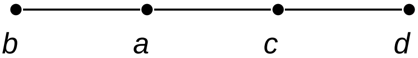
Example : $A =$ 

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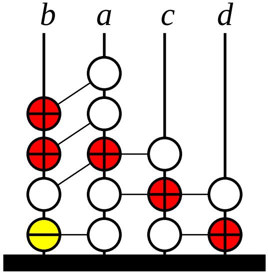
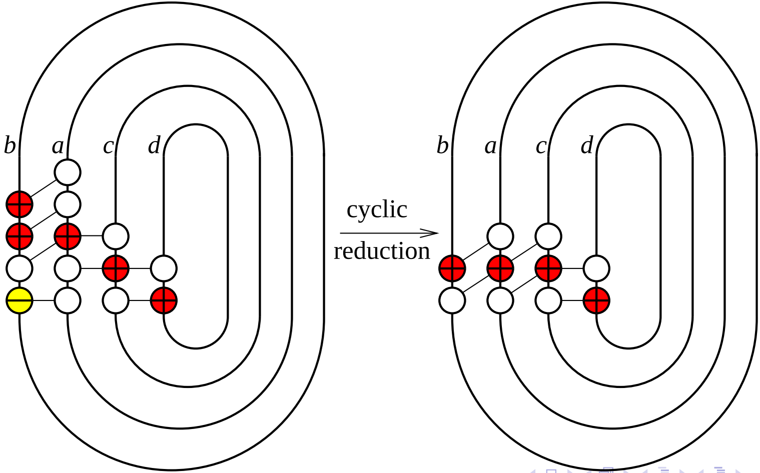


(2) Cyclic reduction



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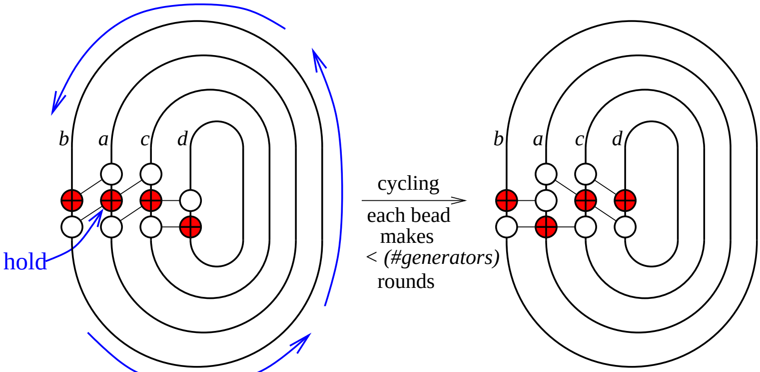
(3) Create a *pyramidal piling* $\widetilde{p}_1$ cyclically equivalent to p_1 :
let a whirlwind act on piling, but hold the lowest *a*-bead in place.



This is the pyramidal piling $\widetilde{p}_1$

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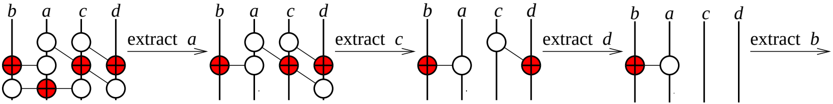
(4) Find a canonical cyclic word $\widetilde{w}_1$ whose piling is $\widetilde{p}_1$.
Algorithm : keep extracting largest letter ($a < b < c < d$)
from bottom of $\widetilde{p}_1$



$$\Longrightarrow \quad \widetilde{w}_1 = acdb$$

Lemma

Suppose

$$\Phi : \underset{\text{cubed}}{X} \longrightarrow \underset{\text{cubed, CAT(0)}}{Y} \quad \text{locally injective, locally convex. Then}$$

- X is locally CAT(0),
- $\widetilde{\Phi}\colon\ \widetilde{X} \longrightarrow \widetilde{Y}$ is an isometric embedding
- $\Phi_*\colon\ \pi_1(X) \to \pi_1(Y)$ is a monomorphism, q.i. embedding.$\square$

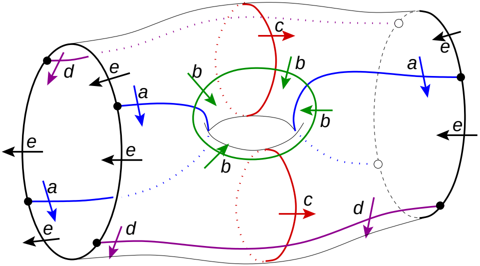
We shall apply lemma in case

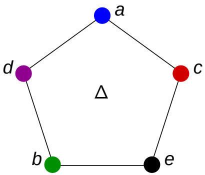
$$\pi_1(Y) \cong A \quad \text{a RAAG,} \ \ Y \text{ the associated cube complex.}$$

Example : surface groups in RAAGs

Theorem [Crisp, Wiest] $\exists$ RAAG A which contains all surface groups $\pi_1(S)$ except $S = \mathbb{R}\mathrm{P}^2, \mathit{KleinB}, S_{\chi=-1}$.

Example : S an orientable surface





Theorem [Crisp, Wiest] The three exceptional surface groups don't embed in any RAAG.

Example : graph braid groups in RAAGs

Definition If Γ a graph, $n \in \mathbb{N}$, define

$$B_n(\Gamma) = \pi_1(\text{discretized config. space of } n \text{ points in } \Gamma)$$

Theorem [Crisp, Wiest] $\forall\Gamma, \forall n, \quad B_n(\Gamma) \hookrightarrow$ some RAAG.

Example For $\Gamma = K_5$ we have

$$B_2(K_5) \hookrightarrow RAAG(\Delta).$$

Amusing fact
(Discretized configuration space of 2 points in K_5) $\cong S_{\chi=-5}$, so

$$B_2(K_5) = \pi_1(S_{\chi=-5}).$$

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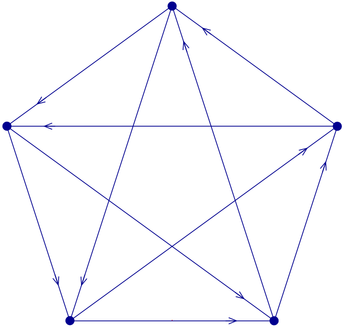

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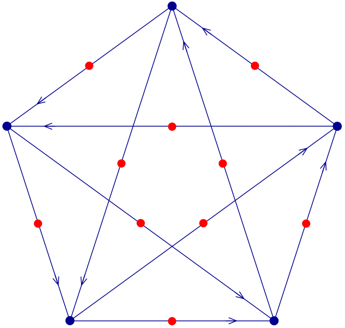

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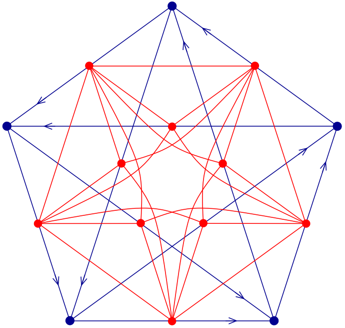

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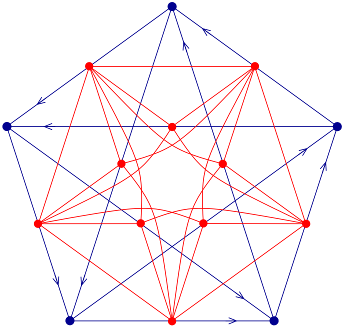

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Suppose $\Phi\colon X \to Y$ as above ($\Longrightarrow \pi_1(X) \hookrightarrow \pi_1(Y) =$RAAG)

Want Linear time solution to conjugacy problem in $\pi_1(X)$.

Wrong theorem If $\alpha, \beta \in \pi_1(X)$,

$$\alpha, \beta \text{ conjugate in } \pi_1(X) \Leftrightarrow \Phi_*(\alpha), \Phi_*(\beta) \text{ conjugate in } \pi_1(Y)$$

Wrong proof Suppose $\Phi(\alpha)$, $\Phi(\beta)$ freely homotopic in Y.

Hypotheses on $\Phi \Longrightarrow$ can pull back free homotopy to X.

$\Longrightarrow \alpha$, β freely homotopic. □

Problem easily remidied While deciding whether $\Phi(\alpha)$, $\Phi(\beta)$ freely homotopic, carry along a finite piece of extra information (a basepoint in X).

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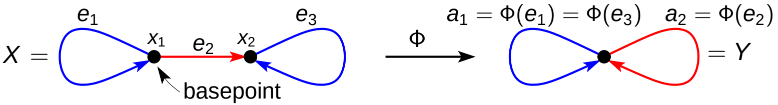


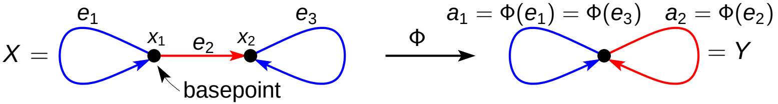
Counterexample to wrong theorem

The loops e_1 and $e_2 e_3 e_2^{-1}$ in X are not freely homotopic but their images a_1 and $a_2 a_1 a_2^{-1} \simeq a_1 a_2^{-1} a_2 \simeq a_1$ are.

Example of our remedy "carrying along a basepoint in X"

The *based words* $x_1 a_1$ and $x_1 a_2 a_1 a_2^{-1} \simeq x_2 a_1 a_2^{-1} a_2 \simeq x_2 a_1$ are not equivalent.

Example where $A = \pi_1(Y) = \mathbb{F}_2$



Counterexample to wrong theorem

The loops e_1 and $e_2e_3e_2^{-1}$ in X are not freely homotopic but their images a_1 and $a_2a_1a_2^{-1} \simeq a_1a_2^{-1}a_2 \simeq a_1$ are.

Example of our remedy "carrying along a basepoint in X"

The *based words* x_1a_1 and $x_1a_2a_1a_2^{-1} \simeq x_2a_1a_2^{-1}a_2 \simeq x_2a_1$ are not equivalent.

1. Conjugacy problem in $\mathbb{F}_n$ and RAAGs
2. Quasiconvex subgroups of RAAGs
3. Example : surface groups in RAAGs
4. Example : graph braid groups in RAAGs
5. Conjugacy problem in subgroups of RAAGs