COMPUTATIONAL EXPLORATIONS IN THOMPSON'S GROUP F

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ABSTRACT. Here we describe the results of some computational explorations in Thompson's group F. We describe experiments to estimate the cogrowth of F with respect to its standard finite generating set, designed to address the subtle and difficult question whether or not Thompson's group is amenable. We also describe experiments to estimate the exponential growth rate of F and the rate of escape of symmetric random walks with respect to the standard generating set.

1. INTRODUCTION

Richard Thompson's group F has attracted a great deal of interest over the last years. The group F is a finitely presented group which arises quite naturally in different contexts, and allows several different, but fairly simple, descriptions – for instance by a presentation, as a diagram group [14], as a group of homeomorphisms of the unit interval, as the geometry group of associativity [7], and as the fundamental group of a component of the loop space of the dunce hat. Cannon, Floyd and Parry [3] give an excellent introduction to F.

The interest in this group stems partly from F's unusual properties, and partly from the fact that some of the basic questions about this group are still open, in particular those related to its cogrowth and growth. It seems clear is that F lies very close to the borderline between different regimes.

Probably the most famous open question is whether or not F is amenable. Also, it is known that F has exponential growth, but the growth rate is unknown. Similarly, the rate of escape of random walks in F is unknown.

The question of amenability is especially intriguing since F is either an example of a finitely presented non-amenable group without free

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non-abelian subgroups, or an example of a finitely presented amenable but not elementary amenable group. Though there are finitely presented examples of groups for each of these phenomena from Grigorchuk [11] and Sapir and Olshanskii [17], those groups were constructed explicitly for those purposes, whereas F is a more "naturally occurring" example to consider – so either answer would be remarkable.

The aim of this paper is to contribute new empirical evidence to the quest to understand cogrowth, growth, and escape rate. This evidence was obtained using large computer simulations.

The structure of this paper is as follows. In Section 2 we recall briefly the definition and those properties of the group F that will be needed in the paper. Moreover, we give the definition of amenability which will be used in our experiments (there are other, equivalent, definitions which are probably more well-known). In Section 3 we describe the algorithms used in our computations relating to amenability. In Section 4 we present the results of our computer experiments, with the aim of obtaining evidence for or against the amenability of F. In Section 5, we describe two computational approaches to estimate the exponential growth rate of F with respect to the standard two-generator generating set, and in Section 6, we describe the results of some computations to measure the average distance from the origin of increasingly-long random walks, known as the rate of escape.

2. Background on Thompson's group F and amenability

Richard J. Thompson's group F is usually defined as the group of piecewise-linear orientation-preserving homeomorphisms of the unit interval, where each homeomorphism has finitely many changes of slope ("breakpoints") which all are dyadic integers and and whose slopes, when defined, are powers of 2. F admits an infinite presentation given by

$$\langle x_1, x_2, x_3, \ldots \mid x_j x_i = x_i x_{j+1} \text{ if } i < j \rangle$$

which is convenient for its symmetry and simplicity, while there is a finite presentation given by

$$\left\langle x_0, x_1 \,|\, [x_0 x_1^{-1}, x_0^{-1} x_1 x_0], [x_0 x_1^{-1}, x_0^{-2} x_1 x_0^2] \right\rangle .$$

Brin and Squier [1] showed that F has no free non-abelian subgroups, and thus the question of the amenability of F is potentially connected to the conjecture of Von Neumann that a group is amenable if and only if it had no free non-abelian subgroups. The conjecture has since been solved negatively, but the problem of the amenability of F is of independent interest and it has been open for at least 25 years.

The usefulness of the infinite presentation is the fact that F admits a normal form based on the infinite set of generators. The relators of the infinite presentation can be used to reorder generators of a given word into an expression of the following form:

$$x_{i_1}^{r_1} x_{i_2}^{r_2} \dots x_{i_n}^{r_n} x_{j_m}^{-s_m} \dots x_{j_2}^{-s_2} x_{j_1}^{-s_1}$$

with

$$i_1 < i_2 < \dots < i_n \qquad j_1 < j_2 < \dots < j_m.$$

This normal form is unique if one requires the following extra condition: if the generators x_i and x_i^{-1} both appear, then either x_{i+1} or x_{i+1}^{-1} must appear as well. Indeed, if neither x_{i+1} nor x_{i+1}^{-1} appeared, then the relator could be applied so as to obtain a shorter word representing the same element. The uniqueness of this normal form can be used to solve the word problem in short time: given a word in the infinite set of generators, find the normal form, which can be done in quadratic time, and the element is the identity if and only if the normal form is empty. This unique normal form is most helpful when the task at hand is to decide whether two words represent the same element of F. If one wishes simply to test whether a given word represents the trivial element of F, it is enough to reorder the generators, but without checking the extra condition for uniqueness.

For an introduction to F and proofs of its basic properties see Cannon, Floyd and Parry [3]. Also, for an excellent introduction to amenability, the interested reader can consult Wagon [18], Chapters 10 to 12.

There are several equivalent definitions of amenability, especially for finitely generated groups. The standard definition is given by the existence of a finitely-additive left-invariant probability measure on the set of subsets of G. If the group is finitely generated, a celebrated characterization due to Følner [8] in terms of the existence of sets with small boundary, has given a special interest to this concept from the point of view of geometric group theory, making it easier to see that amenability is a quasi-isometry invariant.

The numerical criterion we will use extensively in this paper is due to Kesten [15, 16] and it uses the concept of cogrowth.

Definition 2.1. *Let G be a finitely generated group and let*

$$1 \to K \to F_m \to G \to 1$$

be a presentation for G. The cogrowth *of G is the growth of the subgroup K inside F_m. In particular, the* cogrowth function *of G is*

$$g(n) = \#(B(n) \cap K),$$

where $B(n)$ is the ball of radius n in F_m, and the cogrowth rate *of G is*

$$\gamma = \lim_{n\to\infty} g(n)^{1/n}.$$

Kesten's cogrowth criterion for amenability states basically that a group is amenable when it has a large proportion of freely reduced words, for every length n, representing the trivial element; that is, when the cogrowth is large.

Theorem 2.2 (Kesten). *Let G be a finitely generated group, and let X be a finite set of generators, with cardinal m. Let γ be its cogrowth rate. Then G is amenable if and only if $\gamma = 2m - 1$.*

This can also be interpreted in terms of random walks. If the group is nonamenable (that is, if there are very few nontrivial words representing the trivial element of the group), then the probability of a random walk in the group ending at 1 is small. Since our random walks are taken to be non-reduced, we consider the $(2m)^L$ non-reduced words of length L in m generators, and let $T(L)$ be the set of these words which represent the identity in the group G. Then, define

$$p(L) = \frac{\#T(L)}{(2m)^L},$$

that is, we define $p(L)$ to be the proportion of words which are equal to the identity in G. Then, a rewriting of Kesten's criterion for non-reduced words can be given by

Theorem 2.3 (Kesten). *A group is amenable if and only if*

$$\limsup_{L\to\infty} p(L)^{1/L} = 1$$

Roughly speaking, a group is amenable if the probability of a random walk of length L returning to 1 decreases more slowly than exponentially with L. This form of the criterion will be used in the subsequent sections to try to study numerically the amenability of F.

3. Algorithms and programs

The direct approach at finding the numbers $p(L)$ exactly for Thompson's group F fails even at quite small values of L due to the fact that the number of words grows exponentially, so the computation times get large easily. For instance, for a length as small as 14 the number of total words is 4^{14}=268,435,456, out of which there are 1,988,452 representing the neutral element, for a value $p(14)^{1/14} = 0.704423677$. It would be hard to decide whether the sequence approaches 1. A number of improvements can be made to ease the calculation so it becomes more feasible to estimate whether the sequence tends to 1.

First, we take samples of words of a given length instead of the all words of a given length. The number 4^L grows impracticably large even for small values of L, so sampling becomes a necessity. Since the number $p(L)$ is basically a proportion (or a probability), it can be approximated by Monte Carlo methods. One can always take a random non-reduced word in the two generators x_0 and x_1 and check if it is the identity by solving the word problem quickly using the normal form. Repeating this process one can find a reasonably good approximation of the number $p(L)$.

A further improvement can be implemented by taking only *balanced* words. We observe that, since the two relators in G are commutators, a word which represents the identity has to be *balanced*: it has to have total exponent zero in both generators x_0 and x_1. So we consider not *all* random words, but only balanced ones. We remark that the abelianization of F is $\mathbb{Z}^2$, generated by x_0 and x_1, so being balanced is in fact equivalent to representing the trivial element of $\mathbb{Z}^2 = F_{\mathrm{ab}}$. Now we let $C(L)$ be the set of balanced words among the 4^L non-reduced words of length L in F_2, and define

$$\widehat{p}(L) = \frac{\#T(L)}{\#C(L)},$$

the proportion of words representing the identity of F among balanced words of length L. We have

$$\begin{aligned} \sqrt[L]{p(L)} &= \sqrt[L]{\frac{\#T(L)}{4^L}} = \sqrt[L]{\frac{\#T(L)}{\#C(L)}} \cdot \sqrt[L]{\frac{\#C(L)}{4^L}} \\ &= \sqrt[L]{\widehat{p}(L)} \cdot \sqrt[L]{\frac{\#C(L)}{4^L}}. \end{aligned}$$

Moreover, the last factor $\sqrt[L]{\frac{\#C(L)}{4^L}}$ tends to 1 as L tends to infinity, because $\mathbb{Z}^2$ is amenable. Thus F is amenable if and only if we have $\limsup_{L\to\infty} \widehat{p}(L)^{1/L} = 1$.

So in order to decide whether F is amenable, we shall try to find good approximations of $\widehat{p}(L)$, the proportion of words representing 1_F among balanced words of length L, and this for values of L which are as large as possible. Obviously, the algorithm for creating random balanced words must be designed in such a way that all balanced words of length L have the same chance of appearing. The practical advantage of approximating $\widehat{p}(L)$ rather than $p(L)$ is that $\widehat{p}(L)$ is much larger (roughly by a factor $\pi L/2$), so much smaller sample sizes are required.

Yet another improvement, which substantially increases the efficiency of the algorithm, can be made by using a "divide and conquer" strategy. The underlying observation is that if L is even, then the probability that a random word of length L represents the trivial element of F is equal to the probability that two random words of length $L/2$ represent the same element. Thus, the idea of the algorithm is to create a large number N of random words of length $L/2$ (in our implementations, values for N between 15,000 and 200,000 were generally used). Each of the N words is immediately brought into normal form, and these normal forms are stored. In order to decide if two words represent the same element of F, we simply compare their normal forms. Therefore we can consider all $N(N-1)/2$ unordered pairs of words in normal form, and we count how many identical pairs we see. This number, divided by $N(N-1)/2$, is an approximation for the proportion $p(L)$. However, the description just provided is an oversimplification, because as described above, we would like to restrict our sample to *balanced* words. Here we describe the estimation algorithm more precisely:

Each iteration of the algorithm has the following steps. In a preliminary step, we create one random *balanced* word of length L. Then we focus our attention on the first half (the first $L/2$ letters) of this word and we count which element in the quotient $F_{\mathrm{ab}} = \mathbb{Z}^2$ this first half represents —that is, we count the exponent sums of the letters x_0 and x_1 for the first half of the word.

In the second step, we create N random words of length $L/2$ which all represent this same element of the abelianization $F_{\mathrm{ab}} = \mathbb{Z}^2$, in such a way that all possible words of length $L/2$ with the given x_0-balance and x_1-balance have the same chance of appearing. As soon as it is created, each random word is transformed into normal form, and this normal form is stored.

In the third step, we count the proportion of identical pairs among all $N(N-1)/2$ unordered pairs of stored words in normal form.

In this way, each iteration of the algorithm gives an approximation to the true value of $\widehat{p}(L)$. Performing a few thousand iterations, and taking the mean of the proportions obtained in each step, one obtains an approximation to $\widehat{p}(L)$.

The expected value for the result of this algorithm is indeed $\widehat{p}(L)$, which we interpret as the probability that two random words of length $L/2$ represent the same element of F, under the condition that they represent the same element of $F_{\mathrm{ab}} = \mathbb{Z}^2$. It is immediate from the construction of the algorithm that for any pair $(k, l) \in \mathbb{Z}^2$, the proportion of words representing (k, l) in $F_{\rm ab}$ among all words constructed by the algorithm is what it should be —namely the probability that the first half of a balanced random word of length L represents (k, l) in $F_{\rm ab} = \mathbb{Z}^2$.

Then, having fixed some pair (k, l) in $\mathbb{Z}^2$, we restrict our attention to those iterations of the algorithm that deal with words with x_0-balance k and x_1-balance l (and length $L/2$). We have to prove that the expected value for the proportion of identical pairs of words in our algorithm is what it should be – namely the probability that a pair of random words, chosen with uniform probability from the set pairs of words of length $L/2$ representing the element (k, l) of $F_{\rm ab} = \mathbb{Z}^2$, represent the same element of F. That is, we have to prove that our taking words in batches of N and comparing all couples in that batch, rather than taking independent samples of pairs of words, does not distort the result. That, however, follows immediately from the fact that in our algorithm, all pairs of words of length $L/2$ with x_0-balance k and x_1-balance l, appear on average with the same frequency (they have uniform probability). The fact that our $N(N-1)/2$ samples are not independent has no impact on the expected value. It does have an impact on the variation, that is, on the size of the error bars, but even this negative impact becomes negligible when we have, on average, less than one identical pair per batch of N words, as we typically have.

The authors have implemented the last two algorithms in computer programs written in FORTRAN and C. These programs were run for several weeks on the "Wildebeest" 132-processor Beowulf cluster at the City University of New York. The results of these implementations will be shown in the next section.

4. Computational results concerning amenability

The results for the computations of trivial words for F are represented in Table 1. This table contains the following information. For lengths $L = 20, 40, \ldots, 300, 320$, it gives in the second and third columns the sample size (the number of words that were tested) and the number of words among them that were found to represent the trivial element of F; thus the quotient of these two quantities is an approximation of $\widehat{p}(L)$. The fourth column contains the Lth root of this proportion. The last column contains the 20th root of the quotient of the proportions obtained for length L and for length $L - 20$.

In order to clarify the last two columns we remark that the sequences $\sqrt[L]{\widehat{p}(L)}$ and $\sqrt[20]{\widehat{p}(L)/\widehat{p}(L-20)}$ have the same limits – for instance if

length L	sample size	trivial	$\sqrt[L]{\widehat{p}(L)}$	$\sqrt[20]{\frac{\widehat{p}(L)}{\widehat{p}(L-20)}}$
20	$2.000 \cdot 10^{7}$	1 364 638	0.8744	
40	$2.000 \cdot 10^{7}$	82 922	0.8718	0.8693
60	$2.000 \cdot 10^{7}$	6 341	0.8744	0.8794
80	$2.500 \cdot 10^{11}$	7 255 725	0.8776	0.8873
100	$3.125 \cdot 10^{11}$	938 587	0.8806	0.8928
120	$8.750 \cdot 10^{12}$	2 961 321	0.8832	0.8966
140	$1.312 \cdot 10^{13}$	551 480	0.8857	0.9009
160	$1.238 \cdot 10^{13}$	67 542	0.8879	0.9030
180	$2.420 \cdot 10^{13}$	18 618	0.8900	0.9067
200	$1.425 \cdot 10^{14}$	16 040	0.8918	0.9084
220	$1.572 \cdot 10^{15}$	26 596	0.8934	0.9096
240	$2.063 \cdot 10^{16}$	55 941	0.8950	0.9125
260	$2.716 \cdot 10^{16}$	12 162	0.8964	0.9139
280	$7.566 \cdot 10^{15}$	599	0.8976	0.9139
300	$1.343 \cdot 10^{16}$	196	0.8993	0.9221
320	$5.856 \cdot 10^{16}$	148	0.9003	0.9161

TABLE 1. Cogrowth estimates for F.

we had $\widehat{p}(L) \simeq \text{const} \cdot a^L$ then we would obtain

$$\lim_{L\to\infty} \sqrt[L]{\widehat{p}(L)} = \lim_{L\to\infty} \sqrt[20]{\widehat{p}(L)/\widehat{p}(L-20)} = a$$

The difference between the two sequences is that the second one converges much more quickly, but it is also more sensitive to statistical errors related to insufficient sample size.

In summary, the question of amenability comes down to the question whether the numbers in the last two columns converge to 1, or to a smaller number. The numbers in the second to last column converge more slowly, but they are more reliable.

Before we can establish any conclusions, it would be interesting to compare these results with the corresponding results for groups which are known to be amenable or not. As test groups we will take the free group on two generators as a nonamenable example, and the group $\mathbb{Z} \wr \mathbb{Z}$ ($\mathbb{Z}$ wreath $\mathbb{Z}$). The latter group is amenable since it is abelian-by-cyclic, and it appears as a subgroup of F in multiple ways [14, 4]. The group $\mathbb{Z} \wr \mathbb{Z}$ admits the presentation

$$\left\langle a, t \mid \left[a^{t^i}, a^{t^j}\right], i, j \in \mathbb{Z} \right\rangle,$$

	$\mathbb{Z} \wr \mathbb{Z}$			F_2		
L	sample	trivial	$\sqrt[L]{\widehat{p}(L)}$	sample	trivial	$\sqrt[L]{\widehat{p}(L)}$
20	$2.475 \cdot 10^7$	1 802 935	0.8772	$1.000 \cdot 10^7$	655 940	0.8727
40	$2.475 \cdot 10^7$	247 710	0.8913	$1.000 \cdot 10^7$	30 685	0.8653
60	$1.980 \cdot 10^7$	34 658	0.8996	$2.000 \cdot 10^7$	2 888	0.8630
80	$2.475 \cdot 10^7$	9 669	0.9066	$3.000 \cdot 10^7$	230	0.8631
100	$1.980 \cdot 10^7$	2 079	0.9125	$4.000 \cdot 10^8$	159	0.8630
120	$1.095 \cdot 10^8$	3 485	0.9173	$6.975 \cdot 10^{11}$	14 167	0.8628
140	$9.950 \cdot 10^7$	1 035	0.9213	$8.000 \cdot 10^{11}$	819	0.8626
160	$4.990 \cdot 10^8$	1 847	0.9248	$2.400 \cdot 10^{12}$	136	0.8629
180	$2.997 \cdot 10^9$	4 141	0.9278			
200	$4.740 \cdot 10^{10}$	26 919	0.9306			
220	$8.636 \cdot 10^{10}$	20 625	0.9330			
240	$1.859 \cdot 10^{11}$	19 469	0.9352			
260	$4.249 \cdot 10^{11}$	20 112	0.9372			
280	$5.734 \cdot 10^{11}$	12 735	0.9390			
300	$5.844 \cdot 10^{11}$	6 256	0.9407			
320	$4.050 \cdot 10^{12}$	21 229	0.9422			

TABLE 2. Cogrowth estimates for $\mathbb{Z} \wr \mathbb{Z}$ and the free group of rank 2.

and being two-generated it appears to be a good match to compare with F. The results for these two groups are in Table 2.

A graphical representation of comparing these estimates of cogrowth in the three groups F, $\mathbb{Z} \wr \mathbb{Z}$ and $F(2)$ is given in Figures 1.

Do these pictures suggest that F is amenable or non-amenable? It is difficult to discern convergence to 1 or something less than 1 with this data, and it is clear by considering other amenable groups such as iterated wreath products like $\mathbb{Z} \wr \mathbb{Z} \wr \mathbb{Z}$ that the convergence to 1 could be exceptionally slow.

5. Computational results concerning the growth of F

Another family of open questions about Thompson's group F center on the growth of F with respect to its standard generating set $\{x_0, x_1\}$. To study the growth of a group with respect to a generating set, we consider g_n, the number of distinct elements of F of length n and we form the spherical growth series, $g(x) = \sum g_n x^n$. If we consider balls of radius n and the number of elements b_n whose length is less than or equal to n, we have the growth series $b(x) = \sum b_n x^n$. Thompson's group has exponential growth as the submonoid generated by x_0, x_1 and

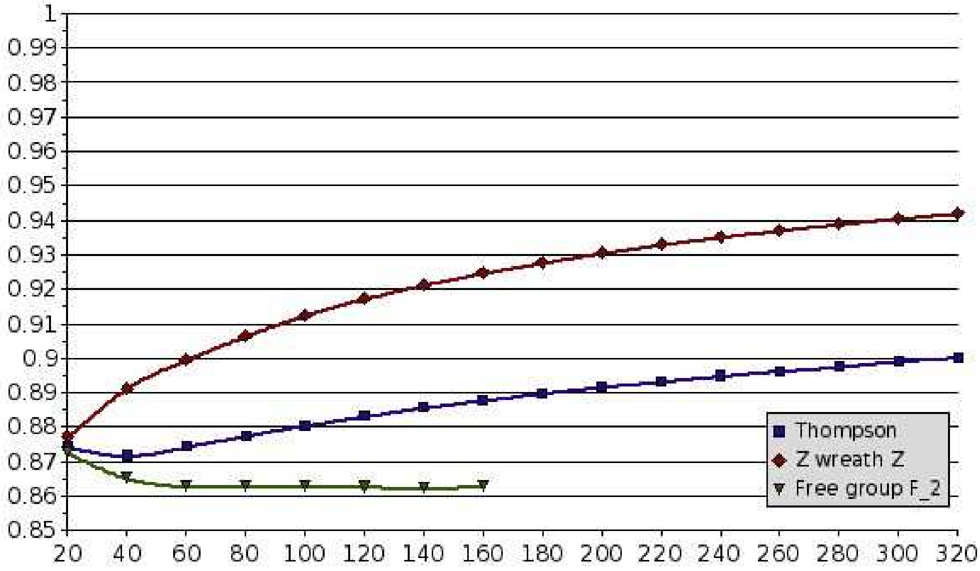

FIGURE 1. Comparing cogrowth estimates $\sqrt[L]{\widehat{p}(L)}$ for three groups.

x_1^{-1} is free (see Cannon, Floyd and Parry [3]). Burillo [2] computed the exact growth function for positive words in F with respect to the standard two generator generating set $\{x_0, x_1\}$ which gives a lower bound for the growth rate of words in the full group as the largest root of $x^3 - 2x^2 - x + 1$, which is about 2.24698. Guba [12] used the normal forms for elements of F developed by Guba and Sapir [13] to sharpen the lower bound of the growth function to $\frac{1}{2}(3 + \sqrt{5})$ which is about 2.61803. Guba conjectures that 2.7956043 is an upper bound by considering the ratio of the ninth and eighth terms in the spherical growth series of F. But the exact growth function of F remains unknown – it is not even known if the growth function is rational, though Cleary, Elder and Taback [5] show that there are infinitely many cone types, which may be evidence that the growth of the full language of geodesics is not rational.

Here, we use a computational approach to estimate the growth function of F. We use two methods both based upon taking random samples of words via random walks. Both of these methods estimate the number of words in successive n-spheres of F. For the first method, we take an element of length n and consider its "inward" and "outward" valence in the Cayley graph. Since the relators of F with respect to the standard finite presentation are all of even length, application of a generator x to an element w of F will either increase or reduce the length by 1. The *inward valence* of w is the number of generators which reduce the word length and the *outward valence* of w is the number of generators which increase word length. If the length of w is n, then the outward valence gives the number of words adjacent to w which lie on the $n+1$ sphere. By taking an average of the outward valence of a large number of elements in the n sphere, we can estimate the ratio of the number of elements in the $n+1$ sphere to the number of elements in the n sphere. Thus we can estimate the rate of growth, as the limit of these ratios (for $n \to \infty$) will be the exponential growth rate for the group.

For the second method, we consider a variation of this approach where instead of looking at the words at distance 1 from w, we look at the words at distance 2 from w and see how many of those words lie in the $n+2$ sphere. This gives an estimate of the ratio of the number of elements in the $n+2$ sphere to the number of elements in the n sphere, and in the limit, we expect the square root of these ratios to approach the exponential growth rate for the group.

We expect both methods to yield overestimates of the true growth rate, but the error should be larger for the first method than for the second one. The raw outward valence method is expected to overestimate because it may count elements in the $n+1$ sphere which are adjacent to more than one element in the n sphere multiple times. An extreme example of this are "dead-end" elements in F, characterized by Cleary and Taback [6]. These dead-end elements have the property that right multiplication by any generator reduces word length. The "outward valence" method includes these dead-end elements in the count of growth – if the randomly selected element in the n sphere is one of the 4 elements in the n sphere which is adjacent to a particular dead-end element in the $n+1$ sphere, it will contribute to the average outward valence at least 1. For the distance two method, however, such elements will not contribute to the growth as there will be no words adjacent to the dead-end element which lie in the $n+2$ ball.

To compute the length of an element of F, we use Fordham's method [10] for measuring word length of elements of F with respect to $\{x_0, x_1\}$. This remarkable method amounts to building the reduced tree pair diagram associated to an element of F, classifying each internal node of the trees diagram into one of seven possible types, and then pairing the nodes and summing a weight function of those node type pairs to get the exact length of the element.

We note that selecting a random element of the n sphere for a predetermined value of n is not feasible given current understanding of the metric balls in F – we do not even know the number of such elements, as

Lengths	Words	Average outward valence	Average num. at dist. 2	Growth estimate from dist 2
0 - 19	5723	2.8440	7.8363	2.7993
20 - 39	629964	2.7334	7.3239	2.7063
40 - 59	1017998	2.7128	7.2521	2.6930
60 - 79	602694	2.6781	7.0389	2.6531
80 - 99	612613	2.6698	7.0041	2.6465
100 - 119	514665	2.6564	6.9256	2.6317
120 - 139	392069	2.6512	6.9074	2.6282
140 - 159	272564	2.6407	6.8529	2.6178
160 - 179	234893	2.6331	6.8057	2.6088
180 - 199	281806	2.6275	6.7779	2.6034
200 - 219	283764	2.6299	6.7897	2.6057
220 - 239	164359	2.6336	6.8234	2.6122
240 - 259	48750	2.6341	6.8431	2.6159
260 - 279	7326	2.6403	6.8756	2.6221
280 - 299	521	2.6430	6.8829	2.6235
300 - 319	17	2.6470	6.8235	2.6122

TABLE 3. Average outward valence of words arising from random walks.

in fact that is what we are trying to estimate. So we construct elements by taking random walks in the group with respect to the standard generating set of a predetermined length n, and then measure the length l of the element obtained. We then compute its outward valence by measuring the lengths of elements adjacent to it in the Cayley graph and we also count the number of elements at distance two from it which lie in the $l+2$ sphere. Thus, we obtain simultaneously estimates of outward valence for elements in a range of balls. Furthermore, we can record the length l of a word obtained by a random walk of length n and use that to estimate crudely the rate of escape of a random walk in F, as described in the next section. The results of the computations concerning growth are presented in Table 3 and Figure 2.

As we can see from the data, and as expected, the estimates using the distance two method are lower than the estimate from the outward valence method. Moreover, for the first experiment, the values lie between the proven lower bound of 2.618... and the conjectured upper bound of 2.763..., for words of length 20 and more. However, other aspects of the computational results are more surprising. Both functions appear to have a minimum at length about 190. Moreover, for

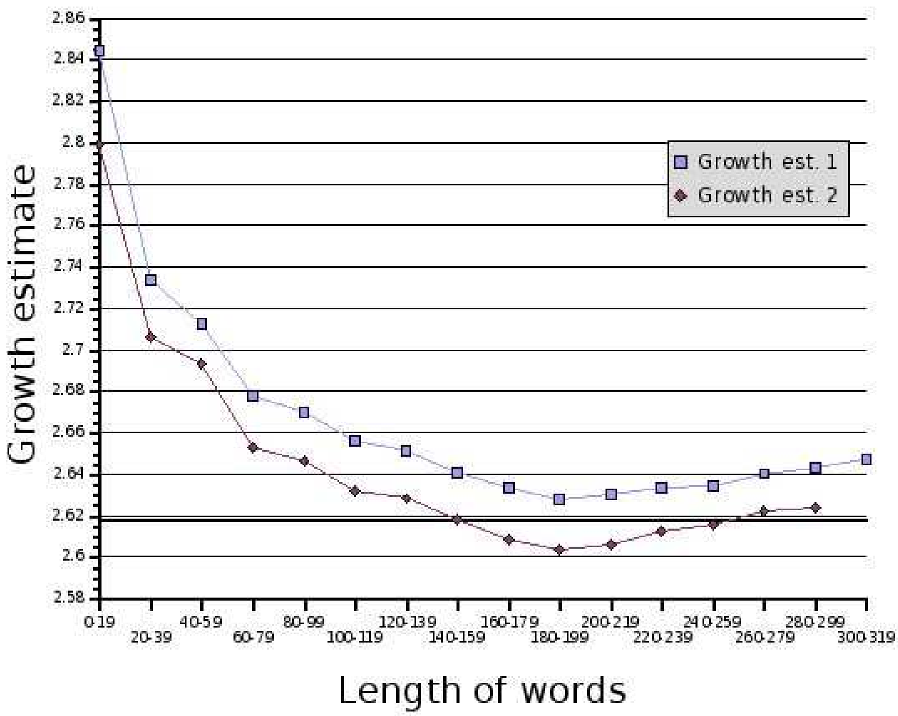

FIGURE 2. Estimates for the exponential growth rate from the data in Table 3

the second experiment, the values obtained lie below the proven lower bound for words of length between 140 and 260, and lie in the expected range before and after that. This data suggests that the rate of growth is close to the proven lower bound or that random walks are not an unbiased method for estimating growth by average outward valence. Of course, since we do not know the growth function, it is difficult to effectively pick a random element, so perhaps random walks tends to bias toward those which have lower outward valence than is representative. The role of "dead-end" elements of outward valence 0 may play a role in this bias and we describe estimates of densities of dead-end elements in the next section. It may be that random walks get stuck near dead-end elements and other low outward valence items and thus random walks may select these elements at a greater proportion than uniform.

Finally, we mention that we have also computed first twelve terms of the exact spherical growth function of F to obtain:

$$\begin{aligned} g(x) &= 1 + 4x + 12x^2 + 36x^3 + 108x^4 + 314x^5 + 906x^6 + 2576x^7 + \\ &\quad +7280x^8 + 20352x^9 + 56664x^{10} + 156570x^{11} + \ldots \end{aligned}$$

Guba [12] had already calculated the first ten terms of this sequence and noticed that the ratios of successive terms of this series appear to decrease and form a natural conjectural upper bound to the growth function. The two additional successive quotients arising from our additional terms continue the decreasing pattern and are 2.7841981 . . . and 2.7631300 . . . and lie well above the experimental estimates of growth described above.

6. Rate of escape of random walks and dead-ends in F

Here we note that as a side effect of the computations described in the previous section to estimate growth, we obtain two pieces of data which are interesting in their own right.

First, since the random elements used to estimate growth are constructed by random walks and we measure their exact lengths using Fordham's method, we are able to see how quickly these random walks leave the origin. Since these are symmetric random walks, there is of course the possibility of backtracking to get non-freely reduced words, so we do not expect a random walk of length 100 to actually reach the sphere of radius 100 with non-negligible probability. Our estimates of the rate of escape of random walks of lengths 100 to 1000 are shown in Table 4 and the rate of escape seems to be decreasing in this range.

Length of random walk	Number of walks	Average length	Standard deviation	Rate of escape
100	4764000	41.18	8.34	0.4118
200	3242898	76.01	12.33	0.3800
300	2700000	109.3	15.51	0.3545
400	1500000	141.8	18.33	0.3544
500	600000	173.8	20.82	0.3476
600	1500000	205.3	23.08	0.3421
700	900000	236.5	25.14	0.3379
800	900000	267.6	27.14	0.3345
900	300000	298.5	29.02	0.3316
1000	300000	329.0	30.86	0.3290

Table 4. Distance from origin (word length) as a function of random walk length

Second, since we compute the outward valence of words to estimate the growth, we can look for words of outward valence zero- these are exactly the "dead-end" elements discovered by Fordham [9] and characterized by Cleary and Taback [6]. Though dead-end elements can occur in any group (with respect to generating sets contrived for that purpose) groups with dead-end elements with respect to natural generating sets are much less common. Geodesic rays from the identity towards infinity cannot pass through dead-end elements, and thus the existence of many dead-end elements tends to reduce the growth of the group. Table 5 shows the observed incidence of dead ends during the course of the growth estimation calculations in Section 5. We see that there are significant numbers of dead ends but that the fraction decreases as the lengths of elements increases.

Range of lengths	Number of words	Number of dead-ends	Fraction
0 - 39	634927	665	0.001047
40 - 79	1620692	1386	0.0008552
80 - 119	1127278	625	0.0005544
120 - 159	665245	239	0.0003593
160 - 199	561502	149	0.0002654
200 - 239	825785	162	0.0001962
240 - 279	689500	114	0.0001653
280 - 319	393643	39	0.00009907
320 - 359	128254	11	0.00008577
360 - 399	20926	1	0.00004779
400 - 439	1193	0	0
440 - 479	21	0	0

TABLE 5. Fractions of dead-ends observed during random walks as a function of resulting word length.

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