# The complexity of braids and the geometry of Teichmüller spaces

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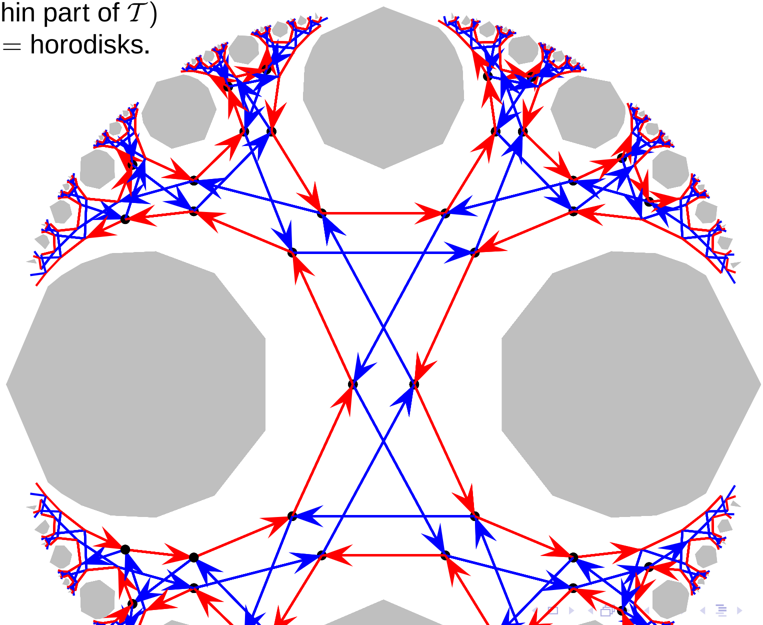
**Recall :** The Teichmüller space of the once-punctured torus

$$(\mathcal{T}(T^2 - pt), d_{\text{Teich}}) = (\mathbb{H}^2, d_{\text{hyp}})$$

(Thin part of $\mathcal{T}$)
$=$ horodisks.



# 3 quasi-equal "complexities" of $f$

Up to a linearly bounded error, the following 3 things are equal : if

$$f = D_{merid}^{k_\ell} \circ \ldots \circ D_{long}^{k_3} \circ D_{merid}^{k_2} \circ D_{long}^{k_1} \in \mathcal{MCG}(T^2 - pt)$$

1. $d_{\mathrm{Teich}}(f(\tau_*)), \tau_*)$, where $\tau_*$ is a base point of $\mathcal{T}$.
2. $\ell_\Delta(f) := \log_2(|k_1| + 1) + \log_2(|k_2| + 1) + \ldots + \log_2(|k_\ell| + 1)$
3. complexity of the curve $f(s_{(1,1)})$, where

   $s_{(p,q)} := (p,q) -$ torus knot, and complexity$(s_{(p,q)}) := \log_2(p+q)$

**Corollary** Up to q.i., a combinatorial model for the thick part of $\mathcal{T}$ :

$$\mathcal{MCG}(T^2 - pt), \text{ with the metric } d(f_1, f_2) := \ell_\Delta(f_1^{-1} f_2)$$

**Fact** Analogue results for all surfaces [Rafi].
We only deal with the case of the $n$-punctured sphere (braid group)

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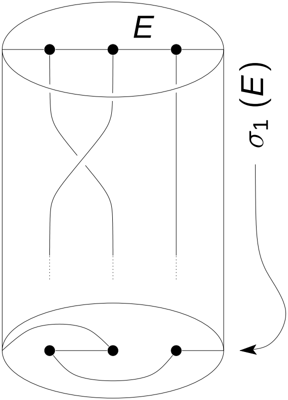
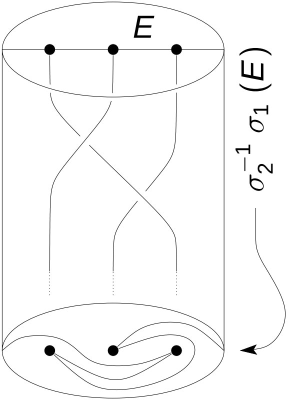
# Curve diagrams

Recall : braid groups $B_n \cong \mathcal{MCG}(D_n)$. Helpful for visualizing homeomorphisms of $D_n$ (i.e. elements of $B_n$) : *curve diagrams*



# Complexity of curve diagrams

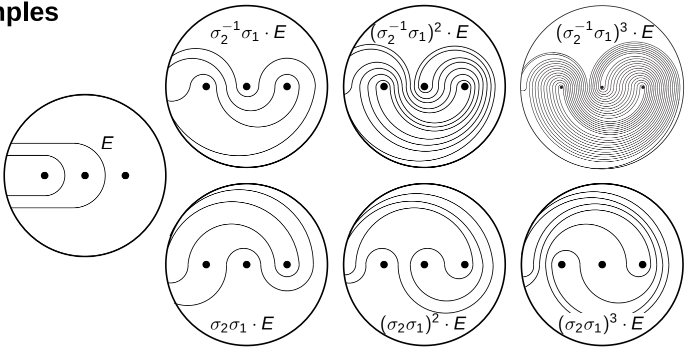
**Definition** If $D$ is a curve diagram,

$$\|D\| := \#\{D \cap \mathbb{R}\}.$$

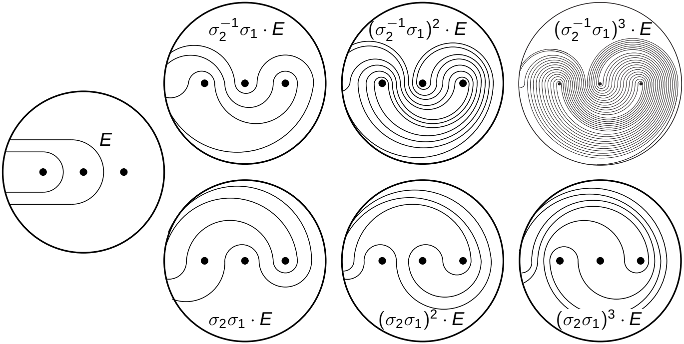
For a braid $\beta$,

$$\text{complexity}(\beta) := \log_2(\|\beta.E\|) - \log_2(\|E\|).$$

**Examples**



**Recall** complexity $c(\beta) = \log_2(\|\beta.E\|) - \log_2(\|E\|)$.



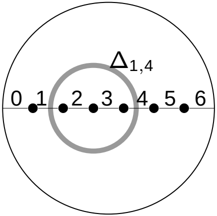

**Key idea :**

- $c$(random word)$\longrightarrow \infty$ *linearly*
- $c(T^n) \longrightarrow \infty$ *logarithmically* (where $T$ = Dehn twist)

# A new "length" of braids

Consider $B_n$, with generators

$$\{\text{Dehn half-twists } \Delta_{i,j} \mid 0 \leqslant i < j \leqslant n\}$$




**Definition** The Δ-length of a word

$$w = \Delta_{*,*}^{a} \cdot \Delta_{*,*}^{b} \cdot \ldots \cdot \Delta_{*,*}^{z}$$

is

$$\ell_\Delta(w) = \log(|a| + 1) + \ldots + \log(|z| + 1)$$

For $\beta \in B_n$, we define

$$\ell_\Delta(\beta) := \text{ the minimal } \Delta\text{-length among all words representing } \beta.$$

# The main results [Dynnikov-W]

**Theorem** There is a bilipschitz relation

$$\text{complexity}(\beta) \;\cong\; \ell_\Delta(\beta)$$

i.e. geometric measure "$=$" algebraic measure.

**Theorem** $\exists$ polynomial-time algorithm with

INPUT : a braid $\beta$

OUTPUT : a representative word of $\beta$ of near-minimal $\Delta$-length.

# A discrete model for $\mathcal{T}\left(S^2 - (n+1 \textbf{ points})\right)$

Define a new (non–geodesic !) metric

**The Δ-metric** on $B_n/\langle\Delta^2\rangle \overset{f.i.}{=} \mathcal{MCG}(S^2 - (n+1 \text{ pts}))$ :

$$d_\Delta(\beta_1, \beta_2) := \ell_\Delta(\beta_1^{-1}\beta_2).$$

**Corollary** Up to quasi-isometry, the group $B_n/\langle\Delta^2\rangle$, equipped with the $d_\Delta$-metric, is a model of

$$\left(\mathcal{T}_{thick}\left(S^2 - (n+1 \text{ points})\right), d_{Lipschitz}\right)$$

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**Theorem** complexity($\beta$) $\overset{\text{bilip}}{=}$ $\ell_\Delta(\beta)$

**Difficult part of the proof**
Given a curve diagram of complexity e.g. 328.67, prove that the diagram can be "relaxed" into the trivial diagram by the action of a braid word

$$w = \Delta_{i_1 j_1}^{k_1} \dots \Delta_{i_\ell j_\ell}^{k_\ell} \quad \text{s.t.} \quad \sum_{m=1}^{\ell} \log_2(|k_m| + 1) \leqslant \mathrm{const} \cdot 328.67.$$

For proving this, the "obvious" strategy is :
prove that for every curve diagram $D$,

$$\exists i, j, k \quad \text{s.t.} \quad \log_2(|k| + 1) \leqslant \mathrm{const} \cdot \left(c(D) - c(\Delta_{ij}^k.D)\right)$$

WRONG !

Good proof : define a complexity function $c_{\mathrm{AHT}}$, which depends on the curve diagram *and the history of the untangling procedure so far.*

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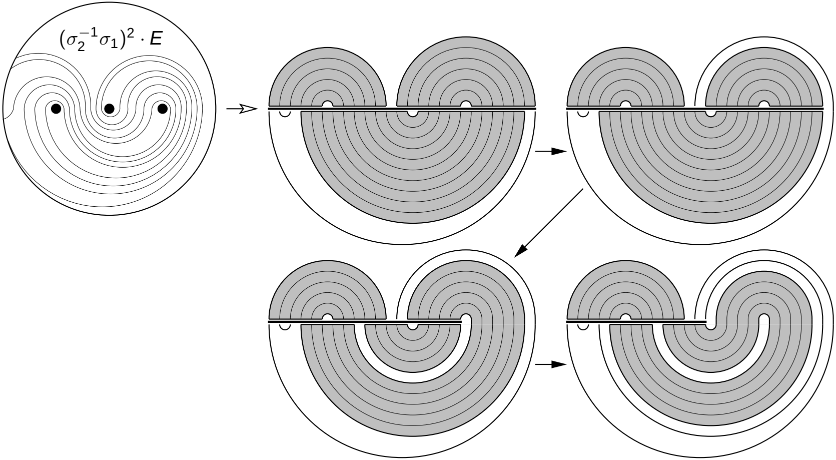
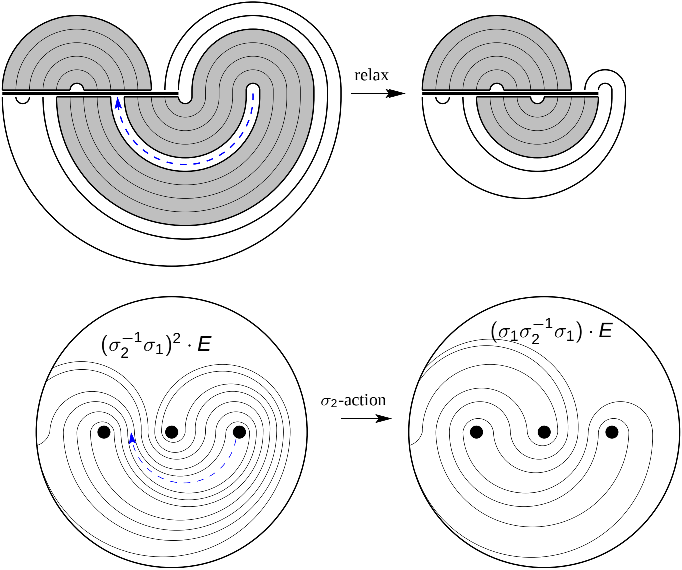


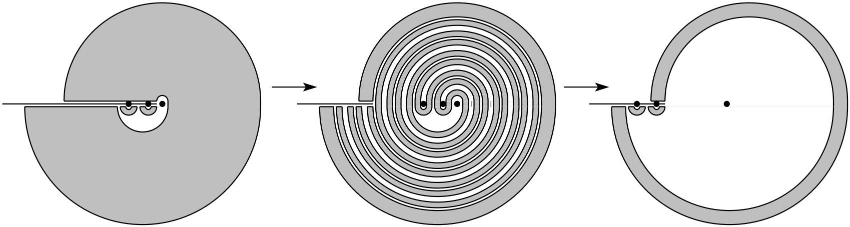

Transform the curve diagram into an IIS (interval identification system), and apply *transmissions* à la Agol-Hass-Thurston.

$$c_{\mathrm{AHT}}(D) := \sum_{\text{strip } S} \log_2(\mathrm{width}(S))$$

Imagine that the strips of the IIS are elastic :



Example of a transmission which decreases a lot $c_{\mathrm{AHT}}$, followed by the action of $\Delta_{ij}^k$, where $k \gg 0$.



**Key observation** If $D, D', D''$ are IISs, and

$$D \xrightarrow{\text{transm}} D' \xrightarrow{\text{relax}} D''$$

where the relaxation is done by the action of a braid $\Delta_{ij}^k$, then

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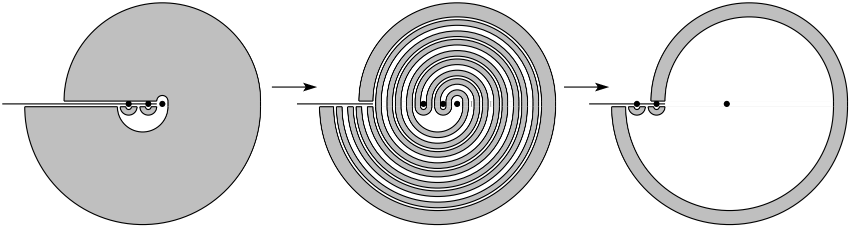

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# It was known already

Kasra Rafi found independently the same model for $(\mathcal{T}_{thick}, d_{Teich})$, using two deep results :

1. The results of [Masur-Minsky2]

2. Y. Minsky's precise description of the thin regions of $\mathcal{T}$ :

**Recall** $\mathcal{T}(torus) = \mathbb{H}^2$ (complex point of view).

**Theorem** (Minsky) Let $\alpha = (\alpha_1, \ldots, \alpha_k)$ be a family of disjoint $s.c.c.$s on a surface $S$. Then the region of $\mathcal{T}$ where all the $\alpha_i$ become simultaneously short is

$$\mathcal{T}(S \setminus \alpha) \times \mathbb{H}^2_1 \times \ldots \times \mathbb{H}^2_k$$

equipped with a metric $d$ which, up to an *additive* constant, is

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# Open questions

1. Generalisations to $\mathcal{MCG}$s of arbitrary surfaces ?

2. Generalisation to $Out(F_n)$, yielding a nice metric on $\mathcal{CV}(F_n)$ ?

3. Consider successive squashings

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   Does our relaxation procedure create paths in $B_n$ which are quasi-geodesics w.r.t. all 3 metrics ?

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# Conclusions

We have found a simple combinatorial model for the thick part of Teichmüller space of a punctured sphere.

**Bad** because :

- Known already, in more generality.
- Previous proofs yield deep geometrical insights.

**Good** because :

- Comparatively low-tech
- Our proofs yield simple, polynomial-time algorithms for calculating quasi-geodesics in $\mathcal{T}_{\text{thick,Teich}}$ (and probably $\mathcal{T}_{\text{WP}}$).

**Help wanted :** Actual algorithmic applications.