CONJUGACY INVARIANTS FOR POWERS OF RIGID BRAIDS: A UNIFORMITY PHENOMENON

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ABSTRACT. Consider an m-strand braid x which is rigid in the sense of Garside-theory. Let $SC(x)$ be the set of rigid conjugates of x – this is a well-known characteristic subset of the conjugacy class of x. We present computational evidence that the sequence $\left(|SC(x^n)|\right)_{n\in\mathbb{N}}$ is not only bounded, but in fact periodic, and that the length of the period can be bounded in terms of the number of strands m. We prove this result in the special case of the 3-strand braid group (where we prove that the sequence is always constant) and of the dual 4-strand braid group.

1. INTRODUCTION

Let G be a Garside group, equipped with a Garside structure – in this paper we will mostly be concerned with the two most well-known examples, namely the m-strand braid group equipped with Garside's original Garside structure, denoted $\mathcal{B}_m$, and the same group equipped with Birman-Ko-Lee's dual Garside structure, denoted $\mathcal{B}_m^*$, but we will also present some calculations in Artin-Tits groups of spherical type, equipped with their classical Garside structures. (Relevant references for the braid groups with the classical structure are [14, 13] and [3] for the dual structure, [4, 12] for the generalisation to Artin-Tits groups of spherical type. For general Garside theory, see [11] (for the first definition), as well as [8, 1], and [20], and finally [10] for a very complete and high-level modern point of view].)

We recall that to any element x of G, we can associate a finite subset of the conjugacy class of x called the sliding circuits set, denoted $SC(x)$ [16]. This subset depends only on the conjugacy class of x; therefore, being able to calculate the sliding circuits set of any given element solves the conjugacy problem in G. If x is rigid (which means, roughly speaking, that x is in Garside normal form as a *cyclic* word), then this subset consists of the set of rigid conjugates.

Conjecture 1.1. *For any rigid element x, the sequence $\left(|SC(x^n)|\right)_{n\in\mathbb{N}}$ is bounded.*

There are at least three motivations for studying this conjecture.

(1) Currently the biggest roadblock to solving the conjugacy problem in braid groups in polynomial time using Garside theory is the following open question: how quickly can $\max_{x\in\mathcal{B}_n, |x|\leq\ell,\ x \text{ pA}} |SC(x)|$ grow, as a function of ℓ (for any fixed value of m)? In particular, it is open whether or not this function is polynomially bounded. It would be valuable to at least understand how the size of the sliding circuit set of powers of a single element behaves.

(2) As will be shown in a future paper, for an element x, the property of having a bounded associated sequence $|SC(x^n)|$ has some very strong consequences. Namely, the element x then either has a "large" centraliser (not virtually equal to $\langle x, \Delta^2\rangle$), or the axis of the element in the Cayley graph of $G/C(G)$ has the Morse property (which, in this case, even implies the very

restrictive *strong contraction*-property [6, 23]). This, together with Conjecture 1.1, would imply that all elements in spherical-type Artin-Tits group with no obvious obstruction to having the Morse property really do have the Morse property; also it would be a rare occasion where the Morse property is not deduced from a loxodromic action on some hyperbolic space.

(3) The centralizer of a pseudo-Anosov braid x is virtually $\mathbb{Z}^2$ [21, 19], virtually generated by x and by Δ^2, and this gives rise to a two-dimensional flat in the Cayley graph containing the axis of x. For reducible braids, the centralizer is larger, and we obtain correspondingly higher-dimensional flats containing the axis. These flats are to be expected. By contrast, it is conceivable that there might be a sequence of bi-infinite geodesics, each of them parallel to the axis of x, coming from rigid conjugates of x^n, with larger and larger powers n and with longer and longer conjugating words. Together, these geodesics would form an *unexpected half-flat*. Conjecture 1.1 would exclude such unexpected half-flats in the Cayley graph bounded by the axis of a rigid element x. What makes this conjecture so elegant is that it applies (in the context of braid groups) not only to pseudo-Anosov braids, but also to reducible ones (as long as they are rigid).

While performing extensive computer experiments in order to test Conjecture 1.1, the authors were surprised to discover that a much stronger result appears to hold. Indeed, there seems to be a uniformity phenomenon: for any fixed number of strands, the power for which the sliding circuit set takes its maximal size appears to be bounded:

Main Conjecture 1.2. *For any Garside group G, equipped with a Garside structure, there exists a finite set of integers $\mathcal{P}_G$ with the following property: for any rigid element x of G, the sequence $(|SC(x^n)|)_{n\in\mathbb{N}}$ is periodic, and the period is an element of $\mathcal{P}_G$.*

This conjecture is the fruit of large computer experiments with various Artin-Tits groups of spherical type. Large numbers of random elements were generated (by writing random words in Artin generators), and for those that happened to have a rigid conjugate, the sizes of the sliding circuit sets of the braids and their first few powers were calculated. For the braid groups equipped with the classical Garside structure, these calculations were based on Juan González-Meneses' C++ program [18]. By contrast, GAP3 [22] (with the package Chevie [7] was used for braid groups with the dual Garside structure, as well as for other Artin-Tits groups (with the classical Garside structure). In the latter cases, we have much less experimental data, since calculations on GAP3/Chevie were substantially slower than on the C++ program [18]. As a result of our calculations, we have the following table with the conjectured sets of possible periods. Here $\mathcal{B}_m$ denotes the m-strand braid group (which is the same as the Artin-Tits group of type A_{m-1}), equipped with the classical Garside structure; $\mathcal{B}_m^*$ denotes the same group, equipped with the dual Garside structure. We did not perform calculations with the dual structure on Artin-Tits groups other than the braid groups.

Group G	$\mathcal{B}_3$	$\mathcal{B}_4$	$\mathcal{B}_5$	$\mathcal{B}_6$	$\mathcal{B}_7$	$\mathcal{B}_8$
$\mathcal{P}_G$	$\{1\}$	$\{1,2\}$	$\{1,2,3\}$	$\{1,2,3,6\}$	$\{1,2,3,4,6\}$	$\{1,2,3,4,6,12\}$

Group G	$\mathcal{B}_3^*$	$\mathcal{B}_4^*$	$\mathcal{B}_5^*$	$\mathcal{B}_6^*$
$\mathcal{P}_G$	$\{1\}$	$\{1,2,3\}$	$\{1,2,3,4,6\}$	$\{1,2,3,4,5,6,?\}$

Group G	B_2	B_3, B_4	B_5	D_4	D_5	F_4	H_3	H_4	$I_2(10)$
$\mathcal{P}_G$	$\{1\}$	$\{1,2\}$	$\{1,2,3\}$	$\{1,2,3\}$	$\{1,2,3,6\}$	$\{1,2,3\}$	$\{1,2\}$	$\{1,2,3\}$	$\{1\}$

Notation 1.3. In the braid group $\mathcal{B}_m$, we will use the notation i instead of the more standard σ_i for the ith Artin half-twist generator. Also, when we write $21|12$, the symbol $|$ indicates that the product of two Garside generators $\sigma_2\sigma_1 \cdot \sigma_1\sigma_2$ is in Garside normal form as written.

Example 1.4. In $\mathcal{B}_4$, for $x = 21|12|2132$, we have $|SC(x^k)| = 6$ if k is odd, and $|SC(x^k)| = 18$ if k is even. Figure 1 shows the "conjugacy graphs" of x and of x^2. Every vertex of a conjugacy graph

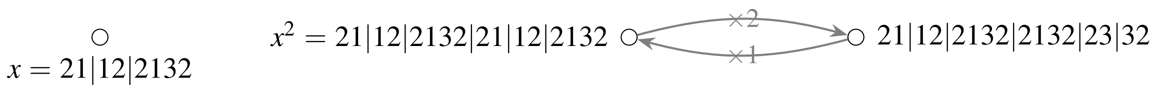

FIGURE 1. For $x = 21|12|2132$, the figure shows on the left the conjugacy graph of x (only one vertex), and on the right the conjugacy graph of x^2.

denotes one orbit in the sliding circuit set under the action of cycling and of τ (i.e. conjugation by Δ). On the left, we see that $SC(x)$ consists only of x, of the two elements obtained from x by cyclically permuting the three factors, and of their images under τ, so it contains only one vertex.

The set $SC(x^2)$, by contrast, is larger: we can conjugate x^2 in two different ways (by 1 or by 3), and we obtain two elements of $SC(x^2)$ which are cyclic conjugates of each other (so they both represent the right-hand vertex). If we want to conjugate back, from one of the elements thus obtained to a rigid braid representing to the left vertex, then we have only one choice: the only simple braid that conjugates $1^{-1} \cdot (x^2) \cdot 1 = 21|12|2132|2132|23|32$ to a cyclic conjugate of x^2 is the braid 3. The arrows are drawn in gray color, because they represent conjugations by a prefix of the complement of the last factor – in the notation of [2], they represent "gray arrows". (By contrast, "black arrows" are conjugations by minimal prefixes of the first factor.)

The next figure shows a part of the Cayley graph, and specifically the conjugation of x^2 by 1 yielding $21|12|2132|2132|23|32$. The blue arcs indicate sequences of generators that are in normal form. It is also intuitive from the red arrows in this figure why $y = 1^{-1} \cdot x \cdot 1$ is not rigid whereas $1^{-1} \cdot x^2 \cdot 1$ is. In Section 2 we will give a detailed explanation of how the calculation of a conjugate along a gray or black arrow works.

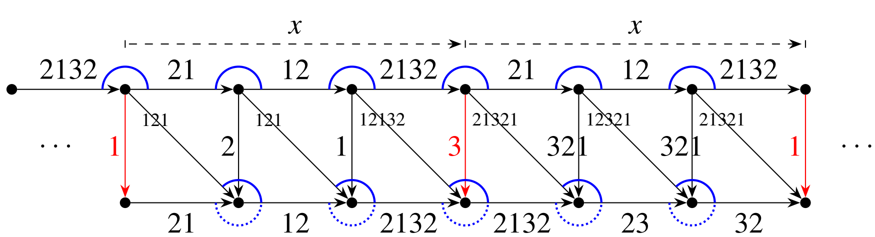

FIGURE 2. The conjugation of x^2 by 1 for $x = 21|12|2132$, seen in the Cayley graph. This diagram can be periodically extended into a bi-infinite strip. This type of diagram is called a domino diagram.

Let us reinterpret our conjectures in light of this example. We have seen that a rigid braid x may very well have a conjugate y which is not rigid, but whose square (living in $SC(x^2)$) is rigid. At first sight, such an element y^2 doesn't look like a square, as its normal form does not consist of some shorter word that is repeated twice. Now it may happen (though it doesn't in this example) that some further, even higher power of y can be conjugated again, yielding yet more rigid conjugates; and there may even be a whole tower of such powers. However, Conjecture 1.1 states that such a tower is necessarily finite, and Conjecture 1.2 claims that there is a bound on the highest power involved, and this bound is uniform in the group.

Notation 1.5. For any element y of G which possesses a rigid power, we denote $r(y)$ the smallest positive integer such that $y^{r(y)}$ is rigid.

We observe that some power y^n is rigid if and only if n is a multiple of $r(y)$. (This is because if q is an integer such that y^q is rigid and if $n \equiv n' \mod q$, then $\iota(y^n) = \iota(y^{n'})$ and $\varphi(y^n) = \varphi(y^{n'})$.)

Introducing some more notation, for any rigid element x of G we define

$$\mathcal{R}(x) = \{r(y) \mid y \text{ is conjugate to } x, \text{ and } y \text{ has a rigid power}\}$$

Conjecture 1.6. *For any Garside group G, there exists a finite subset $\widetilde{\mathcal{P}}(G)$ of $\mathbb{N}$ such that for any rigid element x of G we have $\mathcal{R}(x) \subseteq \widetilde{\mathcal{P}}(G)$.*

We observe that Conjecture 1.6 implies our Main Conjecture 1.2. Indeed, assuming that Conjecture 1.6 holds, the sequence $(|SC(x^n)|)_{n\in\mathbb{N}}$ is periodic of period $lcm(\mathcal{R}(x))$, the least common multiple of all elements of $\mathcal{R}(x)$. The finite set $\widetilde{\mathcal{P}}(G)$ has only finitely many subsets, and in particular only finitely many subsets of type $\mathcal{R}(x)$, so there are only finitely many possible periods of the sequence $|SC(x^n)|_{n\in\mathbb{N}}$, as desired.

We also conjecture that for any rigid element x, the set $\mathcal{R}(x)$ is a lattice with respect to the divisibility relation. If this conjecture holds, then the period $lcm(\mathcal{R}(x))$ is equal to the largest element of $\mathcal{R}(x)$.

Our main theorem is:

Theorem 1.7. *(a) Conjecture 1.6 holds for $\mathcal{B}_3$, the three-strand braid group with the classical Garside structure: for any rigid element x, we have $\mathcal{R}(x) = \{1\}$, and the sequence $|SC(x^n)|$ is constant.*

(b) Conjecture 1.6 holds for $\mathcal{B}_4^*$, the four-strand braid group with the dual Garside structure.

Remark 1.8. Using a detailed combinatorial analysis, it would be possible to prove that in $\mathcal{B}_4^*$, the sequence $|SC(x^n)|$ can only have three possible behaviors: it is either constant, or it is periodic of period 2, or it is periodic of period 3 with $|SC(x^2)| = |SC(x)|$ and $|SC(x^3)| = 4 \cdot |SC(x)|$. Correspondingly, the only possible sets $\mathcal{R}(x)$ are $\{1\}$, $\{1,2\}$, and $\{1,3\}$. We will only prove a weaker result.

2. General framework and preliminary results

We start with some reminders of general Garside theory – the reader can consult [1, 2, 8, 9, 13, 20]. We recall that in a Garside group G, equipped with a Garside structure, every element x is represented by a unique word in *normal form*, taking the shape $x = \Delta^k \cdot x_1|\dots|x_\ell$. Here Δ is the particular element given by the Garside structure (e.g. the half-twist braid in $\mathcal{B}_m$ and the $\frac{2\pi}{m}$-twist braid in $\mathcal{B}_m^*$); each x_i is a *Garside generator* or a *simple element*, and each pair of successive letters $x_i \cdot x_{i+1}$ is *left-weighted* – these terms will be defined shortly – and we indicate this by writing the product $x_i|x_{i+1}$. We denote $k = \inf(x)$ the *infimum*, $k+\ell = \sup(x)$ the *supremum*, and $\ell = \ell_{can}(x)$ the *canonical length* of x. The automorphism $\tau\colon\ G \to G$ is defined by $\tau(x) = \Delta^{-1}x\Delta$.

We recall the *prefix ordering* $\preccurlyeq$ on G, which is a partial ordering given by $g_1 \preccurlyeq g_2$ if and only if $g_1^{-1}g_2 \in G^+$. Thus any two elements g_1, g_2 of G have a meet $g_1 \wedge g_2$, and a join $g_1 \vee g_2$. The Garside generators of G, or *simple elements*, are precisely the elements x satisfying $1 \preccurlyeq x \preccurlyeq \Delta$. The *complement* ∂x of a simple element x is $\partial x = x^{-1}\Delta$. A product of two simple elements $x_1 \cdot x_2$ is said to be *left-weighted*, or *in normal form* if there is no prefix p of x_2 such that $x_1 p$ is simple, or equivalently, if $x_2 \wedge \partial x_1 = 1$; in this case, we write $x_1|x_2$. This completes our definition of the normal form.

For an element x of G with normal form $\Delta^{\inf(x)}x_1|\ldots|x_\ell$ and $\ell > 0$, the *initial* and *final factors* are defined by $\iota(x) = \tau^{-\inf(x)}x_1$ and $\tau(x) = x_\ell$. A element x is said to be *rigid* if it is not a power of Δ and if the product $\varphi(x) \cdot \iota(x)$ is in normal form as written: $\varphi(x)|\iota(x)$. Roughly speaking, saying that x is rigid means that the normal form of x^2 is as expected: twice the power of Δ, followed by the non-Δ factors repeated twice, except that the first half of them may be twisted by some power of τ. If $\inf(x) = 0$, the idea is that the normal form word representing x, regarded as a cyclic word, is still in normal form.

If an element g of G is conjugate to a rigid element, then, following [16], we define $SC(g)$ to be the set of all rigid conjugates of g. (In fact, this is not the original definition of [16], but it is proven in this paper that the original definition – which we won't need – coincides with the one given here in the special case of rigid elements.)

To any rigid element x, we will associate a graph with colored, oriented edges called the *conjugacy graph*, already used in [5], and very similar to the graphs introduced earlier in [2].

Definition 2.1. Let x be a rigid element of G. The *conjugacy graph* of x has one vertex for every orbit in $SC(x)$ under the action of *cycling* (i.e. conjugation by $\iota(x)$) and of τ (conjugation by Δ). The edges, which we will call *arrows*, come in two colors: black and gray. A vertex represented by an element x is connected by a *black arrow* to a vertex represented by y if there is a prefix $c \preccurlyeq \iota(x)$ which conjugates x to y, i.e. if $c^{-1}xc = y$. By contrast, there is a *gray arrow* between these two vertices if the conjugator c satisfies $c \preccurlyeq \partial\varphi(x)$. Any arrow can carry a label of the form "$\times k$" (with $k \geqslant 2$), indicating that there are k different conjugations of one representative of the source vertex, yielding various representatives of the target vertex.

Remark 2.2. Our conjugacy graphs are almost the same as the graphs introduced in [2], but there are some subtle differences: firstly, in the graphs of [2], the vertices are orbits of $SC(x)$ under cycling, but not under the action of τ. Secondly, the graphs of [2] have fewer arrows per vertex than our graphs, because they only contain minimal arrows (i.e. arrows that are not given by the concatenation of two or more arrows of the same color).

Remark 2.3. In this paper, we are drawing two types of diagrams, which should not be confused. On the one hand, we have *domino diagrams* (Figures 2, 3, 4, 8, 9, 10), which live in the Cayley graph of G (with generators = simple elements). In domino diagrams, arrows indicate right multiplication by a simple element, and the little blue arcs connecting the end of one arrow x_i to the start of another arrow x_{i+1} indicate that the word $x_i \cdot x_{i+1}$ is in normal form. On the other hand, we have have pictures of *conjugacy graphs* (Figures 1, 5, 7) from Definition 2.1, where (black or gray) arrows indicate conjugations by simple elements.

Next, we recall the standard method for calculating the conjugate of a rigid braid $x = \Delta^k x_1|\ldots|x_\ell$ (with $\ell > 0$) along a gray arrow, i.e. calculating the normal form of $c^{-1}xc$ for some simple element c such that $x_\ell \cdot c \preccurlyeq \Delta$. We will see that this is essentially the same problem as calculating the normal form of $x_\ell \cdot x \cdot c$ for such an element c. We will first suppose, for simplicity, that $\inf(x) = 0$; the process is illustrated by the domino diagram in Figure 3, in the special case $\ell = 3$. For an explicit example, see Figure 2.

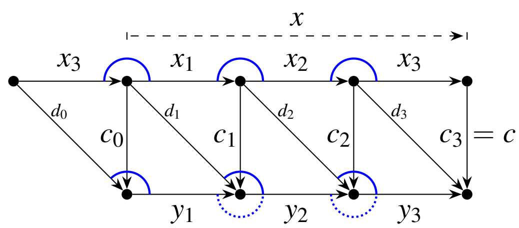

FIGURE 3. A domino diagram showing the calculation of the normal form of $c^{-1}x\,c$ if $\inf(x) = 0$. It is a non-obvious fact that the word $y_1\,y_2\,y_3$ obtained by this calculation is in normal form. Also, under the hypothesis that both x and $c^{-1}x\,c$ are rigid, it is true but non-obvious that $d_0 = d_3$ and $c_0 = c_3 = c$, meaning that $y_1\,y_2\,y_3 = c^{-1}x\,c$, as desired.

We denote $d_\ell = x_\ell \cdot c$, and recall that this is a simple element by hypothesis. The first calculation step consists in determining the normal form of $x_{\ell-1}d_\ell$ – we denote the two factors $d_{\ell-1}$ and y_ℓ, so that $x_{\ell-1} \cdot d_\ell = d_{\ell-1}|y_\ell$. We denote $c_{\ell-1} = x_{\ell-1}^{-1} \cdot d_{\ell-1}$ – this simple element can be interpreted as "the initial part of d_ℓ that can be slid into $x_{\ell-1}$". Then we work our way backwards through the word: $x_{\ell-2}d_{\ell-1} = d_{\ell-2}|y_{\ell-1}$, etc. The last step consists in calculating the normal form $x_\ell \cdot d_1 = d_0|y_1$, and $c_0 = x_\ell^{-1}d_0$. The right domino rule from [9] tells us the non-obvious fact that the word $y_1 \cdot y_2 \cdot \ldots \cdot y_\ell$ thus obtained is in normal form (as indicated by the blue dotted arcs in Figure 3).

Lemma 2.4. *Suppose that x is rigid with $\inf(x) = 0$, that $c \preccurlyeq \varphi(x)$, and that $c^{-1} \cdot x \cdot c$ is also rigid. Suppose the normal form of $x_\ell|x_1|\ldots|x_\ell \cdot c$ is $d_0|y_1|\ldots|y_\ell$. Then $d_0 = d_\ell$, and $c_0 = c_\ell = c$. In particular, $c^{-1}xc = y_1|y_2|\ldots|y_\ell$.*

Proof. This result is contained in the statement and proof of Proposition 2.1 of [15] – in that paper, our elements c_i are denoted u_i. □

We have seen how to calculate the normal form of $c^{-1}xc$, provided that both x and $c^{-1}xc$ belong to $SC(x)$, that $c \preccurlyeq \partial\varphi(x)$ (i.e. the conjugation represents a gray arrow), and that $\inf(x) = 0$.

If $\inf(x) \neq 0$, then we need a slight modification of the previous rule (see Figure 4 and Example 3.7): in the very last step, we do *not* determine the letters d_0 and y_1 by calculating the normal form $d_0|y_1$ of $x_\ell \cdot d_1$; instead, we determine two letters, which we call d_0 and $\tau^{-k}(y_1)$, by calculating the normal form $d_0|\tau^{-k}(y_1)$ of $x_\ell \cdot \tau^{-k}(d_1)$. As previously, we let $c_0 = x_\ell^{-1}d_0$. By the same arguments as above, we have $d_0 = d_\ell$ and $c_0 = c_\ell = c$. Thus, $c^{-1}xc = \tau^{-k}(y_1)\,\Delta^k\,y_2 \ldots y_\ell$. The latter word is almost in normal form: in order to obtain its normal form, it suffices to slide Δ^k to the start, using the rule $\tau^{-k}(y_1)\Delta^k = \Delta^k\,y_1$. Finally, the normal form of $c^{-1}xc$ is $\Delta^k\,y_1|\ldots|y_\ell$.

Lemma 2.5. *Let G be a Garside group. Suppose $x \in G$ is rigid, and n, N are two integers with $n|N$. Denoting $d = \frac{N}{n}$, we have:*

(1) *The function $\pi^d \colon SC(x^n) \longrightarrow SC(x^N)$, $y \mapsto y^d$ is an injection. In particular, $|SC(x^n)| \leqslant |SC(x^N)|$.*

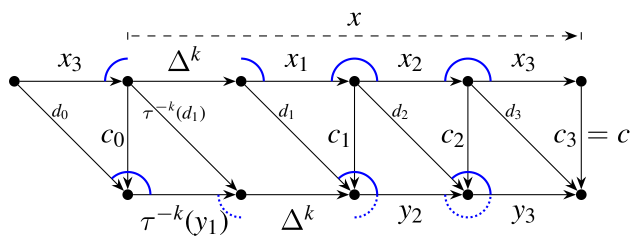

FIGURE 4. A domino diagram showing the calculation of the normal form of $c^{-1}x\,c$, in the case $\inf(x) \neq 0$. Again, the word $\Delta^k y_1\,y_2\,y_3$ obtained by this calculation is in normal form. Also, under the hypothesis that both x and $c^{-1}x\,c$ are rigid, $d_0 = d_3$ and $c_0 = c_3 = c$, meaning that $\Delta^k y_1\,y_2\,y_3 = c^{-1}x\,c$.

(2) *The injection π^d sends each orbit under cycling and τ of $SC(x^n)$ bijectively to an orbit under cycling and τ of $SC(x^N)$.*
(3) *The function π^d gives rise to an inclusion of the conjugacy graph of x^n in the conjugacy graph of x^N.*
(4) *Suppose that the natural injection of commutator subgroups $C(x^n) \hookrightarrow C(x^N)$ is also surjective. Then the image of the inclusion mentioned in (3) of the conjugacy graph of x^n in the conjugacy graph of x^N is an induced subgraph.*

Proof. (1) If y is a rigid element conjugate to x^n, then y^d is a rigid conjugate of x^N, so π^d is well-defined. Next, we claim that the normal form of y can be reconstructed from the normal form of y^d (which implies injectivity). Indeed, since y is rigid, $\inf(y) = \frac{\inf(y^d)}{d}$, and the non-$\Delta$ factors of the normal form of y are the same as the last $\frac{\ell_{can}(y^d)}{d}$ factors of the normal form of y^d.

(2) is a consequence of the fact that π^d commutes with cycling and with τ.

(3) Before proving (3), we warn the reader that minimality of arrows may not be preserved by π^d, i.e. it can happen that a minimal edge in the conjugacy graph of x^n is split into two minimal edges by π^d.

From point (2) we know that the function π^d induces a well-defined map from the vertices of the conjugacy graph of x^n to the conjugacy graph of x^N. Let us now think about the edges.

Suppose in the conjugacy graph of x^n we have a black edge from a vertex represented by an element y of $SC(x^n)$ to a vertex represented by $z = c^{-1}yc \in SC(x^n)$, where $c \preccurlyeq \iota(y)$. Then $c^{-1}y^d c = z^d$, and also $c \preccurlyeq \iota(y^d)$, because (by rigidity of y) we have $\iota(y^d) = \iota(y)$. This means that in the conjugacy graph of x^N, there is still a black edge between the vertices represented by y^d and z^d. The same argument works for gray edges. In summary, π^d induces an inclusion of the conjugacy graph of x^n in the conjugacy graph of x^N.

(4) Suppose that $y, z \in SC(x^n)$ represent distinct vertices of the conjugacy graph of x^n, and that in the conjugacy graph of x^N there is a black edge connecting the vertices represented by y^d and z^d. Let c be the conjugating element: $c \preccurlyeq \iota(y^d)$. We have to prove that there is also a black arrow in the conjugacy graph of x^n from y to z with conjugating element c. For that, we observe that $c \preccurlyeq \iota(y) = \iota(y^d)$, by the rigidity of y; moreover, since y and z are both contained in $SC(x^n)$, there exists an element $\tilde{c}$ which conjugates: $\tilde{c}^{-1}y\tilde{c} = z$. By taking dth powers, we obtain $\tilde{c}^{-1}y^d\tilde{c} = z^d$. Together with the fact

that $c^{-1}y^dc = z^d$ we obtain $\tilde{c}c^{-1}y^dc\tilde{c}^{-1} = y^d$. Since the commutators of y^d and of y coincide, we deduce that $\tilde{c}c^{-1}yc\tilde{c}^{-1} = y$, and conclude that $c^{-1}yc = \tilde{c}^{-1}y\tilde{c} = z$. □

Remark 2.6. Lemma 2.5(4) will not actually be used in the rest of the paper. It is, however, quite a powerful statement: for instance, in the context of braid groups, the hypothesis $C(x^n) = C(x^N)$ is verified for all pseudo-Anosov braids. This is an immediate consequence of the fact (proved in [17] that pseudo-Anosov braids have unique roots: if $y^{-1}x^ly = x^l$, then $(y^{-1}xy)^l = x^l$, and by the uniqueness of roots $y^{-1}x^ky = (y^{-1}xy)^k = x^k$. (The fact that $C(x^k) = C(x^l)$ for a pseudo-Anosov braid x can also be deduced from the results of [19].)

We recall that for any element x of G there is another well-known characteristic subset of the conjugacy class of x, called the *super summit set* $SSS(x)$ which satisfies $SC(x) \subseteq SSS(x)$. We recall the definition from [13]: an element y of G belongs to $SSS(x)$ if it is conjugate to x, if $\inf(y)$ is as large as possible among conjugates of x, *and* if $\sup(y)$ is as small as possible among conjugates of x. It is a non-obvious fact that this subset is always non-empty, and that it coincides with the set of conjugates y of x whose canonical length $\ell_{can}(y)$ is as small as possible among conjugates of x.

Proposition 2.7. *Suppose that $y \in SSS(y)$ is not rigid, but conjugate to a rigid element x. Then no positive power y^n is rigid.*

In other words, if x is rigid and $y \in SSS(x) \setminus SC(x)$, then no power y^n belongs to $SC(x)$. Yet another way of saying this is: for elements possessing a rigid power *and* a rigid conjugate, the property of being in its own SSS is actually equivalent to the (a priori much stronger) condition of being in its own SC.

Proof. For every positive integer n, the element x^n is rigid. Therefore $\ell_{can}(x^n) = n \cdot \ell_{can}(x)$, and no conjugate of x^n has a smaller canonical length than that. In particular, $\ell_{can}(y^n) \geqslant n \cdot \ell_{can}(x)$. On the other hand, $y \in SSS(y) = SSS(x)$, so $\ell_{can}(y) = \ell_{can}(x)$. This implies that $\ell_{can}(y^n) \leqslant n \cdot \ell_{can}(y) = n \cdot \ell_{can}(x)$. We have proven that $\ell_{can}(y^n) = n \cdot \ell_{can}(x)$ for every positive integer n.

This implies that the sequence of initial factors $\iota(y), \iota(y^2), \ldots$ is increasing, in the sense that each is a prefix of the next: $\iota(y) \preccurlyeq \iota(y^2) \preccurlyeq \iota(y^3) \preccurlyeq \ldots$ (see [1](Proof of Lemma 3.28)). In particular, $\iota(y) \preccurlyeq \iota(y^n)$. By a similar argument, $\varphi(y) \succcurlyeq \varphi(y^n)$.

Now by hypothesis, y is not rigid, i.e. the product $\varphi(y) \cdot \iota(y)$ is not left-weighted; this means that $\iota(y)$ has a non-empty prefix i such that $\varphi(y) \cdot i$ is simple. *A fortiori*, the product $\varphi(y^n) \cdot \iota(y^n)$ is not left-weighted, either (as the same element i is still a prefix of $\iota(y^n)$ and $\varphi(y^n) \cdot i$ is still simple). This means that y^n is not rigid. □

Example 2.8. We return to Example 1.4. There, the rigid braid $x = 21|12|2132$, with $\inf(x) = 0$ and $\sup(x) = 3$, has a conjugate $y = 1^{-1}x1 = \Delta^{-1}12132|21321|23|32$ which is not rigid but whose square is rigid. As predicted by Lemma 2.7, y does not even belong to $SSS(x)$, as witnessed by the fact that $\inf(y) = -1$.

3. More examples

In this section we present some examples which we find enlightening. All calculations were performed with the computer programs Cbraid [18] (for $\mathcal{B}_m$) and GAP3 [22, 7] (for $\mathcal{B}_m^*$). Whenever we say that the sequence $|SC(x^n)|_{n\in\mathbb{N}}$ "appears to be" periodic of some period, we mean that this is what our (necessarily finite) calculations indicate.

Example 3.1. In $\mathcal{B}_5$, if we set $x = 213243.34.432$, then the sequence $(|SC(x^n)|)_{n\in\mathbb{N}}$ appears to be periodic of period 3, with $|SC(x)| = |SC(x^2)| = 6$ and $|SC(x^3)| = 42$. The conjugacy graph of x^3 consists of three vertices which are connected by three gray arrows in a cyclic manner.)

Example 3.2. In $\mathcal{B}_6$, if we set $x = 243215432.24$, then the sequence $(|SC(x^n)|)_{n\in\mathbb{N}}$ appears to be periodic of period 6, and $|SC(x)| = 4$, $|SC(x^2)| = 12$, $|SC(x^3)| = 28$, $|SC(x^4)| = 12$, $|SC(x^5)| = 4$ and $|SC(x^6)| = 84$.

Example 3.3. The following example might help for understanding the general case. We consider the 8-strand braid group, equipped with its classical Garside structure, and the element

$$x = 2\,4\,6\,.\,2\,4\,6\,5\,4\,3\,2\,1\,7\,6\,5\,4\,3\,2 \quad \in \mathcal{B}_8$$

Calculations with the computer program [18] indicate that this element x is rigid and pseudo-Anosov, with $\inf(x) = 0$ and $\sup(x) = 2$. Moreover, the sequence $(|SC(x^n)|)_{n\in\mathbb{N}}$ appears to be periodic of period 12, with

n	1	2	3	4	5	6	7	8	9	10	11	12
$\lvert SC(x^n)\rvert$	4	12	40	76	4	120	4	76	40	12	4	760

One possible interpretation is that the 8-strand braid x has some kind of symmetry which only reveals itself in the twelfth power x^{12}. Making this intuition more precise might be a great step towards proving Conjecture 1.2.

Figure 5 shows the conjugacy graph of x^{12}. This graph nicely illustrates the fact that $\mathcal{R}(x) = \{1, 2, 3, 4, 6, 12\}$. The red dot is present in the conjugacy graph of x^n for every n, where it is represented by the braid x^n. The orange dot is present in the conjugacy graph of x^n if n is even: it is represented by y^2 for some non-rigid conjugate y of x with $r(y) = 2$. The yellow dots are in the graph of x^n if $3|n$, the green dots if $4|n$, the blue dots if $6|n$, and the white dots if $12|n$.

Here is an example of an element y which is conjugate to x, which is not rigid, but whose twelfth power is: we take y to be x, conjugated by the braid $c = 1\ 6$

$$y = -6\ -1\ \cdot\ x\ \cdot\ 1\,6 = -1\,.\,2\,4\,6\,5\,4\,3\,2\,1\,7\,6\,5\,4\,3\,2\,1\,.\,4\,6\,.\,6$$

when written in mixed normal form.

As expected from Lemma 2.7, y does not belong to its super summit set – indeed, we have $\inf(y) = -1 = \inf(x) - 1$ and $\sup(y) = 3 = \sup(x) + 1$. Here is the list of infima and suprema of powers of x and y:

n	1	2	3	4	5	6	7	8	9	10	11	12
$\inf(x^n)$	0	0	0	0	0	0	0	0	0	0	0	0
$\inf(y^n)$	-1	-1	-1	0	-1	-1	-1	0	-1	-1	-1	0
$\sup(x^n)$	2	4	6	8	10	12	14	16	18	20	22	24
$\sup(y^n)$	3	5	6	9	11	12	15	17	18	21	23	24

Example 3.4. Here is a generalization of the previous example. In $\mathcal{B}_{2m}$, the element

$$x = (2\,4\,6\,8\ \ldots\ 2m-4\ 2m-2)^2\ 2m-3\ 2m-4\ \ldots\ 3\,2\,1\ 2m-1\ 2m-2\ 2m-3\ \ldots 4\,3\,2$$

is a rigid pseudo-Anosov braid, and it appears that the sequence $(|SC(x^n)|_{n\in\mathbb{N}})$ is periodic of period $m \cdot (m-1)$.

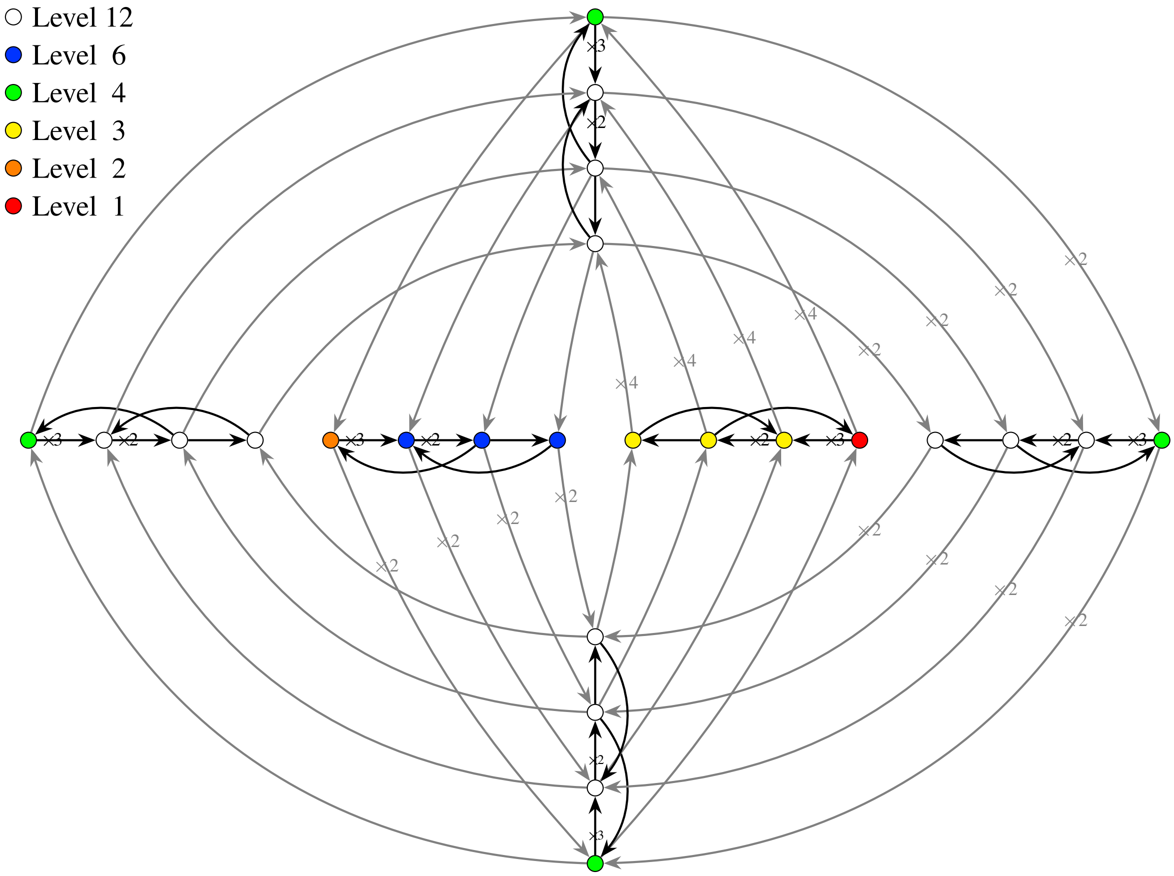

FIGURE 5. The conjugacy graph of x^{12}. However, for the sake of clarity, not *all* black and gray arrows are shown but only the *minimal* ones, i.e. only those arrows that cannot be obtained as the composition of two or more arrows of the same color.

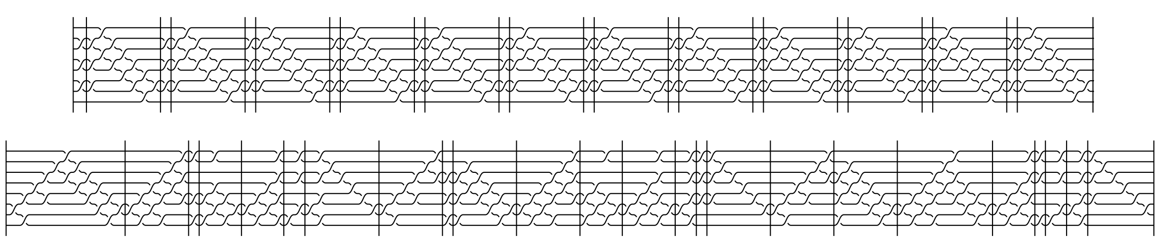

FIGURE 6. The braids x^{12} (above) and y, its conjugate by $\sigma_1\sigma_6$ (below). Thus the upper picture shows a representative of the red dot in Figure 5, and the lower one shows a representative of one of the white dots.

Example 3.5. The aim of the next example is to destroy one possible idea for proving Conjecture 1.6. In the 5-strand braid group with its classical Garside structure, we consider the element $y = \Delta^{-2}$. 1 2 1 3 2 1 4 3 2 . 2 1 3 2 1 4 3 2 1 . 1 2 1 3 2 1 . 2 3 2 1 4 3. It is pseudo-Anosov, it satisfies $\inf(y) = -2$, $\sup(y) = 2$, it is not rigid, but it is conjugate to the rigid braid $x =$ 1 2 3 2 1 . 3 2 1 4 3 with $\inf(x) = 0$ and $\sup(x) = 2$. The powers of y are as follows:

$$\begin{aligned} y &= \Delta^{-2} \,.\, 121321432 \,.\, 213214321 \,.\, 121321 \,.\, 232143 \\ y^2 &= \Delta^{0} \,.\, 12321432 \,.\, 2134321 \,.\, 12 \,.\, 213 \\ y^3 &= \Delta^{-2} \,.\, 121321432 \,.\, 213214321 \,.\, 121321 \,.\, 232143 \,.\, 12321432 \,.\, 2134321 \,.\, 12 \,.\, 213 \\ y^4 &= \Delta^{0} \,.\, 12321432 \,.\, 2134321 \,.\, 12 \,.\, 213 \,.\, 12321432 \,.\, 2134321 \,.\, 12 \,.\, 213 \\ &\vdots \end{aligned}$$

The braid y^n is rigid if and only if n is even. We have $\inf(y^n) = -2$ if n is odd, $\inf(y^n) = 0$ if n is even, and $\sup(y^k) = 2n$.

We observe that the last factors of y and y^3 do not coincide. This example goes to show that the sequence of pairs $(\iota(y^n), \varphi(y^n))$ need not be periodic of period $r(y)$, even when $r(y)$ is the first rigid power of y, and even when the sequence $|SC(x^n)|$ is periodic of period $r(y)$.

The final three examples will illustrate the proof of our main theorem. They take place in $\mathcal{B}_4^*$, the 4-strand braid group equipped with the dual Garside-structure of [3]. Here is a quick reminder of how this structure works. The four punctures are arranged in a circular fashion around the disk, and the Garside element δ is given by a counterclockwise cyclic movement of all four punctures in the disk by an angle of $\frac{\pi}{2}$, giving rise to a cyclic permutation of the punctures. The divisors of δ are in bijection with non-crossing partitions of the four punctures – indeed, given a non-crossing partition, we get a braid by a movement of the punctures which cyclically exchanges the punctures in the same subset. Thus, the divisors of δ in $\mathcal{B}_4^*$ are the trivial element, the six atoms (|:), (‾), (:|), (_), (/) and (\), as well as (◢), (◥), (◤), (◣), (=), (||), and finally (□) $= \delta$. We will call (/) and (\) the "diagonal elements". Here are some examples of relations between these generators: (|:) · (‾) = (◤) (whereas the product (‾) · (|:) cannot be simplified and is in normal form), (/) · (:|) = (◢), (/) · (||) = (□) (◢) · (|:) = (□). We also recall the automorphism $\tau(x) = \delta^{-1} x \delta$, which can be interpreted as a $\frac{\pi}{2}$ counterclockwise rotation of the disk: τ((_)) = (:|), τ((◤)) = (◣) etc.

Example 3.6. In $\mathcal{B}_4^*$: we define $x =$ (\)(/)(‾)(|:)(/). Then the sequence $(|SC(x^n)|)_{n\in\mathbb{N}}$ appears to be periodic of period 2, with $|SC(x)| = 7$ and $|SC(x^2)| = 7 \cdot 20 = 140$.

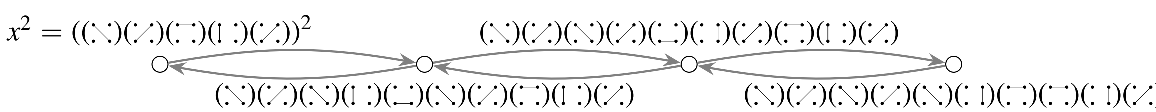

FIGURE 7. The conjugacy graph of x^2 for $x =$ (\)(/)(‾)(|:)(/)

The conjugacy diagram of $SC(x)$ consists of only a single vertex (i.e. the only rigid conjugates of x are those obtained by cyclic permutation of its factors and the action of τ), whereas the diagram of x^2 has four vertices. Each of the rightward-pointing arrows in the diagram represents conjugation by (|:), each of the leftward-pointing arrows represents conjugation by (:|).

We remark that the conjugacy graphs of x^2 can become arbitrarily large, e.g. if we choose x from the family $x =$ (\)(/)(‾)(|:)(/) ((\)(/)(\)(/))^s with $s \in \mathbb{N}$.

Example 3.7. This example is similar to the previous one, but it illustrates the case $\inf(x) \neq 0$. In $\mathcal{B}_4^*$, we consider $x = \delta$(/)(/). Then the sequence $(|SC(x^n)|)_{n\in\mathbb{N}}$ is periodic of period 2, with $|SC(x)| = 4$

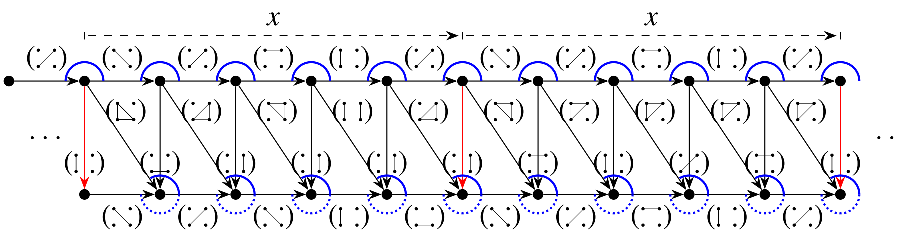

FIGURE 8. The conjugation of x^2 by (| :) for $x =$ (⋱)(⋰)(⊓)(| :)(⋰)

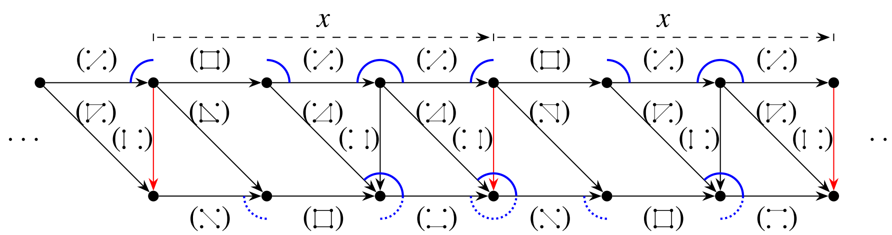

FIGURE 9. For $x = \delta \cdot$ (⋰)(⋰), the conjugation of x^2 by (| :) yields (⋱) δ (⊔)(⋱) δ (⊓) $= \delta^2$ (⋱)(: |)(⋰)(⊓).

and $|SC(x^2)| = 12$. Indeed, the conjugacy graph of x has only one vertex, representing the four obvious elements δ (⋰)(⋰), δ (⋰)(⋱), δ (⋱)(⋱), and δ (⋱)(⋰). The conjugacy graph of x^2 has two vertices: the vertex represented by $x^2 = (\delta$ (⋰)(⋰)$)^2 = \delta^2$ (⋱)(⋱)(⋰)(⋰), and the one represented by (| :)$^{-1}x^2$(| :) $= \delta^2$ (⋱)(: |)(⋱)(⊓). Figure 9 shows the calculation that conjugating x^2 by (| :) yields (⋱) δ (⊔)(⋱) δ (⊓). The latter word is almost in normal form: in order to obtain the normal form, it suffices to slide the letters δ to the start of the word, using the rule $x_i \cdot \delta = \delta \cdot \tau(x_i)$. We find: (| :)$^{-1} \cdot \delta$ (⋰)(⋰) $\cdot$ (| :) $= \delta^2$ (⋱)(: |)(⋰)(⊓).

Example 3.8. Still in $\mathcal{B}_4^*$, we consider $x =$ (⊔)(⊔)(: |)(: |)(⊓)(⊓)(| :)(| :). Then the sequence $(|SC(x^n)|)_{n\in\mathbb{N}}$ appears to be periodic of period 3, with $|SC(x)| = |SC(x^2)| = 3$ and $|SC(x^3)| = 32$.

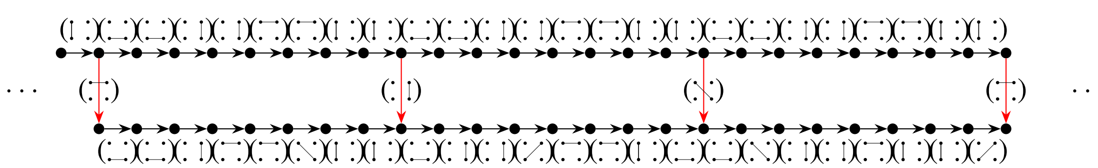

FIGURE 10. The conjugation of x^3 by (⊓) for $x =$ (⊔)(⊔)(: |)(: |)(⊓)(⊓)(| :)(| :)

4. PROOF OF THE MAIN THEOREM

In this last section, we will focus on the special cases of the groups $\mathcal{B}_3$ and $\mathcal{B}_4^*$. These two groups have a very convenient feature in common:

Lemma 4.1. (a) *In the braid group $\mathcal{B}_3$ (with its classical Garside structure) the Garside element Δ is of weight 3.*

(b) *In the braid group $\mathcal{B}_4^*$ (with its dual Garside structure) the Garside element δ is of weight 3.*

We recall that being of *weight* k means being the product of k atoms.

Proof. In $\mathcal{B}_3$, every Garside generator is either a single atom (σ_1 or σ_2), or a product of two atoms ($\sigma_1\sigma_2$ and $\sigma_2\sigma_1$), or it is equal to $\Delta = \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$, a product of three atoms. Similarly, in $\mathcal{B}_4^*$, every Garside generator is either a single atom ((⁞ ⁚), (⁚⁚), (⁚ ⁞), (⁚⁚), (⋱), (⋰)), or a product of two atoms ((◺), (◸), (◹), (◿), (⊏⊐), (⁞ ⁞)), or it is equal to δ, which can be written as a product of three atoms in many different ways: (□) = (⁞ ⁚)(⁚ ⁞)(⋱) = (⁞ ⁚)(⁚⁚)(⁚ ⁞) = (⁚⁚)(⁚ ⁞)(⁚⁚) = ... ☐

Lemma 4.2. *Suppose G is a Garside group, equipped with a Garside structure where the Garside element is of weight 3. Suppose the normal form of a rigid element x of G contains a letter of weight* 1 and *a letter of weight* 2. *Then the conjugacy graph of x (and of x^n for any integer n) consists of only one vertex.*

Proof. Possibly after applying the cycling operation a few times, we can assume that the last letter of x is of weight 2. Now we recall that any gray arrow in the conjugacy graph leaving the vertex represented by x is given by conjugation by an element of

$$\mathcal{C}_x = \{c \in G \mid 1 \prec c \prec \partial\varphi(x)\}.$$

But in our situation, $\partial\varphi(x)$ consists of a single atom, so this set is empty. We conclude that no gray arrow can leave the vertex represented by x.

Similarly, after some cycling, the first non-Δ letter of x is of weight 1. Now, $\iota(x)$ consists of only one atom, and no black arrow can exit the vertex represented by x.

Since the conjugacy graph of x is known to be connected [2], we conclude that it has only one vertex. ☐

Remark 4.3. Because of Lemma 4.2, in Garside groups where the Garside element is of weight 3, we can restrict our attention to rigid elements x whose normal form contains no letters of weight 2, i.e. only Δ and letters of weight 1. (The study of elements whose normal form contains no letters of weight 1 can be reduced to the opposite case – indeed, if x has no letters of weight 1, then x^{-1} has no letters of weight 2, and for any braid x, the sliding circuit sets of x and x^{-1} are in natural bijection.)

We are now ready to prove Theorem 1.7(a), i.e. to prove our Main Conjecture 1.2 in the case of the 3-strand braid group with its classical Garside structure.

By Lemma 4.1(a), the Garside element is of weight 3, and Remark 4.3 applies. In B_3, the *only* rigid elements where all non-Δ factors have weight 1 are of the form $\Delta^{2k}x_1^\ell$ or $\Delta^{2k}x_2^\ell$. It is easy to check by hand that each such element has only two rigid conjugates: itself and its image under τ. In particular, taking a power of x cannot increase the number of rigid conjugates. This completes the proof in the 3-strand case.

In order to deal with the case $\mathcal{B}_4^*$, we need some more theory. The theme of this paper is that, after conjugating a rigid braid along a gray or black arrow, we may obtain a braid which is not rigid, but which becomes rigid if we elevate it to some power. The following lemma places a bound on the required power. Indeed, a bound is given by the number of strict, non-trivial prefixes of the complement of any of the factors of x:

Lemma 4.4 (Limit on rigid power)**.** *Suppose x is a rigid element of G. Consider the set*

$$\mathcal{C}_x = \{c \in G \mid 1 \prec c \prec \partial\varphi(x)\}$$

Suppose that for some element c_* of $\mathcal{C}$ the conjugate $c_*^{-1}xc_*$ has a rigid power. Let ρ be the smallest such power, i.e. $\rho = r(c_*^{-1}xc_*)$ (using Notation 1.5). Then there is a subset $\tilde{\mathcal{C}}$ of $\mathcal{C}_x$ which has precisely ρ elements such that for every c belonging to $\tilde{\mathcal{C}}$, the element $c^{-1}x^\rho c$ is rigid. In particular, $\rho \leqslant |\mathcal{C}_x|$.

There is, of course, an equivalent statement for conjugations of x by prefixes of $\iota(x)$.

Example 4.5. (a) In Example 1.4, there are two different conjugations of x^2, visible as red arrows in Figure 2; indeed, x^2 can be conjugated by 1 to the rigid braid 21.12.2132.2132.23.32, and by 3 to 2132.23.32.21.12.2132, which is a cyclic conjugate of the previous word. These two conjugations are indicated by the label "$\times 2$" on the arrow originating in the vertex x^2 in Figure 1.

(b) In Example 3.6, there are two different conjugations of x^2, one by ( ) and one by ( ) – again, these two are visible as red arrows in Figure 7.

(c) Exactly the same behaviour as in Example 3.6 can be observed in Example 3.7 – see Figure 9.

(d) In Example 3.8, the braid x^3 can be conjugated by three different gray arrows (corresponding to conjugations by ( ), ( ) and ( )) to three braids that are cyclic conjugates of each other.

(e) In Example 3.3 (Figure 5), whenever we have an arrow from a vertex that appears in the conjugacy graph of x^n to a vertex that appears in the conjugacy graph of x^N, with $n|N$, then the corresponding edge is labelled $\times\frac{N}{n}$.

Proof of Lemma 4.4. We look at the domino diagram for calculating the normal form of $c_*^{-1}x^\rho c_*$. Let us denote by $\ell = \ell_{can}(x)$, the canonical length of x.

We start with the $\rho \cdot \ell$ horizontal arrows whose labels spell out the normal form of x^ρ, see Figure 3, and Figure 2 for an example. (We have to take $\rho \cdot (\ell + 1)$ arrows if $\inf(x) \neq 0$, see Figure 4, and Figure 9 for an example.) Also, we have a vertical arrow labelled $c_* = c_{\rho\cdot\ell}$ at the right end of the diagram.

We then construct the full domino diagram from right to left, calculating $c_{\rho\cdot\ell-1}, c_{\rho\cdot\ell-2}$ etc. We will be interested in the subsequence $c_{\rho\cdot\ell}, c_{(\rho-1)\cdot\ell}, \ldots, c_{2\cdot\ell}, c_\ell, c_0$, corresponding to the red arrows in our examples. By minimality of ρ, the terms of this subsequence are pairwise distinct, except that $c_{\rho\cdot\ell} = c_0$. These are our ρ distinct conjugators. We remark that the ρ elements $c_0^{-1}x^\rho c_0, \ldots, c_{\rho\cdot\ell}^{-1}x^\rho c_{\rho\cdot\ell}$ are all in the same orbit under the cycling operation. Since the last of them, $c_{\rho\cdot\ell}^{-1}x^\rho c_{\rho\cdot\ell} = c_*^{-1}x^\rho c_*$, is rigid by hypothesis, they are all rigid. □

Lemma 4.6. *Let x be a rigid element of G. Look at any vertex y in the conjugacy graph of x^n, for any n. Look at the labels of the outgoing gray edges from y, including those labelled "$\times 1$" (for which by convention we didn't write down the label). Then the sum of those labels is at most $|\mathcal{C}_y|$, where $\mathcal{C}_y = \{c \in G \mid 1 \prec c \prec \partial\varphi(y)\}$.*

Proof. This is simply because of the pigeonhole principle: the outgoing edges represent disjoint sets of distinct conjugating elements, which all lie in $\mathcal{C}_y$. □

Let us now concentrate on the braid group $\mathcal{B}_4^*$. The sliding circuit sets and conjugacy diagrams in the case $\mathcal{B}_4^*$ were already studied in [5].

Because of Lemma 4.1(b) and Remark 4.3, when trying to prove our Main Theorem 1.7(b), we can restrict our attention to rigid elements x whose normal form contains no letters of weight 2, but only letters of weight 1, and δ. Any rigid conjugate y of any power of such an element x has the same property: it has no letters of weight 2. In particular, $\iota(y)$ has no strict nontrivial prefixes. Thus

Observation 4.7. Suppose $x \in \mathcal{B}_4^*$ is rigid, and contains no letter of weight 2. Then in the conjugacy graph of x^n, for any n, all vertices correspond to braids which are also rigid and without letters of weight 2. Moreover, there are only gray arrows, no black ones.

Another key observation is that the presence of "diagonal" letters $(\diagdown)$ ou $(\diagup)$ imposes strong restrictions on the possible gray arrows. Indeed, diagonal letters have only two strict nontrivial prefixes to their complement, whereas non-diagonal letters have three. For instance:

$$\{c \in G \mid 1 \prec c \prec \partial(\diagdown)\} = \{(\overline{\;\;}), (\underline{\;\;})\} \text{ whereas } \{c \in G \mid 1 \prec c \prec \partial(|\;)\} = \{(\overline{\;\;}), (\diagdown), (\;|)\}$$

Let us first prove our Main Theorem 1.7(b) in the more subtle case where some rigid conjugate of some power of x contains no diagonal letter:

Lemma 4.8. *Let x be a rigid element of $\mathcal{B}_4^*$ containing no letters of weight 2. Suppose that the normal form of some rigid conjugate of some power of x contains no diagonal letters ($(\diagdown)$ or $(\diagup)$). Then for every integer n, the conjugacy graph of x^n has at most six vertices.*

Proof. By Lemma 2.5, the conjugacy graph of x^n is contained in the conjugacy graph of x^{4n}, whose infimum is congruent to 0 modulo 4. The conjugacy graphs of dual 4-strand braids with infimum 0, all of whose letters are of weight 1 and non-diagonal, were studied in detail in [5]. It turns out that the conjugacy graphs have at most 6 vertices. □

To summarize, if x is a rigid braid with no letters of weight 2 *and no diagonal letters*, then the conjugacy graph has only gray arrows, which Lemma 4.4 can only be labelled "$\times 1$", "$\times 2$", or "$\times 3$"; this means that if y and z are elements of G that represent adjacent vertices of the conjugacy graph, then $r(z) \leqslant 3 \cdot r(y)$. Moreover, the graph has diameter at most 5 (as it has at most 6 vertices). In particular, for any $y \in SC(x)$, we have $r(y) \leqslant 3^5 = 243$.Thus our braid x satisfies Conjecture 1.6. (In fact, a finer analysis would show that the only possible conjugacy graphs are the graph with only one vertex, and the graph with two vertices, an edge labelled $\times 3$ going from one to the other, and an edge labelled $\times 1$ going the other way. We will not prove this result here.)

We now turn to the easier case, where every rigid conjugate of every power of x contains a diagonal letter.

Lemma 4.9. *Let x be a rigid element of $\mathcal{B}_4^*$ containing no letters of weight 2. Suppose that in the conjugacy graph of x^n (for some $n \in \mathbb{N}$) there is a gray arrow emanating from a vertex which represents an element $y \in SC(x^n)$ whose normal form contains a diagonal letter $(\diagdown)$ or $(\diagup)$. Then for any integer d, in the conjugacy graph of $x^{n \cdot d}$, there are no gray arrows emanating from the vertex y^d other than those inherited via π^d from the conjugacy graph of x^n.*

Proof. After cycling, we can assume that the last letter of y is a diagonal letter. In that case, $\partial\varphi(y)$ has only two strict prefixes, so only two gray arrows (counted with multiplicity) can emanate from the vertex y in the conjugacy graph of x^n, or from the vertex y^d in the conjugacy graph of $x^{n \cdot d}$. If d was the smallest integer so that a new gray arrow appeared, emanating from the vertex y^d in the conjugacy graph of $x^{n \cdot d}$, then by Lemma 4.4, there are d copies of this gray arrow. Also, one further gray arrow is inherited via π^d from the conjugacy graph of x^n, by hypothesis. In summary, we'd have $d+1$ gray arrows emanating from the vertex y^d. Thus $d=1$. $\square$

As an immediate consequence of Observation 4.7 and Lemma 4.9 we have:

Lemma 4.10. *Let x be a rigid element of $\mathcal{B}_4^*$ containing no letters of weight 2. Suppose that the normal form of every rigid conjugate of every power of x contains a diagonal letter ((⋰) or (⋱)). Suppose that n and N are two positive integers with $n|N$. If the conjugacy graph of x^n contains more than one vertex, then the conjugacy graph of x^N is isomorphic to the conjugacy graph of x^n.*

We are now ready to prove our Main Theorem 1.7(b) for elements x which are rigid, have no letters of weight 2, and for which every rigid conjugate of any power of x contains a diagonal letter.

Let us fix some power n, and look at the conjugacy graph of x^n. There are two possibilities.

Either the conjugacy graph of x^n has only one vertex. Let d be the smallest integer such that the conjugacy graph of $x^{d\cdot n}$ has more than one vertex. (If there exists none, then the proof is complete.) Let c_* be a conjugating element, representing a gray arrow in that conjugacy graph, such that $c_*^{-1}x^{d\cdot n}c_*$ is rigid. Then $r(c_*^{-1}x^n c_*)=d$. By Lemma 4.4, applied to x^n, we have $d=2$.

The other possibility is that the conjugacy graph of x^n has more than one vertex. Then Lemma 4.10 implies that the conjugacy graph of any further power $x^{d\cdot n}$ is no larger than the conjugacy graph of x^n.

To summarize, for any element y conjugate to x which has some rigid power, we have $r(y)=1$ or $r(y)=2$. This completes the proof of Main Theorem 1.7(b).

This paragraph to be deleted. It is not immediately obvious that this implies that the sequence $(|SC(x^n)|)_{n\in\mathbb{N}}$ is either constant or alternating: if $|SC(x^2)|>|SC(x)|$, it neads a little bit of argument to show that that $|SC(x^3)|=|SC(x)|$. I suggest not do this here.

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