# Pseudo-Anosov braids are generic

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Preliminary version of the paper (in French) at
perso.univ-rennes1.fr/sandrine.caruso/recherche.html

Rolfsenfest, Luminy, July 2013

## Theorem 1 [Caruso-W]

Generic braids are pseudo-Anosov.

More precisely : consider the ball of radius $L$ and center $1$ in the Cayley graph of $B_n$ with generators $= \{$ simple braids $\}$ (Garside's generators). Then

$$\text{proportion of pA elements in this ball} \xrightarrow{L\to\infty} 1$$

(exponentially fast convergence).

**Remark** Maher and Sisto proved this (and much more) if you interpret "generic" as "the result of a long random walk in the Cayley graph".

## Open questions

Generalizations to

- braid group $B_n$ equipped with other generating sets ?
- mapping class groups ?
- groups acting on $\delta$-hyperbolic spaces (analogue of Sisto's result) ?

## Corollary 1

In the Cayley graph there are arbitrarily large balls containing only pA elements.

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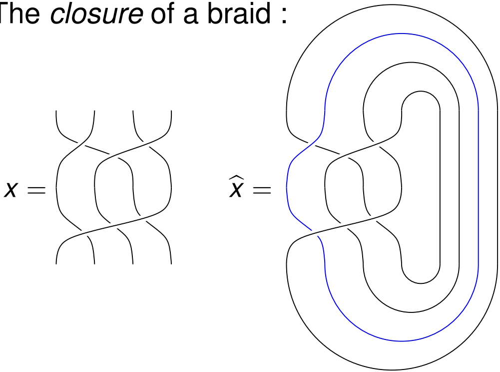
## Corollary 1

In the Cayley graph there are arbitrarily large balls containing only pA elements.

## Corollary 2

The closure of a generic braid is a hyperbolic link.

**Definition** The *closure* of a braid :



**Proof of Corollary 2** uses a theorem of Tetsuya Ito :
the closure of a pA braid with Dehornoy floor $\notin [-2, 2]$
is a hyperbolic link.

**Theorem 2 [Caruso-W]**

Generically, the conjugacy search problem in the braid group $B_n$ can be solved in quadratic time.

I won't talk about this result today.

## Our preferred generators of $B_n$

“Simple braids”, a.k.a. “positive permutation braids” : positive braids, any two strands crossing at most once

$$\Updownarrow$$

Permutations of $\{1, \ldots, n\}$

- **Typical example**

  Simple braid $x \in B_4$ , permutation $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix}$

- **Very special example**

  Half-twist $\Delta \quad \leftrightsquigarrow \quad$ permutation $\begin{pmatrix} 1 & \ldots & n \\ n & \ldots & 1 \end{pmatrix}$

- **Property of $\Delta$ :** “almost commutes” with all braids (and $\Delta^2$ generates $Center(B_n)$ )

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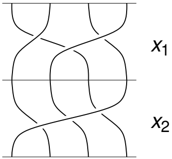
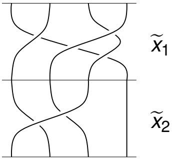
# Left-weighting

**Example**




The product $x_1 \cdot x_2$ is *not* left-weighted ; the product $\widetilde{x}_1 \cdot \widetilde{x}_2$ is.

**Theorem (Thurston, Elrifai–Morton)**

Every $x \in B_n$ has a unique representative of the form

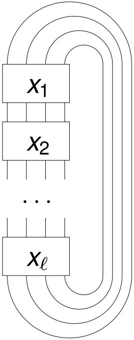
$$\Delta^k \cdot x_1 \cdot \ldots \cdot x_\ell \quad (k \in \mathbb{Z}) \quad \text{with } x_i \cdot x_{i+1} \text{ left-weighted } \forall i$$

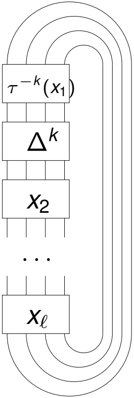
**Notation** $k =$ “infimum of $x$”, $\ell =$“canonical length of $x$”

**Remark** Normal forms are described by a FSA.

## Definition (Rigid braids)

- A braid $x$ with normal form $x_1 \cdot \ldots \cdot x_\ell$ is *rigid* if $x_\ell \cdot x_1$ is left-weighted.
- A braid $x$ with normal form $\Delta^k \cdot x_1 \cdot \ldots \cdot x_\ell$ is *rigid* if $x_\ell \cdot \tau^{-k}(x_1)$ is left-weighted.



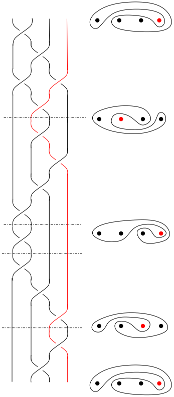



## Theorem (Thurston)

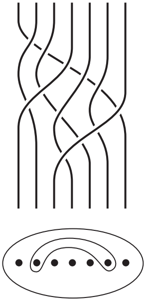
Every braid $x \in B_n$ is exactly one of

- periodic, i.e., $\exists k, m \in \mathbb{Z}$ such that $x^k = \Delta^m$
- reducible (i.e. a curve system is preserved) non-periodic
- pseudo-Anosov

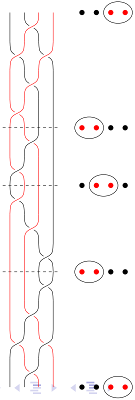
**Examples of reducible braids** (a) nonobviously
(b) rigid braid, almost round curve (c) obviously (round curves)



(a)



(b)



(c)

# Our criterion for being pseudo-Anosov

**Vague hope**
$x$ as ”short, straight and tight” as possible in its conjugacy class
$\stackrel{?}{\Longrightarrow}$ $x$ can only be reducible by being “obviously reducible”

**Theorem (González-Meneses, Wiest)**

If $x \in B_n$ rigid and reducible, then

- either $\exists$ round reducing curve
- or $\exists$ “almost round” reducing curve and interior strands don’t cross (or cross as much as possible in each factor)

## Corollary

If a *rigid* braid wants to be reducible, it must not contain both of the following braids.

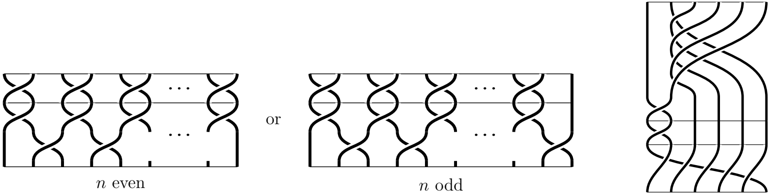



**Figure** (a) Braids sending no round curve to a round curve.
(b) Braid where every pair of strands crosses.

**Remark** A "generic" braid *does* contain both of these subwords with $\mathbb{P} \xrightarrow{L\to\infty} 1$

## Theorem (Caruso)

Among the braids in the $L$-ball of Cayley graph of $B_n$, the proportion of rigid, pA braids $\xrightarrow{L\to\infty} c > 0$.

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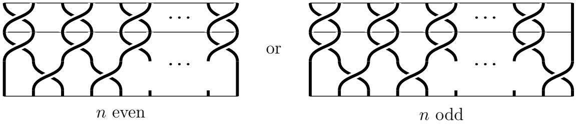


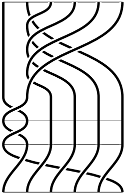

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# Proof of main theorem (Theorem 1)

## Definition (non-intrusive conjugation)

A conjugation

$$x \in B_n \xrightarrow{conjug} \widetilde{x} \in B_n$$

$$x_1 \cdot \ldots \cdot x_\ell \longmapsto \widetilde{x}_1 \cdot \ldots \cdot \widetilde{x}_{\widetilde{\ell}}$$

is *non-intrusive* if the middle third $x_{\frac{1}{3}\ell} \ldots x_{\frac{2}{3}\ell}$ of $x$ also occurs in $\widetilde{x}_1 \cdot \ldots \cdot \widetilde{x}_{\ell}$.

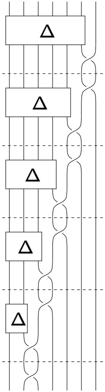
## Claim

Generic braids are non-intrusively conjugate to rigid braids.

I.e., in the ball of radius $L$ of the Cayley graph of $B_n$,
the proportion of braids that have a non-intrusive conjugation to a rigid braid $\xrightarrow{L\to\infty} 1$ exponentially quickly.

## Claim $\Longrightarrow$ Theorem 1

## Example of a blocking braid

$\alpha =$  $\longleftarrow$ last factor : $\sigma_2$



### Defining property

For every braid $X$ such that $X \cdot \alpha$ is in normal form as written, the only simple suffix of $X \cdot \alpha$ is $\sigma_2$.

Picture in Cayley graph of $B_n$
x lifts to bi-infinite path