# The Lawrence-Krammer-Bigelow representation detects the dual braid length

Bert Wiest (Univ. Rennes 1)

joint work with Tetsuya Ito (Univ. Kyoto)

## The *classical* structure : preferred generators of $B_n$

"Simple braids", a.k.a. "positive permutation braids" : positive braids, any two strands crossing at most once

$$\Updownarrow$$

Permutations of $\{1, \ldots, n\}$

- **Typical example**

  Simple braid $x \in B_4$, permutation $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix}$

- **Very special example**

  Half-twist $\Delta \quad \leftrightsquigarrow \quad$ permutation $\begin{pmatrix} 1 & \ldots & n \\ n & \ldots & 1 \end{pmatrix}$

- **Property of $\Delta$ :** "almost commutes" with all braids (and $\Delta^2$ generates $Center(B_n)$)

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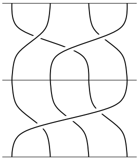
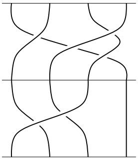
# Left-weighting

**Example**




The product $x_1 \cdot x_2$ is *not* left-weighted ; the product $\widetilde{x}_1 \cdot \widetilde{x}_2$ is.

**Theorem (Thurston, Elrifai–Morton)**

Every $x \in B_n$ has a unique representative of the form

$$\Delta^k \cdot x_1 \cdot \ldots \cdot x_\ell \quad (k \in \mathbb{Z}) \quad \text{with } x_i \cdot x_{i+1} \text{ left-weighted } \forall i$$

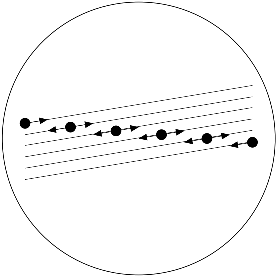
**Notation** $k =$"infimum of $x$", $k + \ell$="supremum of $x$"

**Remark** Normal forms are described by a FSA.

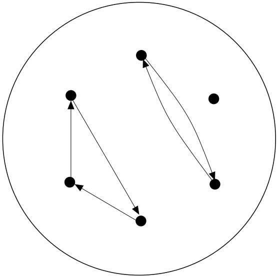
## There is a second Garside structure on the braid group $B_n$

- **Classical** Punctures lined up horizontally. $\Delta\ =$ half-twist.
  Divisors of $\Delta \leftrightarrow$ permutations of the $n$ punctures
  Notation : $\sigma_i =$ exchange adjacent punctures
- **Dual** Punctures on a circle. $\delta\ = \frac{2\pi}{n}$ turn.
  Divisors of $\delta\ \leftrightarrow$ disjoint, non-nested polygons, (possibly degenerate, i.e. having only two vertices)
  Notation : $a_{i,j} =$ exchange punctures $i$ and $j$.

CLASSICAL



DUAL



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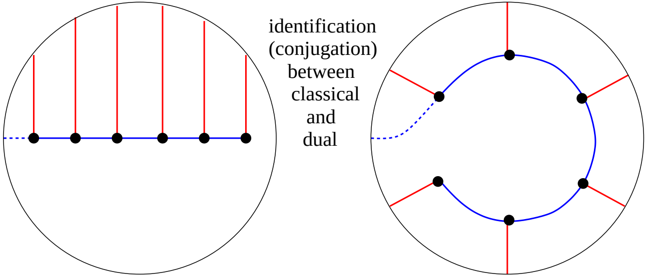

# Historical reminder

**Question**

Is $B_n$ linear, i.e. the subgroup of a matrix group ?

- The *Burau* representation is *not* faithful for $n \geqslant 5$ [Bigelow, Long, Moody], it *is* faithful for $n = 2, 3$, and for $n = 4$ the question is open.
- The LKB–representation (Ruth Lawrence)

$$B_n \xrightarrow{\mathcal{L}} GL\left(\mathbb{Z}\left[q^{\pm 1}, t^{\pm 1}\right], \frac{n(n-1)}{2}\right)$$

**Answer : Yes ! $\mathcal{L}$ is faithful for all $n$.**

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# Explicit formula for the representation $\mathcal{L}$

$$B_n \xrightarrow{\mathcal{L}} GL\left(\mathbb{Z}\left[q^{\pm 1}, t^{\pm 1}\right], \frac{n(n-1)}{2}\right)$$

Denote the basis vectors of $\mathbb{R}^{\frac{n(n-1)}{2}}$ by $F_{i,j}$ (for $1 \leqslant i < j \leqslant n$).
Then $\mathcal{L}(\sigma_k)$ sends

$$F_{i,j} \mapsto \begin{cases} F_{i,j} & k \notin \{i-1, i, j-1, j\} \\ qF_{k,j} + (q^2 - q)F_{k,i} + (1-q)F_{i,j} & k = i-1 \\ F_{i+1,j} & k = i \neq j-1 \\ qF_{i,k} + (1-q)F_{i,j} + (q-q^2)tF_{k,j} & k = j-1 \neq i \\ F_{i,j+1} & k = j \\ -q^2 tF_{i,j} & k = i = j-1 \end{cases}$$

# Krammer's proof that $\mathcal{L}$ is faithful

For any $\beta \in B_n$, consider the maximal and minimal powers of $t$ occurring in the matrix $\mathcal{L}(\beta)$.

**Krammer's main lemma** (which implies faithfulness) :

$$\text{maximal power of } t \text{ in } \mathcal{L}(\beta) = \sup_{ClassicalGarside}(\beta)$$

$$\text{minimal power of } t \text{ in } \mathcal{L}(\beta) = \inf_{ClassicalGarside}(\beta)$$

**Krammer conjectured**

$$\text{maximal power of } q \text{ in } \mathcal{L}(\beta) = 2 \cdot \sup_{DualGarside}(\beta)$$

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**Theorem [Ito,W]** Krammer's conjecture is true.

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**Note** One direction is easy.

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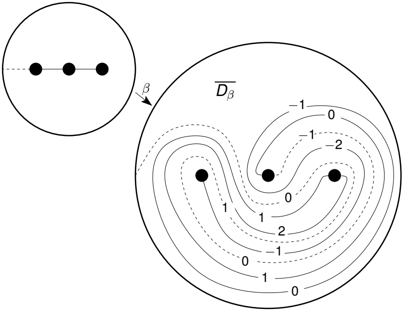
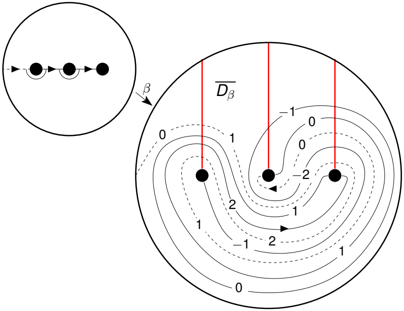
We unsuccessfully tried to also reprove Krammer's result using our techniques.

For any braid $\beta \in B_n$, consider its *curve diagram* $\overline{D_\beta}$ with

Winding number labeling (WNu)




Wall crossing labeling (WCr)




Look at the maximal and minimal labels of the solid arcs.

**Thm 1 [W, 2010]**

$$Min_{WNu}(\beta) = \inf_{Class.Garside}(\beta)$$
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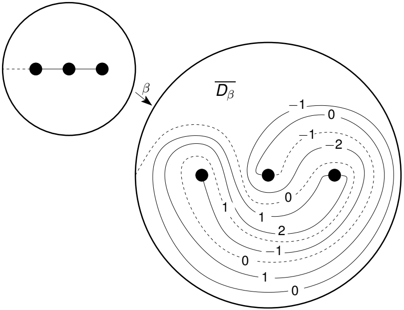


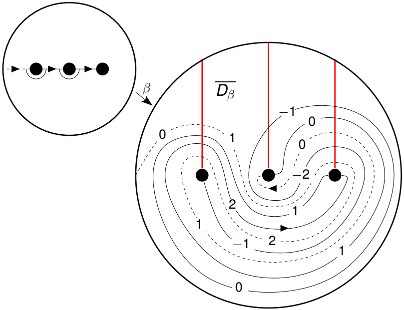


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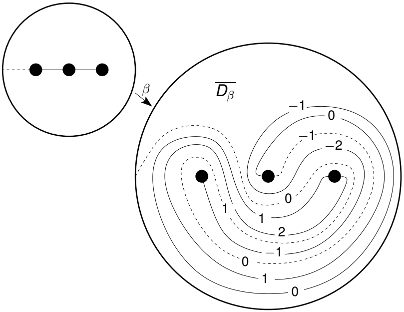


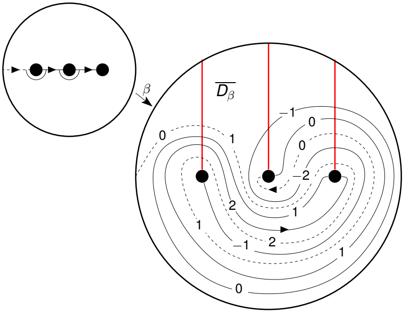


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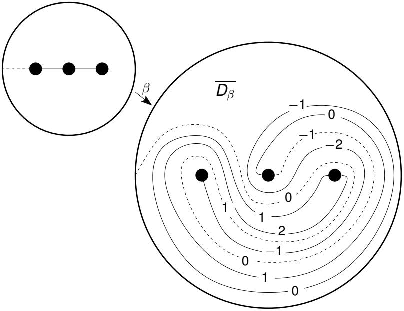
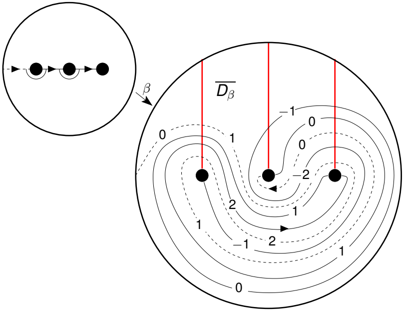
We will use only this result

# Details for this example : $\beta = (\sigma_2^{-1}\sigma_1)^2$

Check the theorems in this special case :




$Min_{WNu}(\beta) = -2,$
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$\Delta^{-2} \cdot \sigma_1\sigma_2 \cdot \sigma_2 \cdot \sigma_2 \cdot \sigma_2\sigma_1$ ✓

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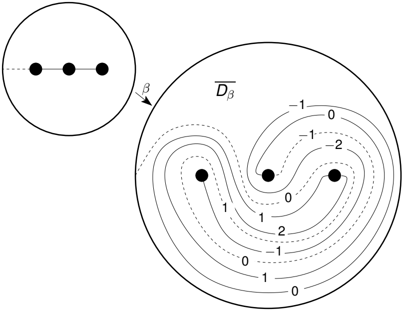


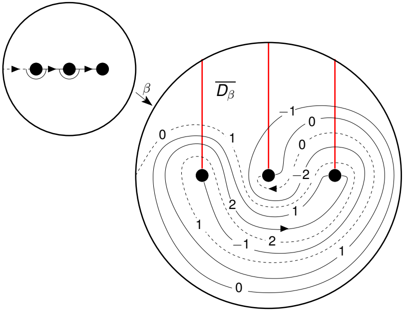


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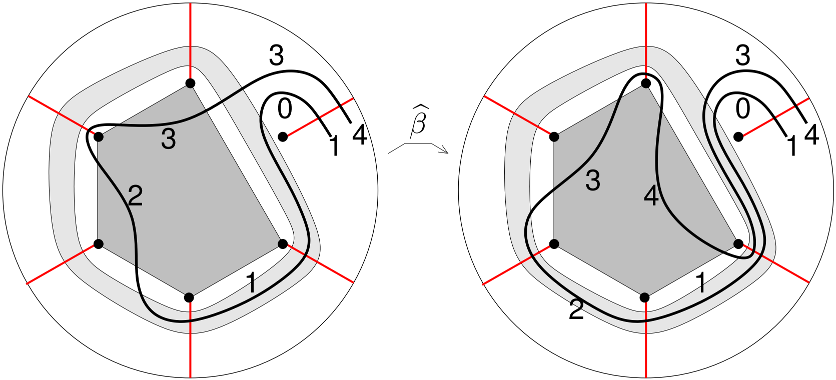
# Proof of Theorem 2 (beginning)

**Theorem 2** Wall crossing labelling detects dual Garside length

**Lemma** Let $\beta \in B_n$, and $\widehat{\beta}$ a divisor of $\delta$. Action of $\widehat{\beta}$ on $D_\beta$ : an arc in $D_\beta$ labelled $k$ gives rise to one or several arcs in $D_{\widehat{\beta}\cdot\beta}$, labelled $k$ or $k+1$.



# Proof of Theorem 2 (end)

**Theorem 2** Wall crossing labelling detects dual Garside length

For simplicity, suppose $Min_{WCr}(\beta) = 0$.

$$\boxed{\text{Need to prove : } \quad Max_{WCr}(\beta) = \sup_{Dual}(\beta)}$$

**Proof of "$\leqslant$"** : follows from Lemma (acting by a divisor of $\delta$ can only increase maximal label by 1).
**Proof of "$\geqslant$"** : by induction on $Max_{WCr}(\beta)$.

1. Construct a collection $P$ of disjoint polygons intersecting all maximally labelled arcs, but none of the minimally labelled ones. Then let $\widehat{\beta}$ be the divisor of $\delta$ corresponding to $P$.
2. Prove that acting on $D_\beta$ by $\widehat{\beta}^{-1}$ decreases the maximal label without decreasing the minimal label :
   - $Max_{WCr}(\widehat{\beta}^{-1}\beta) = Max_{WCr}(\beta) - 1$
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# Lawrence's construction of the representation $\mathcal{L}$

Let $X$ be the configuration space of unordered pairs of points in the $n$-times punctured disk $D_n$ :

$$X = \Big\{ \{x_1, x_2\} \mid x_1, x_2 \in D_n \ , \ x_1 \neq x_2 \Big\}$$

equipped with a basepoint $\{d_1, d_2\}$ (see blackboard).
There is a homomorphism

$$\pi_1(X) \to \mathbb{Z}^2 = \langle q, t \rangle, \ \ \gamma = \{\gamma_1, \gamma_2\} \mapsto q^a t^b$$

where

$a =$ sum of the winding numbers of $\gamma_1$ and $\gamma_2$ around all $n$ punctures

$b = 2 \cdot$ (relative winding number of $\gamma_1$ and $\gamma_2$)

Let $\widetilde{X}$ be the cover corresponding to $\ker\big(\pi_1(X) \to \mathbb{Z}^2\big)$.
Covering group$(\widetilde{X}) = \mathbb{Z}^2 = \langle q, t \rangle$.

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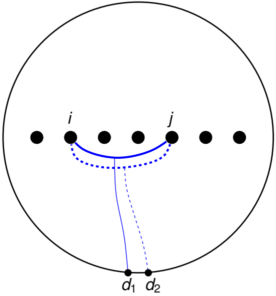
**Note** $B_n \curvearrowright \widetilde{X} \implies B_n \curvearrowright H_2(\widetilde{X}, \partial\widetilde{X}; \mathbb{R}[q^{\pm 1}, t^{\pm 1}])$

**Proposition (Lawrence)** The second homology

$$H_2(\widetilde{X}, \mathbb{R}[q^{\pm 1}, t^{\pm 1}]) \text{ is of dimension } \frac{n(n-1)}{2}$$

with generators $F_{i,j}$ ("forks") as in the following figure.




Moreover, the $B_n$-action on $H_2(\widetilde{X}, \mathbb{R}[q^{\pm 1}, t^{\pm 1}])$ coincides with the representation $\mathcal{L}$.

$*\ *\ *$

# Bigelow's Key Lemma

- Recall $\widetilde{X}$ is a 4-dim. mfd. with covering action by $\mathbb{Z}^2 = \langle q, t\rangle$.
- Recall the "fork" $F_{i,j}$ is a surface (more precisely a square)

in $\widetilde{X}$ representing a generator of $H_2(\widetilde{X}, \mathbb{R}[q^{\pm 1}, t^{\pm 1}])$.

- Let $\widetilde{X}_{0,0}$ be a fund. domain of the $\mathbb{Z}^2$-action containing the

basept. $\{\widetilde{d}_1, \widetilde{d}_2\}$ and all forks $F_{i,j}$. Let $\widetilde{X}_{a,b} = q^a t^b.X_{0,0}$.

**Handwaving version of Bigelow's "Key Lemma"** Let $\beta \in B_n$ and $1 \leqslant i < j \leqslant n$, and consider $\beta(F_{i,j}) \subset \widetilde{X}$. Among the fundamental domains $\widetilde{X}_{a,b}$ intersected by $\beta(F_{i,j}) \subset \widetilde{X}$, select the one $\widetilde{X}_{a_{\max}, b_{\max}}$ with maximal $(a, b)$ (lexicographically). Now in $H_2(\widetilde{X}, \mathbb{R}[q^{\pm 1}, t^{\pm 1}])$ we can write uniquely

$$\beta(F_{i,j}) = \sum_{1 \leqslant i' < j' \leqslant n} P_{i',j'}(q,t) F_{i',j'} \quad (\text{with } P_{i',j'} \in \mathbb{R}[q^{\pm 1}, t^{\pm 1}])$$

Then in one of the polynomials $P_{i',j'}$, the term $q^{a_{\max}} t^{b_{\max}}$ occurs with non-zero coeff. ("contributions do not cancel in homology").

# Bigelow's Key Lemma

- Recall $\widetilde{X}$ is a 4-dim. mfd. with covering action by $\mathbb{Z}^2 = \langle q, t \rangle$.
- Recall the "fork" $F_{i,j}$ is a surface (more precisely a square) in $\widetilde{X}$ representing a generator of $H_2(\widetilde{X}, \mathbb{R}[q^{\pm 1}, t^{\pm 1}])$.
- Let $\widetilde{X}_{0,0}$ be a fund. domain of the $\mathbb{Z}^2$-action containing the basept. $\{\widetilde{d}_1, \widetilde{d}_2\}$ and all forks $F_{i,j}$. Let $\widetilde{X}_{a,b} = q^a t^b . X_{0,0}$.

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# The LKB representation detects $\sup_{DualGarside}(\beta)$

**Proof that** $2 \cdot \sup_{DualGarside}(\beta) =$ **maximal power of $q$ in $\mathcal{L}(\beta)$**

Proof of "$\geqslant$" is easy ($\mathcal{L}$ : divisors of $\delta \mapsto$ matrix of $q$-degree 2).

Proof of "$\leqslant$" :

$2 \cdot \sup_{Dual}(\beta) \overset{\text{Thm 2}}{=} 2 \cdot Max_{WCr}(\beta)$, the max. wall crossing labeling

$\overset{!!!}{=}$ the maximal number $a$ such that $\beta.F_{i,i+1}$ intersects $q^a t^b.X_{0,0}$ for some $i, b$

Now according to Bigelow's Key Lemma, the monomial $q^a t^b$ occurs somewhere in the matrix $\mathcal{L}(\beta)$. $\square$

# Questions

1. Recall Theorem 2 : for a braid $\beta$, we can read the length of $\beta$ from the wall crossing labellings occurring in the curve diagram of $\beta$.

   Question : is there some analogue for $Out(F_n)$, with the sphere system in $(S^1 \times S^2)\#\ldots\#(S^1 \times S^2)$ playing the role of the curve diagram ?

2. Is there a generalization of our main theorem to Artin-Tits groups of finite type ? The question makes sense, as
   - there are dual Garside structures on such groups (Bessis, T.Brady-Watt)
   - there are Lawrence-Krammer-Bigelow type representations on such groups (Digne, Cohen-Wales).

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1. Classical and dual Garside structure on braid groups
2. The Lawrence-Krammer-Bigelow representation
3. Proof : Labellings of curve diagrams
4. Proof : further ideas