The Lawrence-Krammer-Bigelow representation detects the dual braid length

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1. Classical and dual Garside structure on braid groups

2. The Lawrence-Krammer-Bigelow representation

3. Proof: Labellings of curve diagrams

4. Proof: further ideas
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4 Proof: further ideas
The classical structure: preferred generators of $B_n$

“Simple braids”, a.k.a. “positive permutation braids”: positive braids, any two strands crossing at most once

\[ \begin{array}{c}
\vdash \\
\end{array} \]

Permutations of $\{1, \ldots, n\}$

- **Typical example**
  
  Simple braid $x \in B_4$, permutation \[
  \begin{pmatrix}
  1 & 2 & 3 & 4 \\
  3 & 1 & 4 & 2 \\
  \end{pmatrix}
  \]

- **Very special example**
  
  Half-twist $\Delta \leftrightarrow$ permutation \[
  \begin{pmatrix}
  1 & \ldots & n \\
  n & \ldots & 1 \\
  \end{pmatrix}
  \]

- **Property of $\Delta$**: “almost commutes” with all braids (and $\Delta^2$ generates $Center(B_n)$)
The *classical* structure: preferred generators of $B_n$

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Left-weighting

Example

The product $x_1 \cdot x_2$ is *not* left-weighted; the product $\tilde{x}_1 \cdot \tilde{x}_2$ is.

**Theorem (Thurston, Elrifai–Morton)**

Every $x \in B_n$ has a unique representative of the form

$$\Delta^k \cdot x_1 \cdot \ldots \cdot x_\ell \quad (k \in \mathbb{Z})$$

with $x_i \cdot x_{i+1}$ left-weighted $\forall i$

**Notation** $k = \text{“infimum of } x\text{”}$, $k + \ell = \text{“supremum of } x\text{”}$

**Remark** Normal forms are described by a FSA.
There is a second Garside structure on the braid group $B_n$

- **Classical** Punctures lined up horizontally. $\Delta = \text{half-twist.}$ Divisors of $\Delta \leftrightarrow$ permutations of the $n$ punctures
  
  Notation : $\sigma_i = \text{exchange adjacent punctures}$

- **Dual** Punctures on a circle. $\delta = \frac{2\pi}{n}$ turn.
  Divisors of $\delta \leftrightarrow$ disjoint, non-nested polygons, (possibly degenerate, i.e. having only two vertices)
  
  Notation : $a_{i,j} = \text{exchange punctures } i \text{ and } j$. 
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![Diagram of classical and dual punctures](attachment:image.png)
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Historical reminder

Question

Is $B_n$ linear, i.e. the subgroup of a matrix group?

- The *Burau* representation is *not* faithful for $n \geq 5$ [Bigelow, Long, Moody], it *is* faithful for $n = 2, 3$, and for $n = 4$ the question is open.

- The LKB–representation (Ruth Lawrence)

$$B_n \xrightarrow{\mathcal{L}} GL\left(\mathbb{Z}\left[q^{\pm 1}, t^{\pm 1}\right], \frac{n(n-1)}{2}\right)$$

Answer: Yes! $\mathcal{L}$ is faithful for all $n$.

Two proofs: by Daan Krammer and Stephen Bigelow.
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Two proofs : by Daan Krammer and Stephen Bigelow.
Explicit formula for the representation $\mathcal{L}$

$$B_n \overset{\mathcal{L}}{\longrightarrow} GL \left( \mathbb{Z} \left[ q^{\pm 1}, t^{\pm 1} \right], \frac{n(n-1)}{2} \right)$$

Denote the basis vectors of $\mathbb{R}^{\frac{n(n-1)}{2}}$ by $F_{i,j}$ (for $1 \leq i < j \leq n$). Then $\mathcal{L}(\sigma_k)$ sends

$$F_{i,j} \mapsto \begin{cases} 
F_{i,j} & k \not\in \{i-1, i, j-1, j\} \\
qF_{k,j} + (q^2 - q)F_{k,i} + (1 - q)F_{i,j} & k = i - 1 \\
F_{i+1,j} & k = i \neq j - 1 \\
qF_{i,k} + (1 - q)F_{i,j} + (q - q^2)tF_{k,j} & k = j - 1 \neq i \\
F_{i,j+1} & k = j \\
-q^2 tF_{i,j} & k = i = j - 1 
\end{cases}$$
**Krammer’s proof that $\mathcal{L}$ is faithful**

For any $\beta \in B_n$, consider the maximal and minimal powers of $t$ occurring in the matrix $\mathcal{L}(\beta)$.

**Krammer’s main lemma** (which implies faithfulness):

maximal power of $t$ in $\mathcal{L}(\beta) = \sup_{\text{ClassicalGarside}}(\beta)$

minimal power of $t$ in $\mathcal{L}(\beta) = \inf_{\text{ClassicalGarside}}(\beta)$

Krammer conjectured

maximal power of $q$ in $\mathcal{L}(\beta) = 2 \cdot \sup_{\text{DualGarside}}(\beta)$

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**Theorem [Ito,W]** Krammer’s conjecture is true.

The variable $q$ in the LKB-representation detects dual braid length.
Krammer’s proof that $L$ is faithful

For any $\beta \in B_n$, consider the maximal and minimal powers of $t$ occurring in the matrix $L(\beta)$.

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The variable $q$ in the LKB-representation detects dual braid length.

**Note** One direction is easy.
Krammer’s proof that $\mathcal{L}$ is faithful

For any $\beta \in B_n$, consider the maximal and minimal powers of $t$ occurring in the matrix $\mathcal{L}(\beta)$.

Krammer’s main lemma (which implies faithfulness):

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**Theorem [Ito,W]** Krammer’s conjecture is true.

The variable $q$ in the LKB-representntn. detects dual braid length.

We unsuccessfully tried to also reprove Krammer’s result using our techniques.
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3. Proof: Labellings of curve diagrams

4. Proof: further ideas
For any braid \( \beta \in B_n \), consider its \textit{curve diagram} \( \overline{D_\beta} \) with Winding number labeling (WNu) and Wall crossing labeling (WCr).

Look at the maximal and minimal labels of the solid arcs.

\begin{align*}
\text{Thm 1 [W, 2010]} & & \text{Thm 2 [Ito & W, 2011]} \\
\text{Min}_{\text{WNu}}(\beta) &= \inf_{\text{Class.Garside}}(\beta) & \text{Min}_{\text{WCr}}(\beta) &= \inf_{\text{DualGarside}}(\beta) \\
\text{Max}_{\text{WNu}}(\beta) &= \sup_{\text{Class.Garside}}(\beta) & \text{Max}_{\text{WCr}}(\beta) &= \sup_{\text{DualGarside}}(\beta)
\end{align*}
For any braid $\beta \in B_n$, consider its curve diagram $\overline{D_\beta}$ with winding number labeling (WNu) and wall crossing labeling (WCr).

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**Thm 1 [W, 2010]**

$\min_{WNu}(\beta) = \inf_{\text{Class.Garside}}(\beta)$

$\max_{WNu}(\beta) = \sup_{\text{Class.Garside}}(\beta)$

**Thm 2 [Ito & W, 2011]**

$\min_{WCr}(\beta) = \inf_{\text{DualGarside}}(\beta)$

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For any braid $\beta \in B_n$, consider its curve diagram $\overline{D}_\beta$ with:

- **Winding number labeling (WNU)**
- **Wall crossing labeling (WCr)**

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**Thm 2 [Ito & W, 2011]**

- $\text{Min}_{\text{WCr}}(\beta) = \inf_{\text{DualGarside}}(\beta)$
- $\text{Max}_{\text{WCr}}(\beta) = \sup_{\text{DualGarside}}(\beta)$

We will use only this result.
Details for this example: $\beta = (\sigma_2^{-1}\sigma_1)^2$

Check the theorems in this special case:

$\text{Min}_{\text{WNu}}(\beta) = -2$,
$\text{Max}_{\text{WNu}}(\beta) = 2$.

Class. Gars. normal form of $\beta$ is
$\Delta^{-2} \cdot \sigma_1 \sigma_2 \cdot \sigma_2 \cdot \sigma_2 \cdot \sigma_2 \sigma_1$

$\text{Min}_{\text{WCr}}(\beta) = -2$,
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Dual Gars. normal form of $\beta$ is
$\delta^{-2} \cdot a_{2,3} \cdot a_{2,3} \cdot a_{1,2} \cdot a_{1,2}$
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Proof of Theorem 2 (beginning)

**Theorem 2** Wall crossing labelling detects dual Garside length

**Lemma** Let $\beta \in B_n$, and $\hat{\beta}$ a divisor of $\delta$. Action of $\hat{\beta}$ on $D_\beta$: an arc in $D_\beta$ labelled $k$ gives rise to one or several arcs in $D_{\hat{\beta}\cdot\beta}$, labelled $k$ or $k+1$. 

![Diagram](image-url)
Proof of Theorem 2 (end)

**Theorem 2**  Wall crossing labelling detects dual Garside length

For simplicity, suppose $Min_{WCr}(\beta) = 0$.

**Need to prove :** $Max_{WCr}(\beta) = \sup_{Dual}(\beta)$

**Proof of** \(\leq\) : follows from Lemma (acting by a divisor of \(\delta\) can only increase maximal label by 1).

**Proof of** \(\geq\) : by induction on $Max_{WCr}(\beta)$.

1. Construct a collection $P$ of disjoint polygons intersecting all maximally labelled arcs, but none of the minimally labelled ones. Then let $\hat{\beta}$ be the divisor of $\delta$ corresponding to $P$.

2. Prove that acting on $D_\beta$ by $\hat{\beta}^{-1}$ decreases the maximal label without decreasing the minimal label :
   - $Max_{WCr}(\hat{\beta}^{-1}\beta) = Max_{WCr}(\beta) - 1$
   - $Min_{WCr}(\hat{\beta}^{-1}\beta) = Min_{WCr}(\beta) = 0$
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□
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Lawrence’s construction of the representation $\mathcal{L}$

Let $X$ be the configuration space of unordered pairs of points in the $n$-times punctured disk $D_n$:

$$X = \left\{ \{x_1, x_2\} \mid x_1, x_2 \in D_n, x_1 \neq x_2 \right\}$$

equipped with a basepoint $\{d_1, d_2\}$ (see blackboard). There is a homomorphism

$$\pi_1(X) \to \mathbb{Z}^2 = \langle q, t \rangle, \quad \gamma = \{\gamma_1, \gamma_2\} \mapsto q^a t^b$$

where

$a = \text{sum of the winding numbers of } \gamma_1 \text{ and } \gamma_2 \text{ around all } n \text{ punctures}$

$b = 2 \cdot (\text{relative winding number of } \gamma_1 \text{ and } \gamma_2)$

Let $\tilde{X}$ be the cover corresponding to $\ker (\pi_1(X) \to \mathbb{Z}^2)$. Covering group($\tilde{X}$) = $\mathbb{Z}^2 = \langle q, t \rangle$. 
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Let $\tilde{X}$ be the cover corresponding to $\ker (\pi_1(X) \to \mathbb{Z}^2)$.

Covering group($\tilde{X}$) = $\mathbb{Z}^2 = \langle q, t \rangle$. 
Note \( B_n \sim \tilde{X} \implies B_n \sim H_2(\tilde{X}, \partial \tilde{X}; \mathbb{R}[q^\pm, t^\pm]) \)

**Proposition (Lawrence)** The second homology \( H_2(\tilde{X}, \mathbb{R}[q^\pm, t^\pm]) \) is of dimension \( \frac{n(n-1)}{2} \) with generators \( F_{i,j} \) (“forks”) as in the following figure.

Moreover, the \( B_n \)-action on \( H_2(\tilde{X}, \mathbb{R}[q^\pm, t^\pm]) \) coincides with the representation \( \mathcal{L} \).
Bigelow’s Key Lemma

- Recall \( \tilde{X} \) is a 4-dim. mfd. with covering action by \( \mathbb{Z}^2 = \langle q, t \rangle \).
- Recall the “fork” \( F_{i,j} \) is a surface (more precisely a square) in \( \tilde{X} \) representing a generator of \( H_2(\tilde{X}, \mathbb{R}[q^{\pm 1}, t^{\pm 1}]) \).
- Let \( \tilde{X}_{0,0} \) be a fund. domain of the \( \mathbb{Z}^2 \)-action containing the basept. \( \{\tilde{d}_1, \tilde{d}_2\} \) and all forks \( F_{i,j} \). Let \( \tilde{X}_{a,b} = q^a t^b X_{0,0} \).

**Handwaving version of Bigelow’s “Key Lemma”**

Let \( \beta \in B_n \) and \( 1 \leq i < j \leq n \), and consider \( \beta(F_{i,j}) \subset \tilde{X} \). Among the fundamental domains \( \tilde{X}_{a,b} \) intersected by \( \beta(F_{i,j}) \subset \tilde{X} \), select the one \( \tilde{X}_{a_{\text{max}}, b_{\text{max}}} \) with maximal \( (a, b) \) (lexicographically). Now in \( H_2(\tilde{X}, \mathbb{R}[q^{\pm 1}, t^{\pm 1}]) \) we can write uniquely

\[
\beta(F_{i,j}) = \sum_{1 \leq i' < j' \leq n} P_{i',j'}(q, t) F_{i',j'} \quad \text{(with } P_{i',j'} \in \mathbb{R}[q^{\pm 1}, t^{\pm 1}] \text{)}
\]

Then in one of the polynomials \( P_{i',j'} \), the term \( q^{a_{\text{max}}} t^{b_{\text{max}}} \) occurs with non-zero coeff. (“contributions do not cancel in homology”).
Bigelow’s Key Lemma

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Handwaving version of Bigelow’s “Key Lemma” Let $\beta \in B_n$ and $1 \leq i < j \leq n$, and consider $\beta(F_{i,j}) \subset \tilde{X}$. Among the fundamental domains $\tilde{X}_{a,b}$ intersected by $\beta(F_{i,j}) \subset \tilde{X}$, select the one $\tilde{X}_{a_{\text{max}}, b_{\text{max}}}$ with maximal $(a, b)$ (lexicographically). Now in $H_2(\tilde{X}, \mathbb{R}[q^{\pm 1}, t^{\pm 1}])$ we can write uniquely

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Then in one of the polynomials $P_{i', j'}$, the term $q^{a_{\text{max}}} t^{b_{\text{max}}}$ occurs with non-zero coeff. (“contributions do not cancel in homology”).
The LKB representation detects $\sup_{\text{DualGarside}}(\beta)$

Proof that $2 \cdot \sup_{\text{DualGarside}}(\beta) = \text{maximal power of } q \text{ in } \mathcal{L}(\beta)$

Proof of “$\geq$” is easy ($\mathcal{L} : \text{divisors of } \delta \mapsto \text{matrix of } q\text{-degree 2}$).

Proof of “$\leq$”:

$2 \cdot \sup_{\text{Dual}}(\beta) \overset{\text{Thm 2}}{=} 2 \cdot \text{Max}_{\text{WCr}}(\beta)$, the max. wall crossing labeling

the maximal number $a$ such that $\beta.F_{i,i+1}$ intersects $q^a t^b X_{0,0}$ for some $i, b$

Now according to Bigelow’s Key Lemma, the monomial $q^a t^b$ occurs somewhere in the matrix $\mathcal{L}(\beta)$.

$\square$
Questions

1. Recall Theorem 2: for a braid $\beta$, we can read the length of $\beta$ from the wall crossing labellings occurring in the curve diagram of $\beta$.

Question: is there some analogue for $Out(F_n)$, with the sphere system in $(S^1 \times S^2)\# \ldots \#(S^1 \times S^2)$ playing the role of the curve diagram?

2. Is there a generalization of our main theorem to Artin-Tits groups of finite type? The question makes sense, as
   - there are dual Garside structures on such groups (Bessis, T.Brady-Watt)
   - there are Lawrence-Krammer-Bigelow type representations on such groups (Digne, Cohen-Wales).
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   Question: is there some analogue for $\text{Out}(F_n)$, with the sphere system in $(S^1 \times S^2) \# \ldots \# (S^1 \times S^2)$ playing the role of the curve diagram?

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   - there are dual Garside structures on such groups (Bessis, T.Brady-Watt)
   - there are Lawrence-Krammer-Bigelow type representations on such groups (Digne, Cohen-Wales).
1. Classical and dual Garside structure on braid groups

2. The Lawrence-Krammer-Bigelow representation

3. Proof: Labellings of curve diagrams

4. Proof: further ideas