Strongly contracting elements in Garside groups

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Strongly contracting elements in Garside groups

1. Main result: strong contraction in Cay(Garside group)
2. A crash course on Garside theory
3. Proof of the main theorem (ideas)
4. Corollary: Loxodromic action on C_{AL}(Garside group)

Are pA axes in $Mod(S)$ "like" hyperbolic geodesics?

[Duchin & Rafi, 2009] Axes of pA mapping classes in $Cay(Mod(S))$ are contracting, and hence Morse [Behrstock 2006].
Question Are they strongly contracting?

Answer [Rafi & Verberne, 2018] No!
There exists a generating set $\mathcal{K}$ and pA elements in $Mod(\mathbb{S}_5)$ whose axis in $Cay(Mod(\mathbb{S}_5), \mathcal{K})$ is not strongly contracting.

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Theorem (Calvez & W, 2021)

Let B_n = braid group on n strands, and $Z(B_n) = \langle \Delta^2 \rangle$ its center. In the Cayley graph of $B_n/Z(B_n)$ w.r.t. Garside's generating set, the axis of a pA element is strongly contracting.

More generally:

Let G be a Garside group of finite type with cyclic center. In the Cayley graph of $G/Z(G)$ w.r.t. the Garside generating set, the axis of a Morse element is strongly contracting.

Morse

Definition (Morse)

- A quasi-geodesic γ in a metric space X is *Morse* if for every $\Lambda \geqslant 1$, $K \geqslant 0$, there is a number $M_{\Lambda,K}$ such that every (Λ, K)-quasi-geodesic with endpoints on γ remains in a $M_{\Lambda,K}$-neighborhood of γ.
- An infinite order element g in a f.g. $G = \langle S \rangle$ is *Morse* if
  (i) $n \mapsto g^n$ is a quasi-isometric embedding of $\mathbb{Z}$ in $Cay(G, S)$ and
  (ii) the axis $\{g^n \mid n \in \mathbb{Z}\}$ is Morse.

Example

(1) Geodesics in $\mathbb{H}^2$ are Morse. (2) *pAs* in $Mod(S)$ are Morse.

Remark

The Morse property is invariant under quasi-isometry / change of generating set.

Strong contraction

Definition (Strongly contracting)

Let (X, d) be a metric space, and $A \subset X$.
A is *C-strongly contracting* if for every ball B in X disjoint from A, $proj_A(B)$ has diameter $\leqslant C$ (universally bounded).

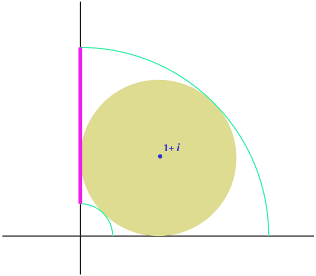
Here, $proj_A(x) = \{a \in A \mid \forall a' \in A,\ d(x,a) \leqslant d(x,a')\}$.

Example

Geodesics in $\mathbb{H}^2$ are $\ln(\frac{\sqrt{2}+1}{\sqrt{2}-1})$-strongly contracting.

Attention

The strong contraction property is **not** invariant under quasi-isometry / change of generating set.

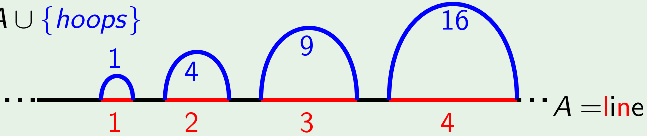


Morse vs. strongly contracting

Recall Strongly contracting $\Rightarrow$ Morse

Example (Morse $\not\Leftarrow$ strongly contracting)

$X = A \cup \{hoops\}$



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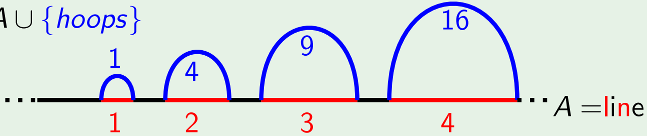

Theorem (Sultan 2014, Cashen 2020)

Suppose $A \subset X$ and X is $CAT(0)$. Then

$$A \text{ Morse} \quad \Rightarrow \quad A \text{ strongly contracting}$$

Thus our theorem ("In Garside, Morse $\Rightarrow$ strongly contracting") says that Garside groups "behave a bit like" $CAT(0)$. Evidence for

Famous conjecture Braid groups are $CAT(0)$

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Our preferred generators of B_n: Garside's generators

"Simple braids", a.k.a. "positive permutation braids":
positive braids, any two strands crossing at most once

$$\Updownarrow$$

Permutations of $\{1, \ldots, n\}$

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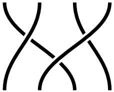
Permutations of $\{1, \ldots, n\}$

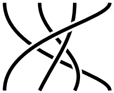
- **Typical example**

  Simple braid $x \in B_4$, permutation $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix}$



- **Very special example**

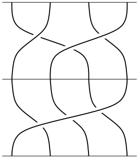
  Half-twist Δ $\quad \leftrightsquigarrow \quad$ permutation $\begin{pmatrix} 1 & \ldots & n \\ n & \ldots & 1 \end{pmatrix}$



- **Property of Δ:** "almost commutes" with all braids (and Δ^2 generates center $Z(B_n)$)

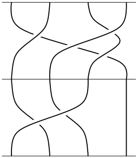
Left-weighting, left normal form

Example



x_1

x_2



$\widetilde{x}_1$

$\widetilde{x}_2$

The product $x_1 \cdot x_2$ is *not* left-weighted; the product $\widetilde{x}_1 \cdot \widetilde{x}_2$ is.

Theorem (Adjan, Thurston, Elrifai–Morton)

Every $x \in B_n$ has a unique representative of the form

$$\Delta^k \cdot x_1 \cdot \ldots \cdot x_\ell \quad (k \in \mathbb{Z}) \quad \text{with } x_i \cdot x_{i+1} \text{ left-weighted } \forall i$$

Notation $k =$ "infimum of x", $k + \ell =$ "supremum of x"

Motivation $\overset{[Garside]}{\rightsquigarrow}$ Solution to word and conjugacy pbm in B_n.

Similarly *Right*-weighted normal form $x'_1 \cdot \ldots \cdot x'_\ell \cdot \Delta^k$

Definition (The prefix ordering)

Partial ordering on B_n:

$$x \preccurlyeq y \quad :\Leftrightarrow \quad \exists \alpha \in B_n^+,\ x \cdot \alpha = y$$

Proposition (Garside)

On B_n^+, the monoid of positive braids, this partial ordering is a lattice ordering: for $x, y \in B_n^+$

$$x \wedge y = g.c.d.(x, y) \quad \text{and} \quad x \vee y = l.c.m.(x, y) \quad \text{exist}$$

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All this works more generally

"Definition" [Dehornoy-Paris] Garside group

A group G is Garside if similar combinatorial machinery works.

Example (Brieskorn-Saito, Deligne, Charney)

Irreducible Artin-Tits groups of spherical type (A_n, B_n, D_n, E_6, E_7, E_8, F_4, H_3, H_4, $I_2(m)$) are Garside.

Bestvina's graph (1999)

Definition (Bestvina's graph)

$\mathcal{X} = Cay(G, S_{Garside})/\langle\Delta\rangle$:

- Vertices $=$ Cosets $g\langle\Delta\rangle$ represented by $\underline{g}$ with $\inf(\underline{g}) = 0$,
- Edge from $g\langle\Delta\rangle$ to $h\langle\Delta\rangle$ if there is $s \in S_{Gars}$ s.t. $\underline{g}s \in h\langle\Delta\rangle$

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Convenient quasi-isometric model for $Cay(G/Z(G), S_{Garside})$:

recall $Z(G) = \langle\Delta^e\rangle$

Lemma

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Want Axes of Morse elements in $\mathcal{X}$ are strongly contracting.

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Proposition (Charney)

Garside normal forms give rise to geodesics in $\mathcal{X}$.

Notation $\mathcal{NF}(g, h) =$ preferred geod. between vertices g and h.

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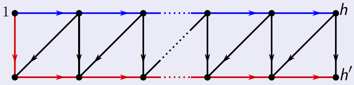
[Charney 1992], [Dehornoy]
If h, h' are adjacent, then $\mathcal{NF}(g, h)$ and $\mathcal{NF}(g, h')$ 1-fellow travel



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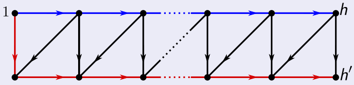
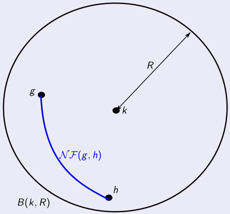


Balls are convex: if $g, h \in B(k,R)$, then $\mathcal{NF}(g,h) \subset B(k,R)$.



Philosophy of the proof

Theorem (reminder)

If G is Garside (fin.type, $Z(G) \cong \mathbb{Z}$), then in $Cay(G/Z(G), S_{Gars}\}$, $axis(g)$ Morse $\Rightarrow$ $axis(g)$ strongly contracting.

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In order to prove strong contraction, one needs excellent control over *geodesics* (not just quasi-geodesics)

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axis(g) *Morse* $\Rightarrow$ *axis*(g) *strongly contracting.*

Difficulty

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In Garside groups, this is not a problem

If G is Garside, then in $\mathcal{X}$ we know a unique preferred geodesic between any pair of vertices (from the Garside normal form). Moreover, these geodesics have good geometric properties, e.g. fellow travelling.

A Garside-theoretical projection $\pi\colon \mathcal{X} \to \text{axis}(x)$

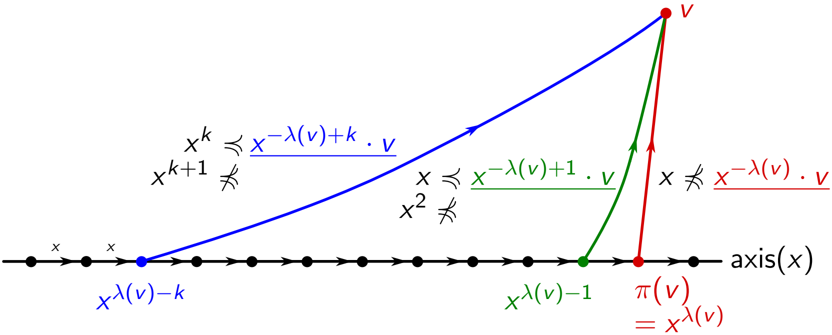
Definition (Projection to *axis*$(x) = \{x^k\langle\Delta\rangle \mid k \in \mathbb{Z}\} \subset \mathcal{X}$)

Let $x \in G$ with $\inf(x) = 0$. Let v be a vertex of $\mathcal{X}$. Define

$$\lambda(v) = -\max\{k \in \mathbb{Z},\ x \not\preccurlyeq \underline{x^k \cdot v}\} \quad \text{and} \quad \pi(v) = x^{\lambda(v)}\langle\Delta\rangle$$

Lemma

Suppose moreover that x is right-rigid. Then this picture holds:



Hyperbolic spaces

Braid groups B_n $\simeq Mod(\mathbb{D}_n)$	$\subset$	Irred. spherical Artin groups A	$\subset$	Garside groups with cyclic center

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$\circlearrowright\ \mathcal{CG}(\mathbb{D}_n)$
curve graph
δ-hyperbolic

[Masur-Minsky]

Hyperbolic spaces

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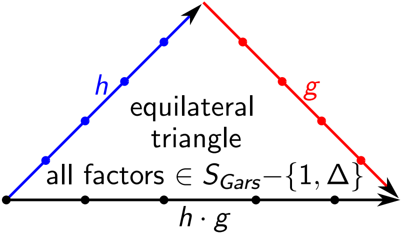
$$\mathcal{CG}(\mathbb{D}_n) \overset{isom}{=} \mathcal{C}_{parab}(B_n)$$

Additional length graph $\mathcal{C}_{AL}$

Definition (Calvez & W, 2017)

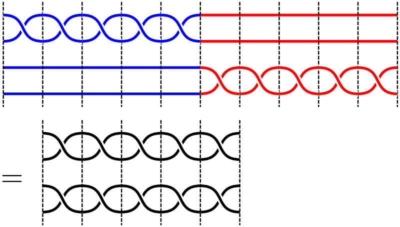
$g \in G$ (Garside group) is *absorbable* if

- $\inf(g) = 0$ or $\sup(g) = 0$ and
- there is some $h \in G$ such that $\inf(hg) = \inf(h)$ and $\sup(hg) = \sup(h)$.



Example (in the braid group B_4)

$g = \sigma_3^{50}$ is absorbable: with $h = \sigma_1^{50}$
we have $hg = (\sigma_1\sigma_3)^{50}$, so
$\inf(hg) = 0 = \inf(h)$ and
$\sup(hg) = 50 = \sup(h)$



$\sigma_1^5\,\sigma_3^5 \;= (\sigma_1\sigma_3)^5$

Definition (Additional length graph – Calvez & W, 2017)

$\mathcal{C}_{AL}(G) = Cay(G, \{\text{Garside genrts}\} \cup \{\text{absorbable elts}\})/\langle\Delta\rangle$

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Corollary (Calvez & W, 2021)

(1) *Suppose G is a Garside group with cyclic center. If $g \in G$ is Morse, then its action on $\mathcal{C}_{AL}(G)$ is loxodromic, WPD.*

(2) *For braid groups B_n:*

- *reducible & finite order braids act elliptically on $\mathcal{C}_{AL}(B_n)$*
- *pA braids act loxodromically, WPD on $\mathcal{C}_{AL}(B_n)$.*