Strongly contracting elements in Garside groups

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1. Main result: strong contraction in Cay(Garside group)

2. Corollary: Loxodromic action on C_{AL}(Garside group)

Are pA axes in $Mod(S)$ "like" hyperbolic geodesics?

[Duchin & Rafi, 2009] Axes of pA mapping classes in $Cay(Mod(S))$ are contracting, and hence Morse [Behrstock 2006].
Question Are they strongly contracting?

Answer [Rafi & Verberne, 2018] No!
There exists a generating set $\mathcal{K}$ and pA elements in $Mod(\mathbb{S}_5)$ whose axis in $Cay(Mod(\mathbb{S}_5), \mathcal{K})$ is not strongly contracting.

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Theorem (Calvez & W, 2021)

Let B_n = braid group on n strands, and $Z(B_n) = \langle \Delta^2 \rangle$ its center. In the Cayley graph of $B_n/Z(B_n)$ w.r.t. Garside's generating set, the axis of a pA element is strongly contracting.

More generally:

Let G be a Garside group of finite type with cyclic center. In the Cayley graph of $G/Z(G)$ w.r.t. the Garside generating set, the axis of a Morse element is strongly contracting.

Morse

Definition (Morse)

- A quasi-geodesic γ in a metric space X is *Morse* if for every $\Lambda \geqslant 1$, $K \geqslant 0$, there is a number $M_{\Lambda,K}$ such that every (Λ, K)-quasi-geodesic with endpoints on γ remains in a $M_{\Lambda,K}$-neighborhood of γ.
- An infinite order element g in a f.g. $G = \langle S \rangle$ is *Morse* if
  (i) $n \mapsto g^n$ is a quasi-isometric embedding of $\mathbb{Z}$ in $Cay(G, S)$ and
  (ii) the axis $\{g^n \mid n \in \mathbb{Z}\}$ is Morse.

Example

(1) Geodesics in $\mathbb{H}^2$ are Morse. (2) *pA*s in $Mod(S)$ are Morse.

Remark

The Morse property is invariant under quasi-isometry / change of generating set.

Strong contraction

Definition (Strongly contracting)

Let (X, d) be a metric space, and $A \subset X$.
A is *C-strongly contracting* if for every ball B in X disjoint from A, $proj_A(B)$ has diameter $\leqslant C$ (universally bounded).

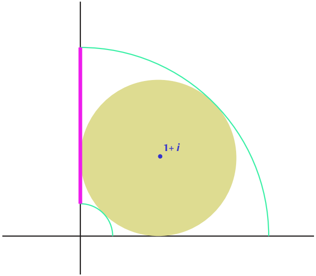
Here, $proj_A(x) = \{a \in A \mid \forall a' \in A,\ d(x,a) \leqslant d(x,a')\}$.

Example

Geodesics in $\mathbb{H}^2$ are $\ln(\frac{\sqrt{2}+1}{\sqrt{2}-1})$-strongly contracting.

Attention

The strong contraction property is **not** invariant under quasi-isometry / change of generating set.

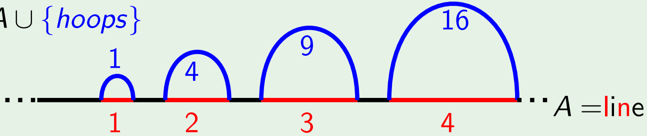


Morse vs. strongly contracting

Recall Strongly contracting $\Rightarrow$ Morse

Example (Morse $\not\Leftarrow$ strongly contracting)

$X = A \cup \{hoops\}$



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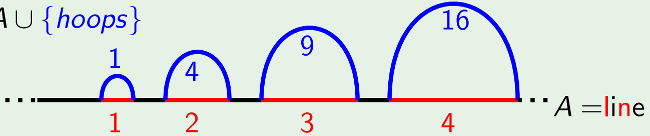

Theorem (Sultan 2014, Cashen 2020)

Suppose $A \subset X$ and X is $CAT(0)$. Then

A Morse $\Rightarrow$ A strongly contracting

Thus our theorem ("In Garside, Morse $\Rightarrow$ strongly contracting") says that Garside groups "behave a bit like" $CAT(0)$. Evidence for

Famous conjecture Braid groups are $CAT(0)$

Proof of the main theorem

Theorem (reminder)

If G is Garside, then in $Cay(G/Z(G), \{Garside\ generators\})$,
$axis(g)$ Morse $\quad\Rightarrow\quad$ *$axis(g)$ strongly contracting.*

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Difficulty

In order to prove strong contraction, one needs excellent control over *geodesics* (not just quasi-geodesics)

In Garside groups, this is not a problem

If G is Garside, then in

$$Cay(G, \text{Garside's generators})$$

we know a unique preferred geodesic between any pair of vertices (coming from the Garside normal form).
Moreover, these geodesics have good geometric properties, e.g. fellow travelling.

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Hyperbolic spaces

Braid groups B_n $\simeq Mod(\mathbb{D}_n)$	$\subset$	Irred. spherical Artin groups A	$\subset$	Garside groups with cyclic center

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$\circlearrowright\ \mathcal{CG}(\mathbb{D}_n)$

curve graph

δ-hyperbolic

[Masur-Minsky]

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$\circlearrowright\ \mathcal{CG}(\mathbb{D}_n)$ curve graph δ-hyperbolic [Masur-Minsky]		$\circlearrowright\ \mathcal{C}_{parab}(A)$ graph of parab. subgps. δ-hyperbolic? [CGGW]	$\overset{q.i.}{\cong}$?	$\circlearrowright\ \mathcal{C}_{AL}(G)$ Additional length graph δ-hyperbolic [Calvez-W 2017]

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Corollary (Calvez & W, 2021)

(1) *Suppose G is a Garside group with cyclic center. If $g \in G$ is Morse, then its action on $\mathcal{C}_{AL}(G)$ is loxodromic, WPD.*

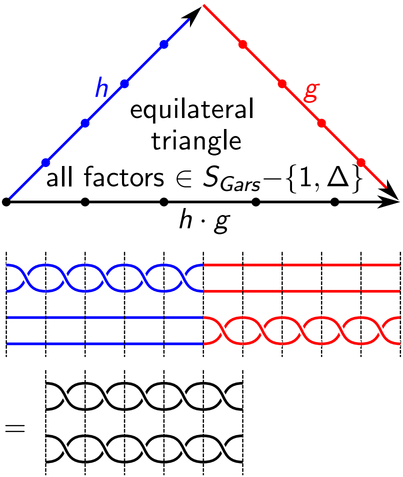
(2) *For braid groups B_n:*
- *reducible & finite order braids act elliptically on $\mathcal{C}_{AL}(B_n)$*
- *pA braids act loxodromically, WPD on $\mathcal{C}_{AL}(B_n)$.*

Additional length graph

Definition (Calvez & W, 2017)

$g \in G$ (Garside group) is *absorbable* if

- $\inf(g) = 0$ or $\sup(g) = 0$ and
- there is some $h \in G$ such that $\inf(hg) = \inf(h)$ and $\sup(hg) = \sup(h)$.



$\color{blue}{\sigma_1^5}\ \color{red}{\sigma_3^5} = (\sigma_1\sigma_3)^5$

Example (in the braid group B_4)

$g = \sigma_3^{50}$ is absorbable: with $h = \sigma_1^{50}$ we have $hg = (\sigma_1\sigma_3)^{50}$, so $\inf(hg) = 0 = \inf(h)$ and $\sup(hg) = 50 = \sup(h)$

Definition (Additional length graph – Calvez & W, 2017)

$\mathcal{C}_{AL}(G) = Cay(G, \{\text{Garside genrts}\} \cup \{\text{absorbable elts}\})/\langle\Delta\rangle$