Fast algorithmic Nielsen-Thurston classification of 4-braids

Matthieu Calvez, Bert Wiest

IRMAR, Université de Rennes 1

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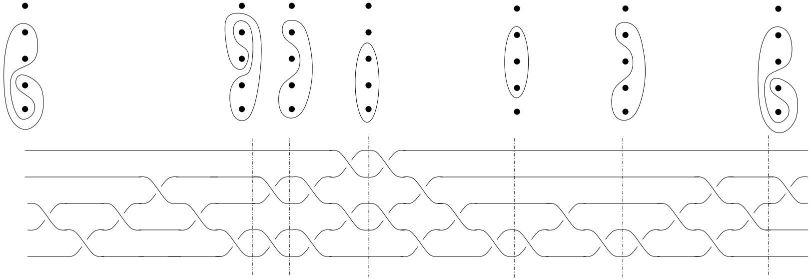
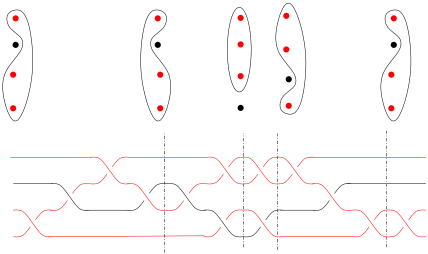
Statement of results...

... on the blackboard.

Examples showing that weaker hypotheses $\Longrightarrow$ wrong statements :





Convention : braid $x \in B_4$,

interior braid $\widehat{x} \in B_3$

Complexity of simple closed curves in D_n

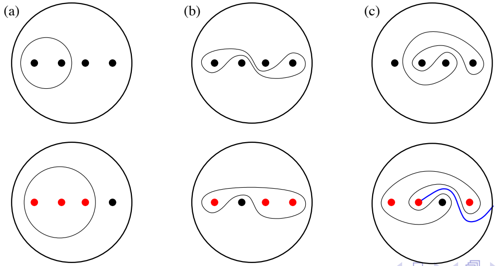
Definition The *complexity* of a (isotopy class of) simple closed curve c in D_n is the minimal Garside length of a braid whose action turns c into a round curve.

Examples

(a) Complexity 0 : round curves

(b) Complexity 1 : almost round curves (projection to x-axis has one max and one min)

(c) Complexity 2 : after an appropriate isotopy which moves points only vertically, the projection to the y-axis has only one max and one min.



Complexity of simple closed curves in D_n

Proposition (V.Gebhardt)

Curve c of complexity k invariant under $x = x_1 \cdot \ldots \cdot x_\ell$, i.e. $c.x = c$

$$\Downarrow$$

Curve $c.x_1 \cdot \ldots \cdot x_i$ is of complexity $\leqslant k$

Conjugating into SSS is fast (quadratic time)

Lemma (Birman-Ko-Lee) $\forall n \in \mathbb{N}$, $\exists$ constant $C_n > 0$,

$$x \in B_n \quad \rightsquigarrow \quad \text{cycle} \;\; C_n \cdot length(x) \;\; \text{times} \quad \rightsquigarrow \quad x' \in SSS(x)$$

Corollary There is an algorithm

- **Input** $x \in B_4$
- **Output** $x' \in SSS(x)$
- **Running time** $O\left((length(x))^2\right)$.

Exercice (N-Th classification in B_3) There is an algorithm

- **Input** $x \in B_3$ (3-strand braid)
- **Output** Information whether x is reducible
- **Running time** $O\left((length(x))^2\right)$.

Proof Exercice ! Idea : (1) cycle into SSS
(2) x was reducible iff resulting braid is of the form $\Delta_3^{2m} \cdot \sigma_i^k$

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3-strand braids

Lemma For $x \in B_3$,

$$x \in SSS \iff x \text{ is rigid}$$

Lemma

If the 3-braid $\Delta_3^m \cdot x_1 \cdot \ldots x_\ell$ is in left-greedy normal form as written

then the braid $x_\ell^{-1} \cdot \ldots \cdot x_1^{-1}$ is in left-greedy normal form as written.

Correction, added after the talk : I meant the reverse braid, not the inverse

(In other words, LGNF $=$ RGNF, except for factors Δ.)

4-strand braids

Lemma

Suppose $x \in B_4^+$ has left greedy normal form

$$x = x_1 \cdot x_2 \cdot \ldots \cdot x_\ell \quad \text{(possibly some initial factors } = \Delta\text{)}$$

Suppose

- $x \in SSS$
- x is reducible with a reduction curve surrounding 3 punctures.

Let $\widehat{x} \in B_3$ be its interior braid, so

$$\widehat{x} = \widehat{x}_1 \cdot \widehat{x}_2 \cdot \ldots \cdot \widehat{x}_\ell \quad \text{where } \widehat{x}_i = (x_i \text{ with 4th strand removed}).$$

Then $\widehat{x} \in SSS$ ($\Longrightarrow$ rigid) and

the left-greedy normal form of $\widehat{x}$ is as written above.

Proof of main theorem (strategy)

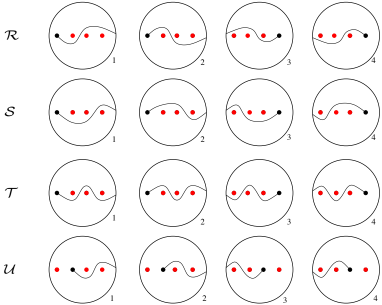
Proof by contradiction : suppose Theorem false.

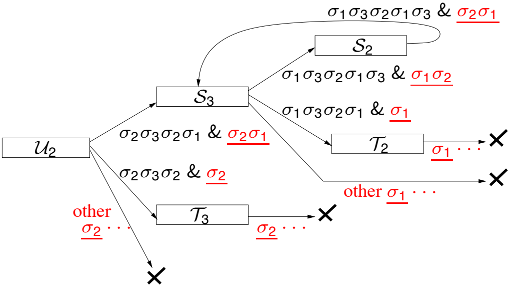
Then $\exists x \in B_4$ s.t.

- $x \in SSS(x)$
- x is reducible, with a reduction curve surrounding 3 punctures *of complexity* $= 2$

Now for every curve of complexity 2 show that it cannot be preserved by x. E.g. : ...

Proof that if $\mathcal{U}_2$ is a reduction curve, then the interior braid has to start with σ_1 : suppose, for a contradiction, that the inner component starts with the letter σ_2. (Bold crosses = curves of complexity > 2).





This picture concerns the case : initial curve = $\mathcal{U}_2$.
We must draw a similar picture for every other curve of complexity 2.

Outlook

Next aim :

solving the conjugacy problem for pA 4-strand-braids in polynomial time.

Fast
Algorithmic N-Th. classification of 4-braids

w/ Matthieu Calvez

This talk: $B_n = MCG(\text{⊙})$, classical Garside structure

Pbm 1 Final algo

Input: $x \in B_n$ of length l

Output: Info whether x is
- periodic (~~fast & easy~~) → fast & easy
- reducible (and if so, describe red. curves)
- pA

In time $O(P_n(l))$ ($P_n \in \mathbb{Z}[l]$)

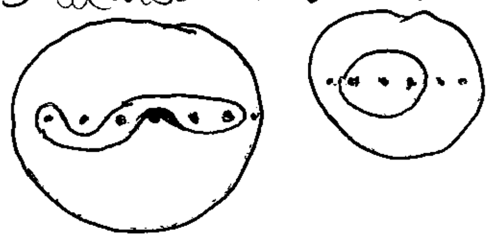
Equiv: detect whether x reducible

| Say One approach: Bestvina Handel tr tr machinery |

Our approach: Garside thy

~~Conj~~ Thm Let $x \in B_4$ ~~B_n~~ be reducible, $\in SSS(\beta_n)$, with red. curve surrounding three punctures

Then reducibility is obvious: $\exists$ reducing curve which is almost round.



Rk Thm $\Longrightarrow$ can solve Pbm 1 for $n=4$ (w/ $P_4(l) = l^2$)

(Pf: Take rnd power of x, squash $D_4 \to S^2_{5\,\text{punct.}}$, ...)