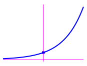
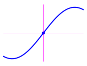
VI Formulaire de développements limités en 0

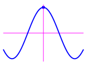
Développements limités classiques en 0, à connaître par cœur. Chaque fonction $\epsilon(x)$ vérifie $\lim_{x\to 0} \epsilon(x) = 0$.



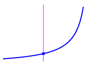
$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!} + x^n\epsilon(x) \qquad = \sum_{k=0}^{n} \frac{x^k}{k!} + x^n\epsilon(x)$$



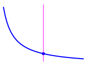
$$\sin(x) = \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + x^{2n+1}\epsilon(x) \qquad = \sum_{k=0}^{n} (-1)^k \frac{x^{2k+1}}{(2k+1)!} + x^{2n+1}\epsilon(x)$$



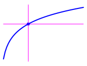
$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots + (-1)^n \frac{x^{2n}}{(2n)!} + x^{2n}\epsilon(x) \qquad = \sum_{k=0}^{n} (-1)^k \frac{x^{2k}}{(2k)!} + x^{2n}\epsilon(x)$$



$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots + x^n + x^n\epsilon(x) \qquad = \sum_{k=0}^{n} x^k + x^n\epsilon(x)$$



$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 + \dots + (-1)^n x^n + x^n\epsilon(x) \qquad = \sum_{k=0}^{n} (-1)^k x^k + x^n\epsilon(x)$$



$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots + (-1)^{n+1} \frac{x^n}{n} + x^n\epsilon(x) \qquad = \sum_{k=1}^{n} (-1)^{k+1} \frac{x^k}{k} + x^n\epsilon(x)$$

$$(1+x)^\alpha = 1 + \frac{\alpha}{1!}x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \dots + \binom{\alpha}{n}x^n + x^n\epsilon(x) \qquad = \sum_{k=0}^{n} \binom{\alpha}{k} x^k + x^n\varepsilon(x)$$

Rappel : $n! = 1 \times 2 \times 3 \times \dots \times n$, et $\binom{\alpha}{n} = \frac{\alpha(\alpha-1)(\alpha-2)\dots(\alpha-(n-1))}{n!}$.
Lien avec les graphiques : le terme constant du DL est la valeur en 0, le terme en x donne la pente de la tangente, le signe du terme en x^2 donne la convexité (courbée vers le haut comme x^2 si le terme en x^2 est positif).

Quelques exemples

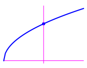
$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + x^5\epsilon(x) \qquad \ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} + x^5\epsilon(x)$$

$$\sin(x) = \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} + x^9\epsilon(x) \qquad \cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} + x^8\epsilon(x)$$

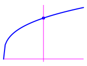
$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + x^5 + x^5\epsilon(x) \qquad \frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - x^5 + x^5\epsilon(x)$$

Développements limités classiques qui ne sont pas à connaître par cœur (mais qu'il faut savoir retrouver)

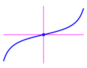
Méthode



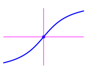
$$\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{5}{128}x^4 + x^4\epsilon(x) \qquad \alpha = \frac{1}{2}$$



$$\sqrt[3]{1+x} = 1 + \frac{1}{3}x - \frac{1}{9}x^2 + \frac{5}{81}x^3 - \frac{10}{243}x^4 + x^4\epsilon(x) \qquad \alpha = \frac{1}{3}$$



$$\tan(x) = x + \frac{x^3}{3} + \frac{2x^5}{15} + x^5\epsilon(x) \qquad \frac{\sin(x)}{\cos(x)}$$



$$\arctan(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} + x^9\epsilon(x) \qquad \int_0^x \frac{dt}{1+t^2}$$

Formules de Taylor-Young (à connaître) :

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \dots + \frac{f^{(n)}(0)}{n!}x^n + x^n\epsilon(x)), \quad \lim_{x\to 0}\epsilon(x) = 0.$$

$$f(a+h) = f(a) + f'(a)h + \frac{f''(a)}{2!}h^2 + \dots + \frac{f^{(n)}(a)}{n!}h^n + h^n\epsilon(h)), \quad \lim_{h\to 0}\epsilon(h) = 0.$$