Perfectoid fields & spaces

- Problems of mixed characteristic
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- 2012

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I. First examples

$$n = a_d p^d + \ldots + a_1 p + a_0 = \sum_{i=0}^{d} a_i p^i \text{ with for all } i, \quad a_i \in [\![0; p-1]\!].$$

$$f(t) = \overline{a_d} t^d + \ldots + \overline{a_1} t + \overline{a_0} = \sum_{i=0}^{d} \overline{a_i} t^i \text{ with for all } i, \quad \overline{a_i} \in \mathbb{F}_p.$$

$$\begin{array}{ccc} \mathbb{N} & \longleftrightarrow & \mathbb{F}_p[t] \\ \sum_{i=0}^{d} a_i p^i & \longmapsto & \sum_{i=0}^{d} \overline{a_i} t^i \end{array}$$

This bijection does not preserve the operations + and x.

I. First examples

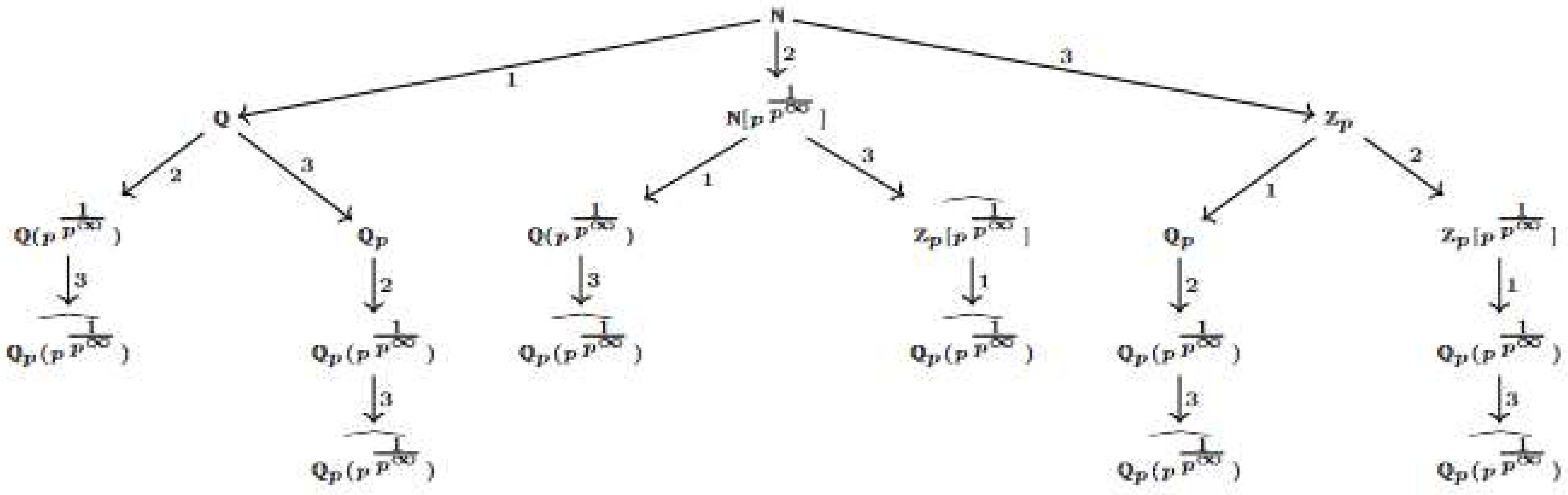
1. Localization
$$\begin{array}{ccc} \mathbb{N}[p^{-1}] & \longleftrightarrow & \mathbb{F}_p[t,t^{-1}] \\ \sum\limits_{i=-e}^{d} a_i p^i & \longmapsto & \sum\limits_{i=-e}^{d} \overline{a_i} t^i \end{array}$$

2. Perfectization
$$\begin{array}{ccc} \mathbb{N}[p^{\frac{1}{p^n}}] & \longleftrightarrow & \mathbb{F}_p[t^{\frac{1}{p^n}}] \\ \sum\limits_{i=0}^{d} a_i p^{\frac{i}{p^n}} & \longmapsto & \sum\limits_{i=0}^{d} \overline{a_i} t^{\frac{i}{p^n}} \end{array}$$

3. Completion
$$\begin{array}{ccc} \mathbb{Z}_p & \longleftrightarrow & \mathbb{F}_p[[t]] \\ \sum\limits_{i=0}^{+\infty} a_i p^i & \longmapsto & \sum\limits_{i=0}^{+\infty} \overline{a_i} t^i \end{array}$$

$$\begin{tikzcd}
\mathbb{Q} & & & & \mathbb{N}[p^{\frac{1}{p^\infty}}] \\
& \mathbb{N}[p^{-1}] \arrow[ul, "1"'] & & \mathbb{N}[p^{\frac{1}{p^n}}] \arrow[ur, "2"] & \\
& & \mathbb{N} \arrow[ul, "1"] \arrow[ur, "2"'] \arrow[d, "3"] & & \\
& & \mathbb{Z}_p & &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{F}_p(t) & & & & \mathbb{F}_p[t^{\frac{1}{p^\infty}}] \\
& \mathbb{F}_p[t,t^{-1}] \arrow[ul, "1"'] & & \mathbb{F}_p[t^{\frac{1}{p^n}}] \arrow[ur, "2"] & \\
& & \mathbb{F}_p[t] \arrow[ul, "1"] \arrow[ur, "2"'] \arrow[d, "3"] & & \\
& & \mathbb{F}_p[[t]] & &
\end{tikzcd}$$

I. First examples



I. First examples

$$
\begin{tikzcd}
 & & \mathbb{F}_p[t] \arrow[dll, "1"'] \arrow[d, "2"] \arrow[drr, "3"] & & & \\
\mathbb{F}_p(t) \arrow[d, "2"] \arrow[dr, "3"] & & \mathbb{F}_p[t^{\frac{1}{p^{\infty}}}] \arrow[d, "1"'] \arrow[dr, "3"] & & \mathbb{F}_p[[t]] \arrow[d, "1"'] \arrow[dr, "2"] & \\
\mathbb{F}_p(t^{\frac{1}{p^{\infty}}}) \arrow[d, "3"] & \mathbb{F}_p((t)) \arrow[d, "2"] & \mathbb{F}_p(t^{\frac{1}{p^{\infty}}}) \arrow[d, "3"] & \mathbb{F}_p\widehat{[[t^{\frac{1}{p^{\infty}}}]]} \arrow[d, "1"] & \mathbb{F}_p((t)) \arrow[d, "2"] & \mathbb{F}_p[[t^{\frac{1}{p^{\infty}}}]] \arrow[d, "1"] \\
\mathbb{F}_p\widehat{((t^{\frac{1}{p^{\infty}}}))} & \mathbb{F}_p((t^{\frac{1}{p^{\infty}}})) \arrow[d, "3"] & \mathbb{F}_p\widehat{((t^{\frac{1}{p^{\infty}}}))} & \mathbb{F}_p\widehat{((t^{\frac{1}{p^{\infty}}}))} & \mathbb{F}_p((t^{\frac{1}{p^{\infty}}})) \arrow[d, "3"] & \mathbb{F}_p((t^{\frac{1}{p^{\infty}}})) \arrow[d, "3"] \\
 & \mathbb{F}_p\widehat{((t^{\frac{1}{p^{\infty}}}))} & & & \mathbb{F}_p\widehat{((t^{\frac{1}{p^{\infty}}}))} & \mathbb{F}_p\widehat{((t^{\frac{1}{p^{\infty}}}))}
\end{tikzcd}
$$

I. First examples

$\mathbb{N}$	$\mathbb{F}_p[t]$
$\mathbb{N}[p^{-1}]$	$\mathbb{F}_p[t,t^{-1}]$
$\mathbb{Q}$	$\mathbb{F}_p(t)$
$\mathbb{N}[p^{\frac{1}{p^n}}]$	$\mathbb{F}_p[t^{\frac{1}{p^n}}]$
$\mathbb{N}[p^{\frac{1}{p^\infty}}]$	$\mathbb{F}_p[t^{\frac{1}{p^\infty}}]$
$\mathbb{Z}_p$	$\mathbb{F}_p[[t]]$
$\mathbb{Q}_p$	$\mathbb{F}_p((t))$
$\mathbb{Q}(p^{\frac{1}{p^\infty}})$	$\mathbb{F}_p(t^{\frac{1}{p^\infty}})$
$\mathbb{Z}_p[p^{\frac{1}{p^\infty}}]$	$\mathbb{F}_p[[t^{\frac{1}{p^\infty}}]]$
$\widehat{\mathbb{Z}_p[p^{\frac{1}{p^\infty}}]}$	$\widehat{\mathbb{F}_p[[t^{\frac{1}{p^\infty}}]]}$
$\mathbb{Q}_p(p^{\frac{1}{p^\infty}})$	$\mathbb{F}_p((t^{\frac{1}{p^\infty}}))$
$\widehat{\mathbb{Q}_p(p^{\frac{1}{p^\infty}})}$	$\widehat{\mathbb{F}_p((t^{\frac{1}{p^\infty}}))}$

We obtain our first two perfectoid fields : $\widehat{\mathbb{Q}_p(p^{\frac{1}{p^\infty}})}$ and its tilt $\widehat{\mathbb{F}_p((t^{\frac{1}{p^\infty}}))}$.

II.A. Valuation

Absolute value : $|.| : R$ (integral domain) $\longrightarrow \mathbb{R}_+$, for all x,y in R :

1. $|x| = 0$ iff $x = 0$.
2. $|xy| = |x||y|$.
3. $|x+y| \leq |x| + |y|$.

Distance : d is defined by for all x,y in R : $d(x,y) = |x - y|$.

Ultrametric absolute value : absolute value $|.|$ with :

3 bis. $|x+y| \leq \max(|x|,|y|)$.

II.A. Valuation

Valuation : $v : R$ (ring) $\longrightarrow \Gamma \cup \{+\infty\}$ (abelian totally ordered group with infinity such that $\forall \gamma \in \Gamma, +\infty \geq \gamma$ and $(+\infty)+\gamma = \gamma+(+\infty) = (+\infty)+(+\infty) = +\infty$), for all x,y in R :

1. $v(0_R) = +\infty$ and $v(1_R) = 0_\Gamma$.

2. $v(xy) = v(x) + v(y)$.

3. $v(x+y) \geq \min(v(x),v(y))$.

Ultrametric absolute value: If R is an integral domain and Γ is the totally ordered group of real number with +, then $|.|_v$, defined by for all x in R, $|x|_v = \exp(-v(x))$, is an ultrametric absolute value.

II.A. Valuation

Valuation ring : The valuation ring of a valued field K is the ring $\{x \text{ in } K \mid v(x) \geq 0\}$.

Rank of a valuation : The rank of a valuation is the Krull dimension of its valuation ring.

A valuation ring is a local ring. So, if K is a valued field, the residue field of K is the residue field of its valuation ring.

II.B. Perfectoid field

Perfectoid field : A perfectoid field K is a complete topological field whose topology is induced by a non-discrete valuation of rank 1, such that the Frobenius endomorphism is surjective on $K^\circ/(p)$ where K° denotes the valuation ring of K where p is the characteristic of the residue field of K.

II.B. Perfectoid field

A field of characteristic $p>0$ is a perfectoid field iff it is complete and perfect.

If L is a finite extension of a perfectoid field K, then L is perfectoid.

If L is an algebraic extension of a perfectoid field K, then $\hat{L}$ is perfectoid.

II.C. Tilt of a perfectoid field

Tilt : Let K be a perfectoid field, the tilt of K, denoted $K^\flat$, is the following inverse limit (projective limit) :

$$K^\flat := \varprojlim_{x\mapsto x^p} K = \{(\dots, x_2, x_1, x_0), x_i \in k, x_{i+1}^p = x_i\}$$

$$\begin{tikzcd}
 & & K^\flat \arrow[dll, bend right, "\pi_0"'] \arrow[dl, bend right, "\pi_1"'] \arrow[d, "\pi_2"] \arrow[dr, bend left] & \\
K \arrow[r, "x\mapsto x^p"] & K \arrow[r, "x\mapsto x^p"] & K \arrow[r, "x\mapsto x^p"] & \dots
\end{tikzcd}$$

$$(\dots, x_2, x_1, x_0)(\dots, y_2, y_1, y_0) = (\dots, x_2y_2, x_1y_1, x_0y_0)$$

$$(\dots, x_2, x_1, x_0) + (\dots, y_2, y_1, y_0) = (\dots, z_2, z_1, z_0)$$

$$z_i = \lim_{n\to+\infty} (x_{i+n} + y_{i+n})^{p^n}$$

II.C. Tilt of a perfectoid field

$\widehat{\mathbb{Q}_p(p^{\frac{1}{p^\infty}})}$ is a perfectoid field and its tilt is $\widehat{\mathbb{F}_p((t^{\frac{1}{p^\infty}}))}$.

$\mathbb{C}_p := \widehat{\overline{\mathbb{Q}}_p}$ is a perfectoid field and its tilt is $\mathbb{C}_p^\flat = \widehat{\overline{\mathbb{F}_p((t))}}$.

II.C. Tilt of a perfectoid field

The tilt of a perfectoid field which has residue field of characteristic $p>0$ is also a perfectoid field and its characteristic is p.

If K is a perfectoid field of characteristic $p>0$, then it is its own tilt.

A perfectoid field K is algebraically closed iff $K^\flat$ is algebraically closed.

<u>Theorem (Fontaine and Wintenberger):</u>

$$\mathrm{Gal}(K^{sep}/K) \simeq \mathrm{Gal}({K^\flat}^{sep}/K^\flat)$$

III.A. Adic space

I-adic topology : Let R be a commutative ring and I an ideal of R. The I-adic topology on the ring R is the unique topology such that I^n, with n in $\mathbb{N}$, form a neighborhood basis of 0.

An I-adic ring is a ring with a I-adic topology.

Huber ring : A Huber ring A is a topological commutative ring which has an open subring which is an I-adic ring R where I is a finitely generated ideal. The open subring is called the definition ring of A and I is called the definition ideal of A.

For example, $\mathbb{Q}_p$ is a Huber ring, $\mathbb{Z}_p$ is its definition ring and (p) its definition ideal.

III.A. Adic space

A power-bounded element ϖ of a topological ring R is an element such that for all neighborhood U of 0, there exists an other neighborhood V such that for all n in $\mathbb{N}$, $\varpi^n V \subset U$.

We denote R° the subring of all power-bounded elements of R.

A topologically nilpotent element ϖ of a topological ring is an element such that ϖ^n tends to 0 when n tends to $+\infty$.

A pseudo-uniformizer ϖ of a topological ring is a topologically nilpotent element which is invertible.

Tate ring : A Tate ring is a Huber ring including a pseudo-uniformizer.

For example, $\mathbb{Q}_p$ is a Tate ring and p is one of its pseudo-uniformizer.

III.A. Adic space

Huber pair : Let R be a commutative ring and I an ideal of R. A Huber pair (R, R^+) is a tuple where is a Huber ring and R^+ is a integrally closed open subring of R^o.

Tate pair : A Tate pair (R, R^+) is a Huber pair where R is a Tate ring.

Two valuations are said to be equivalent if they have the same valuation ring.

Adic spectrum: The adic spectrum Spa(A, A^+) of a Huber pair (A, A^+) is the set of equivalence classes of valuation continuous on A and positive on A^+.

A rational subset of an adic spectrum X of a Huber pair (A, A^+) is a subset :

$$R\left(\frac{f_1, \ldots, f_n}{g}\right) := \{v \in X | \forall i \in [\![1; n]\!], v(f_i) \geq v(g) \neq +\infty\} = \{v \in X | \forall i \in [\![1; n]\!], |f_i|_v \leq |g|_v \neq 0\}$$

with $f_1, \ldots, f_n$ in A generating an open ideal and g in A.

III.A. Adic space

We put on an adic spectrum a topology such that the rational subsets form an open basis.

We define $\mathscr{O}_X = \mathscr{O}_{\mathrm{Spa}(A,A+)}$ as a presheaf of ring on an adic space X such that for all U rational subset $\mathscr{O}_X(U) = A[\widehat{\frac{f_1}{g}, \ldots, \frac{f_n}{g}}]$ and $\mathscr{O}_X(U) = \varprojlim_{V \subset U,\ V \in \mathscr{B}} \mathscr{F}_0(V)$ otherwise with $\mathscr{F}_0$ the $\mathscr{B}$-sheaf with $\mathscr{B}$ the open basis of rational subsets.

This isn't always a sheaf.

III.A. Adic space

Adic space : An affinoid adic space is a topologically locally ringed space such that all stalks are valuation ring, which is isomorphic to an adic spectrum Spa(A, A^+). An adic space is a topologically locally ringed space such that all stalks are valuation ring, admitting an open cover of affinoid adic spaces. If (A, A^+) is a Tate pair, the affinoid adic space is called analytic affinoid adic space.

III.B. Perfectoid space

Perfectoid algebra : Let K be a perfectoid field. A perfectoid K-algebra A is a Banach K-algebra such that A^{o} is open and bounded and the Frobenius endomorphism is surjective on $A^{\circ}/(p)$.

Theorem (Tilting equivalence):
There is equivalence of categories, called the tilting equivalence, between the category of perfectoid K-algebras and the category of perfectoid $K^{\flat}$-algebras.

III.B. Perfectoid space

Perfectoid affinoid algebra : Let K be a field. An affinoid K-algebra (A, A^+) is a Tate pair such that A is a K-algebra.
Let K be a perfectoid field. A perfectoid affinoid K-algebra (A, A^+) is an affinoid K-algebra which is a perfectoid K-algebra.

Perfectoid space : An affinoid perfectoid space over a perfectoid field K is an analytic affinoid adic space which is isomorphic to $\mathrm{Spa}(A, A^+)$ where (A, A^+) is a perfectoid affinoid K-algebra. A perfectoid space over a perfectoid field K is an analytic adic space locally isomorphic to an affinoid perfectoid space.

III.B. Perfectoid space

Theorem :

There is equivalence of categories between the category of perfectoid space over K and the category of perfectoid space over $K^{\flat}$.

III.B. Perfectoid space

<u>Theorem (Almost purity theorem):</u>

Let K be a perfectoid field.

If $X \to Y$ is a finite étale morphism of adic spaces over K and Y is perfectoid, then X also is perfectoid.

A morphism $X \to Y$ of perfectoid spaces over K is finite étale if and only if the tilt $X^\flat \to Y^\flat$ is finite étale over $K^\flat$.